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TENSOR PRODUCTS, INFINITE PRODUCTS, AND PROJECTIVE LIMITS OF CHOQUET SIMPLEXES

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Introduction.

In this paper we prove that the weak tensor product of two simplex spaces is a simplex space, that the projective tensor product of two compact Choquet simplexes is a compact Choquet simplex, and give representations of these tensor products. We then prove that the space of multilinear functions on the infinite cartesian product of a family of compact Choquet simplexes is a simplex space for the natural partial ordering and supremum norm. We then show the existence of projective limits in the category of compact Choquet simplexes and linear homomorphisms, the existence of projective limits in the category of compact Choquet simplexes and linear boundary preserving homomorphisms, and use these results to give a representation of an inductive limit of an inductive family of simplex spaces.

Prerequisites.

An (AL)-space is a Banach lattice V in which norm and order are related by

(i) $\|x+y\|=\|x-y\|$ whenever $x,y \in V$ with $x \wedge y=0$,
(ii) $\|x+y\|=\|x\|+\|y\|$ whenever $0 \leqq x,y \in V$.

An (AL)-space is a boundedly complete vector lattice, and the norm is additive on its positive cone [9, pp. 100, 107]. Choquet [5], [6] and Kendall [14] show that the base

$$\{x \in V:\ 0 \leqq x,\ \|x\|=1\}$$

and the cap

$$\{x \in V:\ 0 \leqq x,\ \|x\| \leqq 1\}$$

of the positive cone are linearly compact Choquet simplexes. If V is a Banach space dual with weak*-closed positive cone, then the cap is weak*-compact, and the base is weak*-compact if it is weak*-closed.

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According to E. G. Effros [11], a *simplex space* is a real partially ordered Banach space A whose dual space is an (AL)-space for the dual partial ordering. A is said to have the Riesz separation property (r.s.p.) if when $f,g \leqq h,k \in A$, there is some $u \in A$ with $f,g \leqq u \leqq h,k$. The following intrinsic characterisation of simplex spaces and its corollary are due to Davies [7], [8].

THEOREM 1. *A is a simplex space if and only if it is a real partially ordered Banach space with the* r.s.p. *and closed positive cone in which norm and order are related by*

(i) $\|f\| \leqq \|g\|$ *whenever* $f,g \in A$ *with* $-g \leqq f \leqq g$;
(ii) *whenever* $f,g \in A$, *there exists* $h \in A$ *with* $f,g \leqq h$, *and such that* $\|h\| \leqq \|f\| \vee \|g\|$.

COROLLARY 1. *If A is a normed linear space satisfying the conditions of the theorem (except that of being a Banach space) then the norm completion $\bar{A}$ of A is a simplex space whose positive cone is the closure in $\bar{A}$ of the positive cone in A.*

If A is a simplex space with dual cone $K(A)$, then the base

$$\{x \in K(A) : \|x\| = 1\}$$

is weak*-compact if and only if A has a norm unit [11]. A *norm unit* is a positive element $e \in A$ such that the unit ball in A is the set

$$\{f \in A : -e \leqq f \leqq e\}.$$

If such an e exists it is unique, and in this case the sets

$$\{x \in K(A) : \|x\| = 1\} \quad \text{and} \quad \{x \in K(A) : \langle e,x\rangle = 1\}$$

are identical. If A is a simplex space we denote by $X(A)$, or simply by X, the weak*-compact Choquet simplex

$$\{x \in A' : 0 \leqq x,\ \|x\| \leqq 1 \text{ and } \langle e,x\rangle = 1 \text{ whenever } e \text{ is a norm unit of } A\}.$$

Therefore X is either a base or a cap for $K(A)$ according as A has a norm unit or not. If X is a base it does not contain the origin. If X is a cap it will contain the origin as an extreme point. *Henceforward a simplex is a compact Choquet simplex in a locally convex Hausdorff topological vector space, which is either disjoint from the origin, or else contains the origin as an extreme point.* An affine function f from a simplex X into a linear space is said to be *linear* if $f(0) = 0$ provided $0 \in X$.

If X is a simplex, and Y is a convex set in a linear topological space, then $\mathscr{A}(X, Y)$ denotes the space of all continuous affine functions from

X to Y, while $\mathscr{A}_0(X, Y)$ denotes the space of continuous linear functions from X to Y:

$$\mathscr{A}_0(X, Y) = \{f \in \mathscr{A}(X, Y) : f(0) = 0\}.$$

So if $0 \notin X$ then $\mathscr{A}_0(X, Y) = \mathscr{A}(X, Y)$, but if $0 \in X$ and $0 \notin Y$, then $\mathscr{A}_0(X, Y)$ is void. When Y is the real field we write $\mathscr{A}(X)$ and $\mathscr{A}_0(X)$. If X is a simplex then X is homeomorphic with $X(\mathscr{A}_0(X))$ under the natural map, and we shall not distinguish between the two simplexes.

If $A_1, \ldots, A_n$ are simplex spaces, then $B(A_1, \ldots, A_n)$ is the space of bounded real multilinear forms on $A_1 \times \ldots \times A_n$, and

$$K(A_1, \ldots, A_n) = \{T \in B(A_1, \ldots, A_n) : T(f_1, \ldots, f_n) \geqq 0 \text{ whenever } 0 \leqq f_i \in A_i,\ i = 1, \ldots, n\}$$

is defined as the positive cone in $B(A_1, \ldots, A_n)$. If $\{X_i : i \in I\}$ is a family of simplexes and Y is a convex subset of a linear space, then a function F from the cartesian product $\times\{X_i : i \in I\}$ to Y is *multiaffine* (*multilinear*) if it is an affine (linear) function in each co-ordinate variable. Thus F is multilinear if it is multiaffine and if $F(\{x_i\}) = 0$ whenever $\{x_i\} \in \times\{X_i : i \in I\}$ has a zero entry. If Y is a convex set in a linear topological space, then $\mathscr{M}\mathscr{A}(\{X_i\} : Y)$ and $\mathscr{M}\mathscr{A}_0(\{X_i\} : Y)$ denote respectively the spaces of continuous multiaffine and multilinear functions from the topological cartesian product $\times\{X_i : i \in I\}$ to Y.

A projective topological tensor product of a family $\{X_i : i \in I\}$ of simplexes is a simplex X together with a function $P \in \mathscr{M}\mathscr{A}_0(\{X_i\}, X)$ such that for every simplex Y, and every $F \in \mathscr{M}\mathscr{A}_0(\{X_i\}, Y)$ there exists a unique $F' \in \mathscr{A}_0(X, Y)$ such that $F = F' \circ P$. A projective topological tensor product of simplexes is unique up to linear homeomorphism and will be denoted by $\hat{\otimes}\{X_i : i \in I\}$. If none of the simplexes $\{X_i : i \in I\}$ contain the origin, and $\hat{\otimes}\{X_i : i \in I\}$ exists, then it is the tensor product of the family $\{X_i : i \in I\}$ considered only as compact convex sets in linear topological spaces [16].

We state here two density results we shall use later.

LEMMA 1. *If X is a simplex, and if L is a subspace of $\mathscr{A}_0(X)$ which contains the constant functions (if any) and separates the points of X, then L is dense in $\mathscr{A}_0(X)$.*

If X is a compact Hausdorff space and L is a subspace of $C(X)$ which contains the constant functions and separates points, then the Choquet boundary of L,

$$\partial_L(X) = \{x \in X : \varepsilon_x \text{ is an extreme point of the unit ball of the dual of } L\}.$$

Here ε_x is the evaluation functional at x. If X is a simplex, then the Choquet boundary of $\mathscr{A}(X)$ is the set of extreme points of X. The following may be deduced from the results of [10].

THEOREM 2. *Let X be a compact Hausdorff space, and let L and M be subspaces of $C(X)$ which contain the constant functions and separate points. If L is a subspace of M with the* r.s.p. *and if $\partial_L(X)=\partial_M(X)$, then L is uniformly dense in M.*

Finite tensor products.

In this section $A_1,\ldots,A_n$ are simplex spaces with positive cones $A_1^+,\ldots,A_n^+$. We write X_i for $X(A_i)$, $i=1\ldots n$.

LEMMA 2.

(i) *If $T\in K(A_1,A_2)$, then*

$$\|T\|=\sup\{T(h_1,h_2) : h_i\in A_i^+,\ \|h_i\|\leqq 1,\ i=1,2\}.$$

(ii) *If $T,U\in K(A_1,A_2)$, then $\|T+U\|=\|T\|+\|U\|$.*

(iii) *If T is a positive real-valued bilinear function on $A_1^+\times A_2^+$ such that*

$$\sup\{T(h,h_2) : h_i\in A_i^+,\ \|h_i\|\leqq 1,\ i=1,2\}<\infty,$$

then T has a unique extension to an element of $K(A_1,A_2)$ again denoted by T.

PROOF. (i) If $T,U\in K(A_1,A_2)$, then given $\varepsilon>0$ there exist $f_i,g_i\in A$ with $\|f_i\|,\|g_i\|\leqq 1$, $i=1,2$, such that

$$\|T\|-\tfrac{1}{2}\varepsilon < T(f_1,f_2), \qquad \|U\|-\tfrac{1}{2}\varepsilon < U(g_1,g_2)\,.$$

By Theorem 1, there exist $h_i\in A_i^+$ with $\pm f_i,\pm g_i\leqq h_i$ and $\|h_i\|\leqq 1$, $i=1,2$. Now

$$T(h_1,h_2)-T(f_1,f_2) \;=\; \tfrac{1}{2}T(h_1+f_1,h_2-f_2)+\tfrac{1}{2}T(h_1-f_1,h_2+f_2)\;\geqq\;0\,.$$

Therefore

$$\|T\|-\tfrac{1}{2}\varepsilon < T(f_1,f_2)\leqq T(h_1,h_2)\leqq \|T\|\,,$$

and

$$\|T\| = \sup\{T(h_1,h_2) : h_i\in A_i^+,\ \|h_i\|\leqq 1\}\,.$$

(ii) Similarly $\|U\|-\tfrac{1}{2}\varepsilon\leqq U(h_1,h_2)$ and

$$\|T\|+\|U\|-\varepsilon < (T+U)(h_1,h_2)\leqq \|T+U\|\,.$$

Since the norm is subadditive, $\|T+U\|=\|T\|+\|U\|$.

(iii) Extend T to the bilinear functional $T, A_1\times A_2^+\to R$ by putting $T(f_1,f_2)=T(g_1,f_2)-T(h_1,f_2)$ whenever $f_1\in A_1$, g_1, $h_1\in A_1^+$ with

$f_1 = g_1 - h_1$, and $f_2 \in A_2^+$. Extend this T to a positive bilinear functional $T\colon A_1 \times A_2 \to R$ by putting

$$T(f_1, f_2) = T(f_1, g_2) - T(f_1, h_2)$$

whenever $f_1 \in A_1$ and $f_2 \in A_2$ with $f_2 = g_2 - h_2$ for $g_2, h_2 \in A_2^+$. Since the positive cone in a simplex space is generating, both of these extensions are well defined and therefore unique. The arguments of (i) show that

$$\sup\{T(f_1, f_2) : f_i \in A_i,\ \|f_i\| \leqq 1\} \leqq \sup\{T(h_1, h_2) : h_i \in A_i^+,\ \|h_i\| \leqq 1,\ i = 1, 2\}$$

so that $T \in K(A_1, A_2)$.

LEMMA 3. *$K(A_1, A_2)$ is a lattice cone.*

PROOF. If $T \in K(A_1, A_2)$ and $f \in A_1$, then $T(f_1, \cdot) \in A_2'$ the dual of A_2. For $T, U \in K$ and $f \in A_1^+$ we put

$$(1) \qquad V(f, \cdot) = \sup\{T(f_2, \cdot) + U(f_2, \cdot) : f_1, f_2 \in A_1^+,\ f_1 + f_2 \leqq f\},$$

the sup being taken in the boundedly complete vector lattice A_2'. Suppose that $f, g, h, k \in A_1^+$ with $h + k \leqq f + g$. Since A_1 has the r.s.p. there exist $f_1, f_2, g_1, g_2 \in A_1^+$ with

$$f_1 + g_1 = h; \quad f_2 + g_2 = k; \quad f_1 + f_2 \leqq f, \quad g_1 + g_2 \leqq g,$$

therefore

$$\begin{aligned} V(f+g, \cdot) &= \sup\{T(h, \cdot) + U(k, \cdot) : h, k \in A_1^+,\ h + k \leqq f + g\} \\ &= \sup\{T(f_1, \cdot) + U(f_2, \cdot) + T(g_1, \cdot) + U(g_2, \cdot) : \\ &\qquad f_1, f_2, g_1, g_2 \in A_1^+,\ f_1 + f_2 \leqq f, g_1 + g_2 \leqq g\} \\ &= V(f, \cdot) + V(g, \cdot), \end{aligned}$$

whenever $f, g \in A_1^+$. Similarly,

$$V(af, \cdot) = a V(f, \cdot)$$

whenever $f \in A_1^+$ and real $a \geqq 0$. If $f_i \in A_i^+$, $i = 1, 2$, then

$$0 \leqq V(f_1, f_2) \leqq (T + U)(f_1, f_2),$$

and V satisfies the conditions (iii) of lemma 2. Therefore V has a unique extension to an element $V \in K(A_1, A_2)$. From (1) we have that $T, U \leqq V$. If $T, U \leqq W \in K(A_1, A_2)$ then whenever $f_1, f_2, f \in A_1^+$ with $f_1 + f_2 \leqq f$ we have

$$T(f_1, \cdot) + U(f_2, \cdot) \leqq W(f_1, \cdot) + W(f_2, \cdot) \leqq W(f, \cdot)$$

and formula (1) implies that $V(f_i, \cdot) \leqq W(f, \cdot)$. Thus V is the least upper bound of T and U, and $K(A_1, A_2)$ is a lattice cone.

Let

$$Y_n = \{T \in K(A_1,\ldots,A_n) : \|T\| \leqq 1 \text{ and } T(e_1,\ldots,e_n) = 1 \text{ whenever } e_i \text{ are norm units of } A_i,\ i = 1,\ldots,n\},$$

and let Y_n be considered as a subset of the locally convex Hausdorff linear topological space $B(A_1,\ldots,A_n)$ endowed with the weak operator topology. If $x_i \in X_i$, $i = 1,\ldots,n$, then the operator

$$x_1 \otimes x_2 \otimes \ldots \otimes x_n \in K(A_1,\ldots,A_n)$$

is defined by the formula

$$x_1 \otimes x_2 \otimes \ldots \otimes x_n(f_1,f_2,\ldots,f_n) = \prod \{\langle f_i,x_i \rangle : i = 1,\ldots,n\}$$

whenever $f_i \in A_i$, $i = 1,\ldots,n$.

THEOREM 3. *Y_2 is a simplex.*

PROOF. The results of Choquet [5], [6] and Kendall [14] show that Y_2 is a linearly compact Choquet simplex. As in the proof of Alaoglu's result [1], Y_2 is homeomorphic with a closed subset of the topological cartesian product of a family of closed intervals, and is therefore compact.

THEOREM 4. *T is an extreme point of Y_2 if and only if $T = x_1 \otimes x_2$ where x_i are extreme points of X_i, $i = 1, 2$.*

PROOF. If T is extreme in Y_2, then either $T = 0$, and we can take either x_1 or $x_2 = 0$, or $\|T\| = 1$. Suppose that $\|T\| = 1$. Let Z be the simplex

$$\{z \in A_2' : 0 \leqq z \text{ and } \|z\| \leqq 1\}.$$

Then Z is the convex hull of X_2 and the origin, and the extreme points of Z are the extreme points of X_2 together with the origin. For $f \in A_1^+$ with $\|f\| \leqq 1$ we have $T(f,\cdot) \in Z$. Let $\mu_{T(f,\cdot)}$ be the unique maximal representing measure of $T(f,\cdot)$ on the simplex Z [15, p. 66], [2]. Let A and B be Borel subsets of Z with $A \cup B = Z$ and $A \cap B = \{0\}$, and define operators T_1 and T_2 by the formulae

$$T_1(f,g) = \int_A g \, d\mu_{T(f,\cdot)}, \qquad T_2(f,g) = \int_B g \, d\mu_{T(f,\cdot)}$$

whenever $f \in A_1^+$ with $\|f\| \leqq 1$ and $g \in A_2$. On their domain of definition, T_1 and T_2 are linear in each variable and may be extended by positive homogeneity and linearity to real positive bilinear functionals on $A_1 \times A_2$, again denoted by T_1, T_2 such that $T_1 + T_2 = T$. By (iii) and (ii) of lemma 2 we have that $T_1, T_2 \in K(A_1, A_2)$ and $\|T_1\| + \|T_2\| = \|T\|$. If T_1 and T_2 are both non-zero we have that

$$T = \|T_1\| \frac{T_1}{\|T_1\|} + \|T_2\| \frac{T_2}{\|T_2\|}$$

is a convex combination of elements of Y_2. Since T is an extreme point of Y_2 we have that $T_1/\|T_1\| = T_2/\|T_2\|$ which is clearly impossible since $A \cap B = \{0\}$. Therefore either T_1 or $T_2 = 0$. Take $f \in A_1{}^+$ with $\|f\| \leqq 1$ such that $T(f, \cdot) \neq 0$. Then for an arbitrary Borel set A we have that $\mu_{T(f,\cdot)}(A \setminus \{0\})$ is either 0 or $\mu_{T(f,\cdot)}(Z \setminus \{0\})$. Therefore the support of $\mu_{T(f,\cdot)}$ contains one point x_2 other than the origin. Since $\mu_{T(f,\cdot)}$ is a maximal measure, x_2 is an extreme point of X. If we put $A = \{x_2\} \cup \{0\}$ and $B = Z \setminus \{x_2\}$, then for this A and B, $T_1 \neq 0$ so that $T_2 = 0$. Therefore

$$T(f,g) = \int_{\{x_2\} \cup \{0\}} g \, d\mu_{T(f,\cdot)} = \varphi(f) \langle g, x_2 \rangle, \qquad f \in A_1{}^+,\ \|f\| \leqq 1,\ g \in A_2$$

where φ is a positive affine functional on $\{f \in A_1{}^+ : \|f\| \leqq 1\}$ with $\varphi(0) = 0$. By linearity, φ may be extended to an element $x_1 \in X_1$ such that

$$T(f,g) = \langle f, x_1 \rangle \langle g, x_2 \rangle \quad \text{whenever} \quad f \in A,\ g \in A_2 .$$

It follows immediately that x_1 is an extreme point of X_1.

The converse is straightforward, and will be omitted.

THEOREM 5. *$A_1 \check{\otimes} A_2$ is isometrically isomorphic with $\mathscr{A}_0(Y_2)$, and is therefore a simplex space in the induced partial ordering. Moreover, the positive cone in $A_1 \check{\otimes} A_2$ is the closed convex hull of the set*

$$\{f_1 \otimes f_2 : f_i \in A_i{}^+,\ i = 1, 2\} .$$

PROOF. Let $u = \sum_{j=1}^m f_j \otimes g_j$, $f_j \in A_1$, $g_j \in A_2$, $j = 1, \ldots, m$, be an arbitrary element of $A_1 \otimes A_2$. Then the crossnorm λ given by the formula

$$\begin{aligned} \lambda(u) &= \sup \{ |\textstyle\sum_{j=1}^m \langle f_j, x_1 \rangle \langle g_j, x_2 \rangle| : x_i \in A_i',\ \|x_i\| \leqq 1,\ i = 1, 2 \} \\ &= \sup \{ |\textstyle\sum_{j=1}^m \langle f_j, x_1 \rangle \langle g_j, x_2 \rangle| : x_i \in X_i,\ i = 1, 2 \} \end{aligned}$$

is the least crossnorm on $A_1 \otimes A_2$ whose associate is a crossnorm [12], [9, p. 65]. The weak tensor product $A_1 \check{\otimes} A_2$ is the completion of $A_1 \otimes A_2$ for λ [9], [12]. The formula

$$(\theta[u])(T) = \sum_{j=1}^m T(f_j, g_j), \qquad T \in Y_2,$$

defines a linear map θ of $A_1 \otimes A_2$ into $\mathscr{A}_0(Y_2)$. The supremum norm

$$\begin{aligned} \|\theta[u]\| &= \sup\{|(\theta[u])(T)| : T \in Y_2\} \\ &= \sup\{|(\theta[u])(T)| : T \text{ extreme in } Y_2\} \quad [9, \text{p. } 65] \\ &= \sup\{|\textstyle\sum_{j=1}^m \langle f_j, x_1\rangle\langle g_j, x_2\rangle : x_i \text{ extreme in } X_i\} = \lambda(u) . \end{aligned}$$

Therefore θ is a well defined linear isometry. If $T \neq U \in Y_2$, then there exist $f \in A_1$ and $g \in A_2$ such that $T(f,g) \neq U(f,g)$ so that $\theta(f\otimes g)$ separates T and U. By lemma 1, $A_1 \otimes A_2$ is isometrically isomorphic with a dense subspace of $\mathscr{A}_0(Y_2)$ so that $A_1 \check{\otimes} A_2$ is isometrically isomorphic with $\mathscr{A}_0(Y_2)$. Let H be the convex hull of

$$\{\theta(f_1 \otimes f_2) : f_i \in A_i{}^+,\ i=1,2\} .$$

Let φ be in the dual of $\mathscr{A}_0(Y_2)$. Then there exist positive scalars a,b and elements T and U of Y_2 such that

$$\langle f, \varphi\rangle = af(T) - bf(U) \quad \text{whenever} \quad f \in \mathscr{A}_0(Y_2) .$$

If φ is positive on H, then

$$aT(f_1,f_2) - b\,U(f_1,f_2) \geqq 0 \quad \text{whenever} \quad f_i \in A_i{}^+,\ i=1,2 .$$

It follows that

$$aT - b\,U \in K(A_1,A_2) \quad \text{and} \quad \varphi \geqq 0 .$$

If H is not dense in the positive cone of $\mathscr{A}_0(Y_2)$, then, by the Hahn–Banach theorem there exist $0 \leqq f \in \mathscr{A}_0(Y_2)$ and a φ in the dual of $\mathscr{A}_0(Y_2)$ such that $\varphi(H) \geqq 0$ and $\varphi(f) < 0$. Then φ is not positive, which contradicts the above remarks. Therefore H is dense in the positive cone of $\mathscr{A}_0(Y_2)$.

From theorems 4 and 5 we obtain the following corollary.

COROLLARY 2. *$A_1 \check{\otimes} A_2$ has a norm unit if and only if both A_1 and A_2 have one.*

THEOREM 6. *$A_1 \check{\otimes} A_2$ is isometrically and order isomorphic with $\mathscr{M}\mathscr{A}_0(X_1 \times X_2)$.*

PROOF. Let $\mathscr{M} = \mathscr{M}\mathscr{A}_0(X_1 \times X_2)$, let $\mathscr{M}_1$ be the linear span of $\mathscr{M}$ and the constant functions, and put

$$W = \{(x_1,x_2) \in X_1 \times X_2 :\ x_1 \text{ or } x_2 = 0\} .$$

Then $\mathscr{M}_1$ separates points of $X_1 \times X_2 \setminus W$, and separates W from the points of $(X_1 \times X_2) \setminus W$. The functions in $\mathscr{M}_1$ are constant on W. Let S be the quotient space of $X_1 \times X_2$ obtained by identifying the points of W, and let the positive isometry $\psi : \mathscr{M}_1 \to C(S)$ be the natural embedding of $\mathscr{M}_1$ in $C(S)$. Then $M_1 = \psi(\mathscr{M}_1)$ contains the constant functions and separates points of S, moreover

$$M = \psi(\mathscr{M}) = \{f \in M_1 : f(W) = 0\}.$$

Define the map $\Phi : \mathscr{A}(Y_2) \to \mathscr{M}_1$ by the formula

$$\Phi(f)(x_1,x_2) = f(x_1 \otimes x_2), \qquad f \in \mathscr{A}(Y_2),\ (x_1,x_2) \in X_1 \times X_2 .$$

As in the proof of theorem 5, it follows from the definition of the norm in $\mathscr{A}(Y_2)$ that Φ is a positive isometry. If $L_1 = \psi \circ \Phi(\mathscr{A}(Y_2))$, then

$$L = \psi \circ \Phi(\mathscr{A}_0(Y_2)) = \{f \in L_1 : f(W) = 0\}.$$

Moreover $L_1 \subset M_1$ and L_1 separates the points of S. The Choquet boundary of $\mathscr{A}(Y_2)$ is the set of extreme points of Y_2. It follows from theorem 4, that $s \in \partial_{L_1}(S)$ if and only if either $s = W$ or $s = (x_1,x_2)$ with x_i extreme in X_i, $i = 1,2$. Since $L_1 \subset M_1$ we have that $\partial_{L_1}(S) \subset \partial_{M_1}(S)$. If

$$(x_1,x_2) \in \partial_{M_1}(S) \setminus \partial_{L_1}(S) \quad \text{and} \quad x_2 = \tfrac{1}{2}(y_2 + z_2)$$

where $x_2, y_2 \in X_2$, then

$$\varepsilon_{(x_1,x_2)|M_1} = \tfrac{1}{2}(\varepsilon_{(x_1,y_2)|M_1} + \varepsilon_{(x_1,z_2)|M_1}).$$

Since M_1 separates points of S we have that $(x_1,y_2) = (x_1,z_2) = (x_1,x_2)$ so that x_2 is an extreme point of X_2. Similarly x_1 is extreme in X_1, and $(x_1,x_2) \in \partial_{L_1}(S)$. Since L_1 has the r.s.p., we have that L_1 is uniformly dense in M_1 by theorem 2. Since L_1 and M_1 are uniformly closed, $L_1 = M_1$. Therefore $L = M$. Since Φ and ψ are both isometric order isomorphisms the result follows.

COROLLARY 3. *If X_1 and X_2 are simplexes, then $\mathscr{A}(X_1) \check{\otimes} \mathscr{A}(X_2)$ is isometrically and order isomorphic with $\mathscr{A}(X_1 \times X_2)$. The boundary of $\mathscr{M}\mathscr{A}(X_1 \times X_2)$ is the cartesian product of the boundaries of $\mathscr{A}(X_1)$ and $\mathscr{A}(X_2)$.*

The above results have natural extensions to the product of n simplex spaces which we state below. The proofs of these results are by induction on n, and we omit them.

THEOREM 7. *$K(A_1,\ldots,A_n)$ is a lattice cone, Y_n is a simplex and T is an extreme point of Y_n if and only if $T = x_1 \otimes \ldots \otimes x_n$, where x_i are extreme points of X_i, $i = 1,\ldots,n$.*

If

$$u = \sum_{j=1}^{m} f_{1j} \otimes \ldots \otimes f_{nj}; \qquad f_{ij} \in A_i,\ i = 1,\ldots,n,\ j = 1,\ldots,m,$$

is an arbitrary element of $\otimes\{A_i : i=1,\ldots,n\}$, then the formulae

$$\theta(u)(T) = \sum_{j=1}^{m} T(f_{1j},\ldots,f_{nj}), \qquad T \in Y_n ,$$

$$\Gamma(u)(x_1,\ldots,x_n) = \sum_{j=1}^{m} \langle f_{1j},x_1\rangle\cdot\ldots\cdot\langle f_{nj},x_n\rangle, \qquad x_i \in X_i,\ i=1,\ldots,n ,$$

define maps

$$\theta : \otimes\{A_i : i=1,\ldots,n\} \to \mathscr{A}_0(Y_n),$$
$$\Gamma : \otimes\{A_i : i=1,\ldots,n\} \to \mathscr{M}\mathscr{A}_0(\{X_i\}_{i=1}^n) .$$

THEOREM 8. *The range of* θ *is dense in* $\mathscr{A}_0(Y_n)$, *the range of* Γ *is dense in* $\mathscr{M}\mathscr{A}_0(\{X_i\}_{i=1}^n)$. *The maps* θ *and* Γ *are isometries which extend by continuity to isometries of* $\check{\otimes}\{A_i : i=1,\ldots,n\}$ *onto* $\mathscr{A}_0(Y_n)$ *and* $\mathscr{M}\mathscr{A}_0(\{X_i\}_{i=1}^n)$. *If* $\check{\otimes}\{A_i : i=1,\ldots,n\}$ *has the partial ordering induced by* θ, *then the positive cone is the closed convex hull of*

$$\{f_1\otimes\ldots\otimes f_n : f_i \in A_i^+,\ i=1,\ldots,n\} ,$$

and Γ *is an order isomorphism.*

THEOREM 9. Y_n *is a projective tensor product of* $\{X_i : i=1,\ldots,n\}$.

PROOF. Define $P \in \mathscr{M}\mathscr{A}_0(\{X_i\}, Y_n)$ by the formula

$$P(x_1,\ldots,x_n) = x_1\otimes\ldots\otimes x_n, \qquad x_i \in X_i,\ i=1,\ldots,n .$$

If Y is a simplex, and $F \in \mathscr{M}\mathscr{A}_0(\{X_i\}, Y)$, then for $f \in \mathscr{A}_0(Y)$,

$$f\circ F \in \mathscr{M}\mathscr{A}_0(\{X_i\}) ,$$
$$\Gamma^{-1}(f\circ F) \in \check{\otimes}\{A_i : i=1,\ldots,n\} ,$$
$$\theta\Gamma^{-1}(f\circ F) \in \mathscr{A}_0(Y_n) .$$

Then the formula

$$\varphi_T(f) = \theta\Gamma^{-1}(f\circ F)(T), \qquad T \in Y_n,\ f \in \mathscr{A}_0(Y) ,$$

defines for each $T \in Y_n$ an element $\varphi_T \in Y$. This is because

(i) φ_T is linear,
(ii) $\varphi_T(e)=1$ whenever e is a norm unit in $\mathscr{A}_0(Y)$,
(iii) $\varphi_T(f) \geqq 0$ whenever $0 \leqq f \in \mathscr{A}_0(Y)$,
(iv) $|\varphi_T(f)| \leqq 1$ whenever $f \in \mathscr{A}_0(Y)$, and $\|f\| \leqq 1$.

Furthermore, if T_α is a net in Y_n converging to $T \in Y_n$, then $\varphi_{T_\alpha}(f)$ converges to $\varphi_T(f)$ for each $f \in \mathscr{A}_0(Y)$, and φ_{T_α} converges to φ_T. Therefore the map $T \to \varphi_T$ defines a function $F' \in \mathscr{A}_0(Y_n, Y)$, which is unique since it is uniquely defined on the boundary of Y_n. We see immediately that $F = F' \circ P$, so that Y_n is indeed a projective tensor product of $\{X_i : i = 1, \ldots, n\}$.

We remark that P is an injection if and only if A_i has a norm unit for each $i = 1, \ldots, n$. In this case Y_n is a tensor product of $\{X_i : i = 1, \ldots, n\}$ in the usual sense [16]. Combining theorems 8 and 9 we have the following representation:

THEOREM 10. *If $\{X_i : i = 1, \ldots, n\}$ is a family of simplexes, then*

$$\check{\otimes}\{\mathscr{A}_0(X_i) : i = 1, \ldots, n\}$$

is isometrically isomorphic with

$$\mathscr{A}_0(\hat{\otimes}\{X_i : i = 1, \ldots, n\}).$$

Moreover, the positive cone of

$$\check{\otimes}\{\mathscr{A}_0(X_i) : i = 1, \ldots, n\}$$

for the induced partial ordering is the closed convex hull of

$$\{f_1 \otimes \ldots \otimes f_n : f_i \in A_i{}^+, \ i = 1, \ldots, n\}.$$

Infinite tensor products.

In this section $\{X_i : i \in I\}$ is a family of simplexes.

LEMMA 4. *If $\mathscr{F}$ is the subspace of $\mathscr{M}\mathscr{A}_0(\{X_i\})$ which consists of those functions which depend only on a finite number of co-ordinate variables, then $\mathscr{F}$ is uniformly dense in $\mathscr{M}\mathscr{A}_0(\{X_i\})$.*

PROOF. If $h \in \mathscr{M}\mathscr{A}_0(\{X_i\})$ and $\varepsilon > 0$, then there exists an open cover $\{V_j : j = 1, \ldots, k\}$ of $X\{X_i : i \in I\}$ such that

$$(2) \quad |h(\{x_i\}) - h(\{y_i\})| < \varepsilon \quad \text{whenever} \quad \{x_i\}, \{y_i\} \in V_j, \ j = 1, \ldots, k.$$

Moreover, we may assume that each V_j is the intersection of cylinder sets:

$$V_j = U_{1(j)} \times \ldots \times U_{n(j)} \times \{X_i : i \in I, \ i \neq 1(j), \ldots, n(j)\},$$

where $U_{r(j)}$ is open in $X_{r(j)}$, $r(j) = 1(j), \ldots, n(j)$, $j = 1, \ldots, k$. Fix $p_i \in X_i$ with $p_i = 0$ whenever $0 \in X_i$, $i \in I$. Define $f \in \mathscr{M}\mathscr{A}_0(\{X_i\})$ by the formulae

$$f(\{x_i\}) = h(\{y_i\}) \quad \text{whenever} \quad \{x_i\} \in \times\{X_i : i \in I\}$$

and

$$y_i = \begin{cases} x_i & \text{whenever } i = 1(j), \ldots, n(j), \\ p_i & \text{otherwise}. \end{cases}$$

Then $\{y_i\} \in V_j$ whenever $\{x_i\} \in V_j$, $j = 1, \ldots, k$, and by (2), $\|f - h\| < \varepsilon$.

If $h \geqq 0$, then $f \geqq 0$, if there are an infinite number of X_i which contain the origin, then $f = 0$, and we obtain the following corollaries.

COROLLARY 4. *The positive cone in* $\mathscr{M}\mathscr{A}_0(\{X_i\})$ *is the closure of* $\{f \in \mathscr{F} : f \geqq 0\}$.

COROLLARY 5. *If more than a finite number of* X_i *contain the origin, then* $\mathscr{M}\mathscr{A}_0(\{X_i\})$ *contains only the zero function.*

THEOREM 11. $\mathscr{M}\mathscr{A}_0(\{X_i\})$ *is a simplex space.*

PROOF. It is enough to consider the case where only $X_{j(1)}, \ldots, X_{j(k)}$ contain the origin. Let $\mathscr{F}(i_1, \ldots, i_n)$ be the set of $f \in \mathscr{F}$ which depend on no co-ordinate variable other than $i_1, \ldots, i_n$. Then

$$\mathscr{F} = \bigcup \{\mathscr{F}(i_1, \ldots, i_n) : i_1, \ldots, i_n \in I, \ (j_1, \ldots, j_k) \subset (i_1, \ldots, i_n)\} .$$

If $i_1, \ldots, i_n$ is an arbitrary finite subset of I containing $j_1, \ldots, j_k$, then there is a canonical isometric and order isomorphism

$$\sigma : \mathscr{M}\mathscr{A}_0(\{X_{i_1}, \ldots, X_{i_n}\}) \quad \text{onto} \quad \mathscr{F}(i_1, \ldots, i_n)$$

given by

$$\sigma(g)(\{x_i\}) = g(x_{i_1}, \ldots, x_{i_n}), \quad x_i \in X_i, \ i \in I, \qquad g \in \mathscr{M}\mathscr{A}_0(X_{i_1}, \ldots, X_{i_n}) .$$

Thus $\mathscr{F}(i_1, \ldots, i_n)$ is a simplex space for the natural partial ordering. Since any four elements of $\mathscr{F}$ are contained in some such $\mathscr{F}(i_1, \ldots, i_n)$, it follows that $\mathscr{F}$ has the r.s.p., and satisfies conditions (i) and (ii) of corollary 1. Moreover, the positive cone in $\mathscr{F}$ is closed in $\mathscr{F}$. By corollary 2, the completion of $\mathscr{F}$, namely $\mathscr{M}\mathscr{A}_0(\{X_i\})$ is a simplex space.

DEFINITION 1. If $x_i \in X_i$, $i \in I$, we shall use $\varepsilon_{\{x_i\}}$ to mean the restriction of the evaluation functional at $\{x_i\}$ to $\mathscr{M}\mathscr{A}_0(\{X_i\})$. Further we define

$$Y_\infty = \{x \in \mathscr{M}\mathscr{A}_0(\{X_i\})' : 0 \leqq x, \ \|x\| \leqq 1, \text{ and } \langle e, x\rangle = 1 \text{ whenever } e \text{ is a norm unit of } \mathscr{M}\mathscr{A}_0(\{X_i\})\} .$$

LEMMA 5. *The non-zero extreme points of* Y_∞ *are*

$$\{\varepsilon_{\{x_i\}} : 0 \neq x_i \textit{ is extreme in } X_i, \ i \in I\} .$$

PROOF. If $\varepsilon_{\{x_i\}}$ is extreme in Y_∞, and $x_j = \frac{1}{2}(y_j + z_j)$ where $y_j, z_j \in X_j$ for some $j \in I$, then $\varepsilon_{\{x_i\}} = \frac{1}{2}(\varepsilon_{\{y_i\}} + \varepsilon_{\{z_i\}})$, where

$$y_i = \begin{cases} y_j, & i = j, \\ x_i & \text{otherwise}, \end{cases} \qquad z_i = \begin{cases} z_j, & i = j\ , \\ x_i & \text{otherwise}\ . \end{cases}$$

Therefore x_i is extreme in X_i for each $i \in I$. Conversely, suppose $x_i \neq 0$ is extreme in X_i, $i \in I$. If $\varepsilon_{\{x_i\}} = \frac{1}{2}(y + z)$ with $y, z \in Y_\infty$ then, by theorem

7, the restrictions of y, z, and $\varepsilon_{\{xi\}}$ to $\mathscr{F}(i_1,\ldots,i_n)$ agree for arbitrary $i_1,\ldots,i_n \in I$. Therefore

$$Y_{|\mathscr{F}} = Z_{|\mathscr{F}} = \varepsilon_{\{xi\}|\mathscr{F}}\,.$$

Since $\mathscr{F}$ is uniformly dense in $\mathscr{M}\mathscr{A}_0(\{X_i\})$, $y=z=\varepsilon_{\{xi\}}$ is extreme in Y_∞.

Let Φ be the natural isometric isomorphism of $\mathscr{A}_0(Y_\infty)$ onto $\mathscr{M}\mathscr{A}_0(\{X_i\})$.

THEOREM 12. *Y_∞ is a projective topological tensor product of $\{X_i : i \in I\}$.*

PROOF. Define $P \in \mathscr{M}\mathscr{A}_0(\{X_i\}, Y_\infty)$ by putting $P(\{x_i\}) = \varepsilon_{\{xi\}}$ whenever $x_i \in X_i$, $i \in I$. P is 1-1 on those $\{x_i\}$ containing no zero entry. If Y is a simplex, and $F \in \mathscr{M}\mathscr{A}_0(\{X_i\}, Y)$, then $f \circ F \in \mathscr{M}\mathscr{A}_0(\{X_i\})$ and $\Phi^{-1}(f \circ F) \in \mathscr{A}_0(Y_\infty)$ whenever $f \in \mathscr{A}_0(Y)$. For $T \in Y_\infty$, define $\varphi_T \in Y$ by putting $\varphi_T(f) = \Phi^{-1}(f \circ F)(T)$ whenever $f \in \mathscr{A}_0(Y)$. As in the proof of theorem 9, the map $T \to \varphi_T$ defines a unique $F' \in \mathscr{M}\mathscr{A}_0(Y_\infty, Y)$ such that $F' \circ P = F$. Therefore Y_∞ is a projective topological tensor product of $\{X_i : i \in I\}$.

Projective limits of simplexes.

From the proof of lemma 4, we observe that $\mathscr{M}\mathscr{A}_0(\{X_i\})$ is the norm closure of an inductive family of simplex spaces. We are therefore led to consider inductive systems of simplex spaces, and their dual systems, projective systems of simplexes. Throughout this section $(\Lambda, <)$ is a directed set.

DEFINITION 2. *A projective system of simplexes directed by Λ*, is a family $\{X_\alpha : \alpha \in \Lambda\}$ of simplexes together with a family of continuous linear maps $p_{\alpha\beta} : X_\beta \to X_\alpha$ defined whenever $\alpha < \beta \in \Lambda$ such that

(i) $p_{\alpha\alpha}$ is the identity on X_α, $\alpha \in \Lambda$,
(ii) $p_{\alpha\beta} \circ p_{\beta\gamma} = p_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma \in \Lambda$,

and is denoted by $\{X_\alpha, p_{\alpha\beta}\}$. By $\{X_\infty, p_\alpha\}$ we mean that X_∞ is a simplex, and $\{p_\alpha\}$ is a family of continuous linear maps from X_∞ to X_α which satisfy the relation $p_{\alpha\beta} \circ p_\beta = p_\alpha$ whenever $\alpha < \beta \in \Lambda$. We call $\{X_\infty, p_\alpha\}$ a *projective limit* of $\{X_\alpha, p_{\alpha\beta}\}$ if, given $\{X_\infty', p_\alpha'\}$, there exists a continuous linear map $p : X_\infty' \to X_\infty$ such that $p_\alpha' = p_\alpha \circ p$ whenever $\alpha \in \Lambda$.

A projective limit is unique up to linear homeomorphism. In what follows, none of the simplexes contain the origin.

DEFINITION 3. *An inductive system of simplex spaces directed by Λ* is a family $\{E_\alpha : \alpha \in \Lambda\}$ of simplex spaces together with a family of positive linear contractions $q_{\beta\alpha} : E_\alpha \to E_\beta$ defined whenever $\alpha < \beta \in \Lambda$, and such that

(i) $q_{\gamma\beta} \circ q_{\beta\alpha} = q_{\gamma\alpha}$ whenever $\alpha < \beta < \gamma \in \Lambda$

(ii) $q_{\alpha\alpha}$ is the identity on E_α, $\alpha \in \Lambda$,

and is denoted by $\{E_\alpha, q_{\beta\alpha}\}$. By $\{E_0, q_\alpha\}$ we mean a simplex space E_0 together with a family of positive linear contractions $q_\alpha : E_\alpha \to E_0$ such that $q_\beta \circ q_{\beta\alpha} = q_\alpha$ whenever $\alpha < \beta \in \Lambda$. We call $\{E_0, q_\alpha\}$ an *inductive limit* of $\{E_\alpha, q_{\beta\alpha}\}$ if, given $\{E_0', q_\alpha'\}$, there exists a positive linear contraction $g' : E_0 \to E_0'$ such that $q_\alpha' = q' \circ q_\alpha$ for all $\alpha \subset \Lambda$. An inductive limit of $\{E_\alpha, q_{\beta\alpha}\}$ is unique up to positive isometry.

Suppose that $p : X \to Y$ is a continuous linear map of the simplex X into the simplex Y, then p is Borel measurable. If $m(X)$ denotes the space of real-valued functions on X, then the formula

$$(qf)(x) = f(px), \qquad f \in m(Y),\ x \in X ,$$

defines a positive linear map $q : m(Y) \to m(X)$. Moreover q maps Borel measurable functions to Borel measurable functions, convex functions to convex functions, and upper semicontinuous functions to upper semicontinuous functions. We call q the *dual* map of p. If X and Y are canonically embedded in the spaces $\mathscr{A}(X)'$ and $\mathscr{A}(Y')$ with the weak*-topologies, then the adjoint of the restriction of q to $\mathscr{A}(X)$ agrees with p on X. We therefore denote the adjoint of $q : \mathscr{A}(X) \to \mathscr{A}(Y)$ by p. Similarly, we consider X and Y canonically embedded in $C(X)'$, $C(Y)'$ and also denote by p the adjoint of $q : C(Y) \to C(X)$. Then p maps probability measures to probability measures, and

$$\int_B f \, d(p\mu) = \int_{P^{-1}(B)} q(f) \, d\mu$$

whenever f is a Borel measurable function on Y and B is a Borel subset of Y.

The following lemma has been proved independently by F. Jellett in the case of onto maps, and also occurs in [10].

LEMMA 6. *If p is a continuous affine map of the simplex X into the simplex Y, then $p(\partial X) \subset \partial Y$ if and only if p maps maximal measures to maximal measures.*

PROOF. If q is the dual map of p, then q maps convex functions to convex functions, and u.s.c. functions to u.s.c. functions. Suppose $h \in C(Y)$ is convex, then $\hat{h}$ is u.s.c. and affine, and agrees with h on ∂Y. If $p(\partial X) \subset \partial Y$, then $q(\hat{h})$ is u.s.c. and affine, and agrees with $q(h)$ on ∂X. It follows that

$$q(\hat{h}) = q(h) = \inf \{q(f) : h \leqq f \in \mathscr{A}(Y)\} .$$

If μ is maximal then [14, prop. 9,3],

$$\int q(h)\,d\mu = \int q(\hat{h})\,d\mu = \inf\left\{\int q(f)\,d\mu : h \leqq f \in \mathscr{A}(Y)\right\},$$

$$\int h\,dp(\mu) = \inf\left\{\int f\,dp(\mu) : h \leqq f \in \mathscr{A}(Y)\right\} = \int \hat{h}\,d(p\mu)\,,$$

so that $p(\mu)$ is maximal [14, prop. 9,3]. The converse is trivial.

COROLLARY 6. *If $p(\partial X) \subset \partial Y$, then the range of p is a face of Y.*

PROOF. Let Z be the range of p. Then Z is a compact subset of Y. If $p(x) \in Z$, and μ is the maximal representing measure of $x \in X$, then $p(\mu)$ is the maximal representing measure of $p(x)$ on Y. Moreover, $\operatorname{supp} p(\mu) \subset Z$. Suppose $p(x) = \frac{1}{2}(y+z)$ with $y,z \in Y$. Let λ, ϑ be the maximal representing measures of y and z respectively. Then $p(\mu) = \frac{1}{2}(\lambda + \vartheta)$ so that $\operatorname{supp}\lambda, \operatorname{supp}\vartheta \subset Z$. Therefore $y,z \in Z$ and Z is a face of Y.

THEOREM 13. *If $\{X_\alpha, p_{\alpha\beta}\}$ is a projective system of simplexes directed by Λ, then a projective limit $\{X_\infty, p_\alpha\}$ exists.*

PROOF. If $\mathscr{A}(X_\alpha)'$, $\alpha \in \Lambda$, has the weak*-topology, then the cartesian product $E = \times\{\mathscr{A}(X_\alpha)' : \alpha \in \Lambda\}$ is a locally convex Hausdorff linear topological space. The cone

$$K = \{\{x_\alpha\} \in E : 0 \leqq x_\alpha,\ \alpha \in \Lambda\}$$

is a boundedly complete lattice ordered cone, where lattice suprema are taken co-ordinatewise. K is the cone of all positive linear functionals on

$$F = \sum\{\mathscr{A}(X_\alpha) : \alpha \in \Lambda\}\,,$$

where $0 \leqq \{f_\alpha\} \in F$, if and only if $0 \leqq f_\alpha$ for all $\alpha \in \Lambda$. For $\alpha < \beta \in \Lambda$ we denote by $q_{\beta\alpha}$ the dual map of $p_{\alpha\beta}$. If $\beta \in \Lambda$, we define the positive linear map $T_\beta : F \to F$ by putting

$$T_\beta(\{f_\alpha\}) = \{g_\alpha\}, \qquad \text{where } g_\alpha = \begin{cases} f_\alpha & \text{whenever } \alpha \not< \beta\,, \\ q_{\beta\alpha}(f_\alpha) & \text{whenever } \alpha < \beta\,. \end{cases}$$

We denote by t_β the restriction to K of the adjoint of T_β. This t_β is given by the formula

$$t_\beta(\{x_\alpha\}) = \{y_\alpha\}, \qquad y_\alpha = \begin{cases} x_\alpha & \text{whenever } \alpha \not< \beta\,, \\ p_{\alpha\beta}(x_\beta) & \text{whenever } \alpha < \beta\,. \end{cases}$$

Let

$$L = \left\{\{x_\alpha\} \in K : t_\beta(\{x_\alpha\}) = \{x_\alpha\} \text{ for all } \beta \in \Lambda\right\}$$

be the cone of invariant elements of K. Then $\{x_\alpha\} \in L$ if and only if $0 \leqq x_\alpha \in \mathscr{A}(X_\alpha)'$ and $p_{\alpha\beta}(x_\beta) = x_\alpha$ whenever $\alpha, \beta \in \Lambda$ with $\alpha < \beta$. If

$$X_\infty = \{\{x_\alpha\} \in L : x_\alpha \in X_\alpha \text{ for all } \alpha \in \Lambda\},$$

then X_∞ is a closed subset of the cartesian product $\times\{X_\alpha : \alpha \in \Lambda\}$, and is compact for the relative topology. This coincides with the relative topology of X_∞ as a subset of E, so that X_∞ is a compact subset of a locally compact Hausdorff topological linear space. If $\{x_\alpha\} \in L$, then there exists a unique net $\{a_\alpha : \alpha \in \Lambda\}$ of positive real numbers such that $x_\alpha \in a_\alpha X_\alpha$, $\alpha \in \Lambda$. If $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, then $x_\alpha = p_{\alpha\beta}(x_\beta) \in a_\beta X_\alpha$ so that

$$a_\beta = a_\alpha = a \quad \text{for all} \quad \alpha, \beta \in \Lambda .$$

Therefore $\{ax_\alpha\} \in X_\infty$, and X_∞ is a base for L. We now show that L is a lattice cone.

Suppose that $x, y \in L$, and that $x \vee y$ is their least upper bound in K. For $\alpha \in \Lambda$ we have that

$$x \vee y \leqq t_\alpha(x \vee y) .$$

If $\alpha, \beta \in \Lambda$ with $\alpha, \beta < \mu \in \Lambda$, then $t_\mu t_\alpha = t_\mu t_\beta = t_\mu$ so that

$$t_\alpha(x \vee y),\ t_\beta(x \vee y) \leqq t_\mu(x \vee y)$$

and the net $\{t_\alpha(x \vee y) : \alpha \in \Lambda\}$ is directed up. Define $x \veebar y$ as the least upper bound of $\{t_\alpha(x \vee y) : \alpha \in \Lambda\}$. Since this set is directed up we have [4, p. 29, formula (2)]

$$\begin{aligned}\langle f, x \veebar y\rangle &= \sup\{\langle f, t_\alpha(x \vee y)\rangle : \alpha \in \Lambda\} \\ &= \sup\{\langle f, t_\alpha(x \vee y)\rangle : \alpha > \beta \in \Lambda\} \\ &= \sup\{\langle f, t_\beta t_\alpha(x \vee y)\rangle : \alpha > \beta \in \Lambda\} \\ &= \sup\{\langle T_\beta f, t_\alpha(x \vee y)\rangle : \alpha > \beta \in \Lambda\} \\ &= \langle T_\beta f, x \veebar y\rangle = \langle f, t_\beta(x \veebar y)\rangle ,\end{aligned}$$

whenever $0 \leqq f \in F$. Since β is arbitrary, $x \veebar y \in L$. If $x \vee y \leqq z \in L$, then $t_\alpha(x \vee y) \leqq t_\alpha(z) = z$, $\alpha \in \Lambda$, so that $x \veebar y$ is the least upper bound of x and y in L, and L is a lattice cone. Therefore X_∞ is a simplex. Define $p_\alpha : X_\infty \to X_\alpha$ by $p_\alpha(\{x_\alpha\}) = x_\alpha$ whenever $\{x_\alpha\} \in X_\infty$, then $\{X_\infty, p_\alpha\}$. Suppose that $\{X_\infty', p_\alpha'\}$ and define $p' : X_\infty' \to X_\infty$ by putting $p'x' = \{p_\alpha' x'\}$ whenever $x' \in X_\infty'$. It is immediate that p' is continuous, linear, and that $p_\alpha' = p_\alpha \circ p$ whenever $\alpha \in \Lambda$, so that $\{X_\infty, p_\alpha\}$ is a projective limit of $\{X_\alpha, p_{\alpha\beta}\}$.

We note that the projective limit X_∞ obtained above coincides with the projective limit of the projective system of compact Hausdorff spaces, $\{X_\alpha, p_{\alpha\beta}\}$ as defined by Bourbaki [3]. Theorem 13 shows the existence of the projective limit in the category of simplexes and continuous linear transformations, the following shows that the projective limit exists in the category of simplexes and continuous linear boundary preserving maps.

THEOREM 14. *Let $\{X_\alpha, p_{\alpha\beta}\}$ be a projective system of simplexes directed by Λ such that $p_{\alpha\beta}(\partial X_\beta) = \partial X_\alpha$ whenever $\alpha, \beta \in \Lambda$ with $\alpha < \beta$. Then $\{\partial X_\alpha : p_{\alpha\beta}\}$ is a projective system of topological spaces. If $\{X_\infty, p_\alpha\}$ is a projective limit of $\{X_\alpha, p_{\alpha\beta}\}$, then ∂X_∞ is homeomorphic with the topological projective limit $\varprojlim \{\partial X_\alpha : p_{\alpha\beta}\}$.*

PROOF. Let X_∞ be as in the proof of theorem 13. If $\{x_\alpha\} \in X_\infty$ is such that $x_\alpha \in \partial X_\alpha$ for each $\alpha \in \Lambda$, then it is immediate that $\{x_\alpha\} \in \partial X_\infty$. To prove the converse take $\{x_\alpha\} \in X_\infty$, and choose an arbitrary $\alpha \in \Lambda$. Let μ_α be the unique maximal representing measure of x_α. By lemma 6, $p_{\alpha\beta}(\mu_\beta) = \mu_\alpha$ whenever $\alpha < \beta \in \Lambda$. Fix $\alpha \in \Lambda$ and let B be an arbitrary Borel subset of X_α. For $\alpha < \beta \in \Lambda$, define $y_\beta \in \mathscr{A}(X_\beta)'$ by the formula

$$\langle f_\beta, y_\beta \rangle = \int_{p_{\alpha\beta}^{-1}(B)} f_\beta \, d\mu_\beta, \qquad f_\beta \in \mathscr{A}(X_\beta) .$$

If $\beta < \gamma \in \Lambda$, then by lemma 6,

$$p_{\beta\gamma}(\mu_\gamma) = \mu_\beta ,$$

so that

$$\langle q_{\gamma\beta}(f_\beta), y_\gamma \rangle = \int_{p_{\beta\gamma}^{-1} \circ p_{\alpha\beta}^{-1}(B)} q_{\gamma\beta}(f_\beta) \, d\mu_\gamma = \int_{p_{\alpha\beta}^{-1}(B)} f_\beta \, d\mu_\beta = \langle f_\beta, y_\beta \rangle ,$$

so that $y_\beta = p_{\beta\gamma} y_\gamma$. For arbitrary $\beta \in \Lambda$ choose $\gamma \in \Lambda$ with $\alpha, \beta < \gamma$ and put $y_\beta = p_{\beta\gamma} y_\gamma$. It may be verified that y_β is well defined by this formula and that $y_\beta = p_{\beta\gamma} y_\gamma$ whenever $\beta < \gamma \in \Lambda$. Define z_α, similarly, by replacing B by $X_\alpha \setminus B$ in the above construction. Then $\{y_\beta\}, \{z_\beta\} \in L$, and $\{y_\beta\} + \{z_\beta\} = \{x_\beta\}$. If $\{x_\beta\}$ is extreme in X_∞, then there exist real positive numbers a, b such that $y_\alpha = a x_\alpha$ and $z_\alpha = b x_\alpha$. That is

$$b \int_B f_\alpha \, d\mu_\alpha = a \int_{X_\alpha \setminus B} f_\alpha \, d\mu_\alpha \quad \text{whenever} \quad f_\alpha \in \mathscr{A}(X_\alpha) .$$

If follows that $\mu_\alpha(B)$ is either 0 or 1 whenever B is a Borel subset of X_α. Therefore $\mu_\alpha = \varepsilon_{x_\alpha}$ for some $x_\alpha \in \partial X_\alpha$. Since α is arbitrary, $x_\alpha \in \partial X_\alpha$ for all $\alpha \in \Lambda$. Therefore [3]

$$\partial X_\infty = \lim_{\leftarrow}\{\partial X_\alpha : p_{\alpha\beta}\}.$$

Moreover, the relative topology on ∂X_∞ is the projective limit topology and the theorem is proved.

If some of the X_α contain the origin, then we choose $0 \neq z_\alpha \in X_\alpha$, put $Y_\alpha = X_\alpha + z_\alpha$; and define $\overline{p_{\alpha\beta}} : Y_\beta \to Y_\alpha$ by

$$\overline{p_{\alpha\beta}}(x_\alpha + z_\beta) = p_{\alpha\beta}(x_\beta) + z_\alpha$$

whenever $x_\beta \in X_\beta$ and $\alpha < \beta \in \Lambda$. If $\{Y_\infty, \overline{p_\alpha}\}$ is the projective limit of the system constructed in theorem 13, then we define $X_\infty = Y_\infty - \{z_\alpha\}$ and put $p_\alpha(\{y_\alpha\}) = y_\alpha - z_\alpha$, $\alpha \in \Lambda$. It follows that $\{X_\infty, p_\alpha\}$ is a projective limit of $\{X_\alpha, p_{\alpha\beta}\}$, and that theorems 13 and 14 are true in this case.

We now consider the dual question of inductive limits. Suppose that $\{E_\alpha, q_{\beta\alpha}\}$ is an inductive family of simplex spaces, and let $X_\alpha = X(E_\alpha)$. Then $E_\alpha = \mathscr{A}_0(X_\alpha)$ and we may define $p_{\alpha\beta} : X_\beta \to X_\alpha$ by the formula

$$\langle f_\alpha, p_{\alpha\beta}(x_\beta)\rangle = \langle q_{\beta\alpha}(f_\alpha), x_\beta\rangle, \qquad x_\beta \in X_\beta,\ f_\alpha \in E_\alpha .$$

It is immediate that $\{X_\alpha, p_{\alpha\beta}\}$ is a projective family of simplexes with a projective limit $\{X_\infty, p_\alpha\}$. If, for $\alpha \in \Lambda$, we define $q_\alpha : E_\alpha \to \mathscr{A}_0(X_\infty)$ by putting

$$\langle q_\alpha f_\alpha, \{x_\alpha\}\rangle = \langle f_\alpha, x_\alpha\rangle, \qquad f_\alpha \in E_\alpha,\ \{x_\alpha\} \in X_\infty ,$$

then q_α is a continuous linear contraction. Therefore $\{\mathscr{A}_0(X_\infty), q_\alpha\}$. Suppose that $\{E_0', q_\alpha'\}$ and define the transformation $q' : \mathscr{A}_0(X_\infty) \to E_0'$ as the dual of the natural map $p' : X_0' \to X_\infty$. Then $q_\alpha' = q' \circ q_\alpha$ for all $\alpha \in \Lambda$, and $\{\mathscr{A}_0(X_\infty), q_\alpha\}$ is an inductive limit of $\{E_\alpha, q_{\beta\alpha}\}$.

We have shown the existence of the inductive limit for which we now give a representation. Consider the direct sum

$$F = \sum\{E_\alpha : \alpha \in \Lambda\} = \sum\{\mathscr{A}_0(X) : \alpha \in \Lambda\},$$

and let the map $\theta : F \to \mathscr{A}_0(X_\alpha)$ be defined by the formula

$$\theta(\{f_\alpha\})(\{x_\alpha\}) = \sum_{\alpha \in \Lambda} \langle f_\alpha, x_\alpha\rangle, \qquad \{f_\alpha\} \in F,\ \{x_\alpha\} \in X_\infty .$$

If $\{x_\alpha\}$, $\{y_\alpha\}$ are distinct points of X_∞, then for some $\beta \in \Lambda$ we have $x_\beta \neq y_\beta$. Define $\{f_\alpha\} \in F$, putting $f_\alpha = 0$ for $\alpha \neq \beta$ and $f_\beta = g_\beta$, where g_β separates x_β and y_β. Then $\theta(\{f_\alpha\})$ separates $\{x_\alpha\}$ and $\{y_\alpha\}$, and by lemma 3, the set $\theta(F)$ is uniformly dense in $\mathscr{A}_0(X_\infty)$. Similarly, arguing as in the proof of theorem 5, the positive elements in $\theta(F)$ are uniformly dense in the positive cone of $\mathscr{A}_0(X_\infty)$. Let $\|\cdot\|_\infty$ and $\|\cdot\|_\alpha$ denote the norms in $\mathscr{A}_0(X_\infty)$ and $\mathscr{A}_0(X_\alpha)$, respectively, and let N_∞ be the semi-norm induced on F by θ. That is:

$$\begin{aligned}
N_\infty(\{f_\alpha\}) &= \|\theta\{f_\alpha\}\|_\infty \\
&= \sup\{|\Sigma\langle f_\alpha, x_\alpha\rangle| : \{x_\alpha\} \in X_\infty\} \\
&= \sup\Big\{|\Sigma\langle q_{\beta\alpha}(f_\alpha), x_\beta\rangle| : \beta \in \Lambda,\ f_\alpha \neq 0 \Rightarrow \alpha < \beta, \text{ and} \\
&\qquad\qquad x_\beta \in \bigcap\{p_{\beta\gamma}X_\gamma : \beta < \gamma \in \Lambda\}\Big\} \\
&= \inf_{\beta<\gamma} \sup\{|\Sigma_{\alpha\in\Lambda}\langle q_{\beta\alpha}(f_\alpha), x_\beta\rangle| : \beta \in \Lambda,\ f_\alpha \neq 0 \Rightarrow \alpha < \beta,\ x_\beta \in p_{\beta\gamma}X_\gamma\} \\
&= \lim_{\beta\in(\Lambda, <)} \{\|\Sigma_{\alpha<\beta} q_{\beta\alpha}(f_\alpha)\|_\beta\}
\end{aligned}$$

whenever $\{f_\alpha\} \in F$. The third equality follows from [3, Ch. 1, appendix 2, Theorem 1]. For $f \in \mathscr{A}_0(X_\alpha)$ we put

$$m_\alpha(f) = \min\{f(x) : x \in X_\alpha\} .$$

If $\{f_\alpha\} \in F$, then it may be shown in a similar manner that $\theta(\{f_\alpha\}) \geqq 0$ if and only if

$$\lim_{\beta\in(\Lambda, <)} \{m_\beta(\Sigma_{\alpha<\beta}\, q_{\beta\alpha}(f_\alpha))\} \geqq 0 .$$

Let E be the quotient space $F/N_\infty^{-1}(0)$, and let $\{f_\alpha\}^\sim$ denote the equivalence class of $\{f_\alpha\} \in F$. We denote by K the cone

$$\left\{\{f_\alpha\}^\sim \in E : \lim_{\beta\in(\Lambda, <)} \{m_\beta(\Sigma_{\alpha<\beta}\, q_{\beta\alpha}(f_\alpha))\} \geqq 0\right\}.$$

THEOREM 15. *Let E_0 be the completion of E for the quotient norm*

$$\|\{f_\alpha\}^\sim\| = \lim_{\beta\in(\Lambda, <)} \|\Sigma_{\alpha<\beta}\, q_{\beta\alpha}(f_\alpha)\|_\beta, \qquad \{f_\alpha\}^\sim \in E ,$$

and let K_0 be the closure of K for this norm. Then E_0 is a simplex space, with positive cone K_0. Moreover $\{E_0, q_\alpha\}$ is an inductive limit of $\{E, q_{\beta\alpha}\}$, where $q_\alpha : E_\alpha \to E_0$ is given by the formula

$$q_\alpha(f) = \{f_\alpha\}^\sim,$$

where $f_\alpha = f$ and $f_\beta = 0$ if $\alpha \neq \beta$, whenever $f \in E_\alpha$.

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