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AFFINE PRODUCTS OF SIMPLEXES

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1. Introduction.

Let S_1, S_2 be two compact Hausdorff spaces and denote by $C(S_1, C(S_2))$ the space of all continuous functions on S_1 having values in $C(S_2)$. If $C(S_1, C(S_2))$ is endowed with the supremum norm, then it is well known that this Banach space is isometrically isomorphic to $C(S_1 \times S_2)$, the space of all continuous functions on the cartesian product $S_1 \times S_2$ with the usual norm. It turns out that a similar situation is valid for simplexes too. Suppose that K_1, K_2 are simplexes and let $A(K_1, A(K_2))$ be the space of all continuous and affine functions on K_1 with values in $A(K_2)$, the space of all real valued continuous and affine functions on K_2 with the supremum norm. $A(K_1, A(K_2))$ normed by $\|f\| = \sup_{k_1 \in K_1} \|f(k_1)\|$ has the finite binary intersection property for balls (cf. [13, Theorem 3.3]). Since it is clear that the closed unit ball of $A(K_1, A(K_2))$ has at least one extreme point, it follows from results of Lindenstrauss [14] and Semadeni [16] that there exists a simplex K such that $A(K) = A(K_1, A(K_2))$. Moreover, by Theorem 2.3 below, this simplex is uniquely determined up to an affine homeomorphism. We investigate here the properties of the simplex K, called by us the affine product of K_1 and K_2. Actually we deal with the affine product of an arbitrary family of simplexes. Since the above construction is cumbersome for infinite families of simplexes we start from another representation of $A(K_1, A(K_2))$: this space is isometrically isomorphic to the space of those real-valued continuous functions on $K_1 \times K_2$ which are affine with respect to each variable in part (endowed also with the supremum norm).

Section 2 of the paper contains two general results on simplexes which are used in the sequel. The first one gives conditions for a continuous function defined on the extreme points of a simplex K to admit a continuous and affine extension to K and is a generalization for non-metrizable simplexes of a theorem of Alfsen [2], [3]. The section ends with an adaptation for simplexes of the Banach–Stone theorem. In Section 3 we define the affine product of simplexes and study some of its general

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properties. In Section 4 the facial structure of the affine product is analysed. To this end we use maps of the affine product onto its factors which remind the projections of a cartesian product of compact Hausdorff spaces.

The affine product of simplexes was independently defined and investigated by Davies and Vincent-Smith [7], using a tensorial approach. We did not omit those of our results which overlap with theirs (e.g., Theorem 3.2), since we thought it preferable to have a self-contained alternative way of treatment.

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Let K be a compact convex subset of a locally convex linear topological space. A subset $F \subset K$ is called a face of K if it is convex and satisfies

$$0 < \lambda < 1,\ k_1, k_2 \in K,\ \lambda k_1 + (1-\lambda) k_2 \in F \ \Rightarrow\ k_1, k_2 \in F .$$

The set of the extreme points of K (one-point faces) is denoted by ∂K. If X is a Banach space, we denote by $A(K,X)$ the space of all continuous and affine functions on K having their values in X and we consider it equipped with the norm $\|f\| = \sup_{k \in K} \|f(k)\|$. We denote $A(K,(-\infty,\infty))$ by $A(K)$. Let $S(K)$ be the cone of all continuous concave functions on K. The upper and the lower envelopes of a bounded real valued function f given on a subset M of K containing ∂K are defined as follows (cf. [6]):

$$\begin{aligned} \hat{f}(k) &= \inf\{g(k) : g \in S(K),\ g_{|M} \geqq f\}, && k \in K , \\ \check{f}(k) &= \sup\{g(k) : g \in -S(K),\ g_{|M} \leqq f\}, && k \in K . \end{aligned}$$

If μ is a Radon probability measure on the compact convex set K, then the point $k \in K$ is called the barycenter of μ if $f(k) = \int_K f d\mu$ for each $f \in A(K)$. The probability Radon measures on K are ordered by

$$\mu_1 \succ \mu_2 \ \Leftrightarrow\ \int_K f\, d\mu_1 \leqq \int_K f\, d\mu_2,\ \forall f \in S(K)$$

(Choquet's ordering). For every $k \in K$ there exists a measure μ on K, maximal in this ordering, whose barycenter is k ([6], [15, p. 24]). The compact convex set K is called a simplex if for each $k \in K$ the corresponding maximal measure is unique. It is easily seen that a closed face of a simplex is itself a simplex. One can embed the simplex K into $A^*(K)$ by an affine homeomorphism in a natural way:

$$T(k)(f) = f(k), \qquad k \in K,\ f \in A(K) .$$

We shall make no distinction between K and its image in $A^*(K)$. The simplex K is called an r-simplex if $\overline{\partial K} = \partial K$ (cf. [1]).

A normed space X is said to have the finite binary intersection property (F.2.I.P.) if every finite collection of mutually intersecting closed balls in X has a non-void intersection (cf. [14]).

Let H be a space of functions on a set. The space H has the Riesz separation property if, whenever $f_1, f_2, g_1, g_2 \in H$, $f_1 \vee f_2 \leqq g_1 \wedge g_2$, there exists $h \in H$ such that $f_1 \vee f_2 \leqq h \leqq g_1 \wedge g_2$.

Let S be a compact Hausdorff space and H a subspace of $C(S)$ which separates the points of S and contains the constants. The Choquet boundary of H consists of those points $s \in S$ such that the only positive linear functional $u \in C^*(S)$ which satisfies $u(f) = f(s)$ for all $f \in H$ is the evaluation at s.

2. Affine extensions and affine homeomorphisms.

Alfsen [3] (see also [2]) gave necessary and sufficient conditions for a continuous map defined on the extreme points of a metrizable compact convex set K (valued in a locally convex linear topological space) to admit an affine continuous extension to the whole of K. We are going to show that his characterization is valid for non-metrizable simplexes as well. A similar result was obtained by Effros [11].

LEMMA 2.1. *Let K be a simplex and $f\colon K \to (-\infty, \infty)$ a bounded continuous function. There exists an affine and continuous extension of f to K if and only if $\hat{f}$ and $\check{f}$ are continuous on $\overline{\partial K}$. Such an extension, if it exists, is unique.*

PROOF. The conditions are obviously necessary since an affine continuous real function is the upper and the lower envelope of itself.

Suppose that $\hat{f}$ and $\check{f}$ are continuous on $\overline{\partial K}$. As remarked by Alfsen [3, p. 4] this implies that

$$\hat{f}(k) = \check{f}(k) = f(k), \qquad k \in \partial K . \tag{2.1}$$

Hence, f can be extended to a continuous function on $\overline{\partial K}$.

The upper envelope $\hat{f}$ is the pointwise limit of a decreasing net of continuous concave functions on K. By Dini's theorem the convergence of the net is uniform on $\overline{\partial K}$. Consequently, for every natural number n there is a continuous concave function g_n on K such that

$$\hat{f}(k) \leqq g_n(k) \leqq \hat{f}(k)+1/n, \qquad k \in \overline{\partial K}\,. \tag{2.2}$$

Similarly, there is a continuous convex function h_n on K such that

$$\check{f}(k)-1/n \leqq h_n(k) \leqq \check{f}(k), \qquad k \in \overline{\partial K}\,. \tag{2.3}$$

By (2.2), (2.3) and Bauer's maximum principle [4] we infer that $h_n \leqq g_n$. From a theorem of Edwards [9] it follows that there exists $f_n \in A(K)$ with $h_n \leqq f_n \leqq g_n$. From (2.1), (2.2), (2.3) and the last inequality it follows that

$$|f(k)-f_n(k)| \leqq 1/n, \qquad k \in \partial K\,,$$

that is, the sequence $\{f_n\}_{n=1}^{\infty} \subset A(K)$ converges uniformly to f on ∂K. By applying again Bauer's maximum principle we conclude that this sequence is uniformly convergent on K. Clearly, its limit, which coincides with f on ∂K, is the desired extension.

The uniqueness of the extension is an immediate consequence of the Krein–Milman theorem.

REMARK. The continuity on $\overline{\partial K}$ of only one of the envelopes is not sufficient for the existence of a continuous and affine extension. Indeed, suppose that K is a simplex, $\overline{\partial K} \neq \partial K$ and let f be a continuous concave function on K such that for a certain point $k_0 \in \overline{\partial K} \setminus \partial K$ it satisfies $f(k_0) > \mu(f)$, μ being a probability Radon measure on K having its barycenter in k_0. Then $\hat{f}=f$ but $f_{|\partial K}$ has no continuous and affine extension to K.

THEOREM 2.2. *Let K be a simplex and K_1 a compact convex subset of a locally convex linear topological space E. A continuous map $T\colon \partial K \to K_1$ can be extended to a continuous and affine map of K into K_1 if and only if for each $f^* \in E^*$ the real functions*

$$\widehat{f^* \circ T} \quad \textit{and} \quad \widecheck{f^* \circ T}$$

are continuous on ∂K. Such an extension, if it exists, is unique.

PROOF. Let $A'(K_1)$ be the subspace of $A(K_1)$ formed by the restrictions to K_1 of all continuous and affine functionals on E. By the hypothesis and the preceding lemma it follows that if $\varphi \in A'(K_1)$, then $\varphi \circ T$ has a unique continuous and affine extension to K. Let $U'\colon A'(K_1) \to A(K)$ be the map which puts in correspondence the function $\varphi \in A'(K)$

with this extension of $\varphi \circ T$. It is readily seen that U' is a bounded linear operator. Since $A'(K_1)$ is norm dense in $A(K_1)$ (see [15, p. 31]), U' has a bounded linear extension $U: A(K_1) \to A(K)$. The adjoint U^* of U maps K into K_1 (K, K_1 are considered canonically embedded into $A^*(K)$, $A^*(K_1)$, respectively) since $U^* \geqq 0$ and $U(1_{K_1}) = 1_K$. Obviously $U^*_{|K}$ is the required extension of T.

The uniqueness of the extension follows from the fact that $A'(K_1)$ separates the points of K_1.

REMARK. As pointed out by Alfsen [3], if K is an r-simplex, then any continuous map $T: \partial K \to K_1$ fulfills the conditions of the theorem and therefore admits a continuous and affine extension to K. For real valued maps this fact was proved by Bauer [5].

Now we turn to the generalization of the Banach–Stone theorem.

THEOREM 2.3. *Let K_1, K_2 be simplexes. The spaces $A(K_1)$, $A(K_2)$ are isometrically isomorphic if and only if K_1 and K_2 are affinely homeomorphic.*

PROOF. The "if" part of the theorem is trivial. Assume that T is a linear isometry of $A(K_1)$ onto $A(K_2)$. Obviously $T^*(K_2)$ is a simplex affinely homeomorphic to K_2 contained in the closed unit ball of $A^*(K_1)$. It is easy to see that $F^+ = K_1 \cap T^*(K_2)$ is a closed face of K_1 and $T^*(K_2)$. Similarly, $F^- = K_1 \cap T^*(-K_2)$ is a closed face of K_1 and $T^*(-K_2)$. The extreme points of K_1 belong to $T^*(K_2) \cup T^*(-K_2)$. Hence

$$K_1 = \overline{\mathrm{conv}}\,(F^+ \cup F^-) = \mathrm{conv}\,(F^+ \cup F^-)$$

by the Krein–Milman theorem and [8, p. 79–80]. Moreover, since K_1 is a simplex, the representation of each point $k \in K_1$ as

$$k = \lambda k' + (1-\lambda)k''$$

with $0 \leqq \lambda \leqq 1$, $k' \in F^+$, $k'' \in F^-$ is unique. Define $\sigma: K_1 \to T^*(K_2)$ by

$$\sigma(k) = \lambda k' - (1-\lambda)k''$$

where $k \in K_1$ is represented as above. Clearly σ is affine and a compactness argument proves that it is also continuous. Since $T^*(K_2) = \mathrm{conv}\,(F^+ \cup (-F^-))$ and $T^*(K_2)$ is a simplex, σ is an one-to-one map onto $T^*(K_2)$. The map $T^{*-1} \circ \sigma$ is an affine homeomorphism of K_1 onto K_2 and the proof is complete.

REMARK. It is worth noting that, as remarked in [3], two non-affinely homeomorphic simplexes may have homeomorphic sets of extreme points.

3. The affine product.

Let $\{E_i\}_{i\in I}$ be a family of linear spaces and $K_i \subset E_i$ a convex set for each $i \in 1$. Denote by p_i the projection of $\times\{K_i : i \in I\}$ onto K_i. A map f from $\times\{K_i : i \in I\}$ into a linear space is called multi-affine if it is affine with respect to each variable in part, that is, if for each $i_0 \in I$, from

$$k, k^1, k^2 \in \times\{K_i : i \in I\},$$
$$p_i(k) = p_i(k^1) = p_i(k^2), \quad i \in I \setminus \{i_0\},$$
$$p_{i_0}(k) = \lambda p_{i_0}(k^1) + (1-\lambda) p_{i_0}(k^2), \quad 0 \leqq \lambda \leqq 1,$$

it follows that

$$f(k) = \lambda f(k^1) + (1-\lambda) f(k^2).$$

From here throughout this paper $K = \{K_i : i \in I\}$ will be a family of simplexes. By H we denote the space of all continuous and multi-affine real valued functions on $\times\{K_i : i \in I\}$ with the supremum norm. This space will be used to define the affine product of K but before doing this we must take one more step.

THEOREM 3.1. *H has the* F.2.I.P.

PROOF. If I is finite, the statement of the theorem may be proved by induction relying on Theorem 3.3 of [13]. Suppose that I is infinite. Denote by B the subspace of H defined as follows: $f \in B$ if and only if there exists a finite set $I_f \subset I$ such that f is constant with respect to the variables $k_i \in K_i$, $i \in I \setminus I_f$, that is,

$$p_i(k^1) = p_i(k^2), \ \forall i \in I_f \ \Rightarrow \ f(k^1) = f(k^2).$$

According to [14, p. 34] the assertion will be proved if we show that B has the F.2.I.P. and is dense in H.

Let us prove that B has the F.2.I.P. Clearly, it suffices to show that any finite set $\{f_1, f_2, \ldots, f_n\} \subset B$ is contained in a subspace of B having the F.2.I.P. By the definition of B there is a finite set $I' \subset I$ such that $f_1, f_2, \ldots, f_n$ are constant with respect to the variables $k_i \in K_i$, $i \in I \setminus I'$. Denote

$$B' = \{f \in H : k^1, k^2 \in \times K_i, \ p_i(k^1) = p_i(k^2) \ \forall i \in I \setminus I' \ \Rightarrow \ f(k^1) = f(k^2)\}.$$

Then $\{f_1, f_2, \ldots, f_n\} \subset B' \subset B$. Since B' is isometrically isomorphic to the space of all continuous and multi-affine functions on $\times\{K_i : i \in I'\}$ and this space has the F.2.I.P. as remarked at the beginning of the proof, our claim about B is proved.

Now we pass to prove that B is dense in H. Let $f \in H$ and $\varepsilon > 0$. Cover $\times\{K_i : i \in I\}$ by neighborhoods $V_1, V_2, \ldots, V_m$ from the usual basis of the cartesian product such that the oscillation of f on each of them is less than ε. Denote

$$J = \{i \in I : \exists V_l, p_i(V_l) \neq K_i,\ 1 \leqq l \leqq m\}.$$

Choose $k_i^0 \in K_i$, $i \in I \setminus J$, and define $f' : \times\{K_i : i \in I\} \to (-\infty, \infty)$ by $f'(k) = f(k')$ where

$$p_i(k') = \begin{cases} p_i(k), & i \in J, \\ k_i^0, & i \in I \setminus J. \end{cases}$$

Since J is finite we have $f' \in B$. Clearly, if $k \in \times\{K_i : i \in I\}$, then

$$|f'(k) - f(k)| = |f(k') - f(k)| < \varepsilon$$

$(k \in V_l \Rightarrow k' \in V_l)$. Thus $\bar{B} = H$ and this completes the proof of the theorem.

Since the function identically equal to 1 on $\times\{K_i : i \in I\}$ belongs to H (and is an extreme point of the closed unit ball of this space) it follows from [14, Theorem 4.7, Theorem 6.1] and [16] that there exists a simplex K such that $H = A(K)$. By Theorem 2.3 this simplex is uniquely determined up to an affine homeomorphism and we call it the affine product of the family K. It will also be denoted in the sequel by $\prod\{K_i : i \in I\}$. The order in H given by [14, Theorem 4.7] coincides with the natural order ($f \geqq 0 \Leftrightarrow f(k) \geqq 0$, $\forall k \in \times K_i$), so one may take as K the positive face of the unit sphere of H^* when the order in H^* is dual to that of H. It is not hard to see that the defined affine product of simplexes is commutative and associative.

We are now going to identify the extreme points of $\prod\{K_i\, i \in I\}$. If $k \in \times\{K_i : i \in I\}$, then $\tilde{k} \in H^*$ denotes the functional given by $\tilde{k}(f) = f(k)$, $f \in H$. It is easily seen that $k \to \tilde{k}$ is a homeomorphism between $\times\{K_i : i \in I\}$ and the subset $\{\tilde{k} : k \in \times K_i\}$ of $\prod\{K_i : i \in I\}$.

THEOREM 3.2 *A functional $f^* \in H^*$ is an extreme point of $\prod\{K_i : i \in I\}$ if and only if $f^* = \tilde{k}$ for a certain $k \in \times\{\partial K_i : i \in I\}$.*

PROOF. Let $k \in \times\{\partial K_i : i \in I\}$. If I is finite, one can prove that $\tilde{k} \in \partial \prod\{K_i : i \in I\}$ by using Theorem 3.2 of [13] and an induction argument. Let I be infinite and assume that

$$\tilde{k} = \tfrac{1}{2}(l_1 + l_2), \qquad l_1, l_2 \in \prod\{K_i : i \in I\}.$$

It is enough to show that l_1 and l_2 take the same values on B, where B is the space defined in the proof of Theorem 3.1. Pick $f \in B$ and let $I' \subset I$ be a finite set satisfying the condition

$$p_i(k^1)=p_i(k^2),\ \forall i \in I \setminus I' \Rightarrow f(k^1)=f(k^2)\,.$$

Denote, as before,

$$B' = \{g \in H : k^1, k^2 \in \times K_i,\ p_i(k^1)=p_i(k^2),\ \forall i \in I \setminus I' \Rightarrow g(k^1)=g(k^2)\}\,.$$

We have already remarked that B' is isometrically isomorphic to the space of all continuous and multi-affine functions on $\times\{K_i : i \in I'\}$. Thus, by the validity of the theorem for finite affine products, the functional $\tilde{k}_{|B'}$ is an extreme point of the closed unit ball of B'^*. Hence $\tilde{k}(f)=l_1(f)=l_2(f)$, that is, $\tilde{k} \in \partial \prod\{K_i : i \in I\}$.

The proof of the "only if" part is almost identical with that of the corresponding part of Theorem 3.2 in [13] so we omit it.

The next result establishes the relationship between the cartesian product of compact Hausdorff spaces and the affine product of simplexes defined by us. If S is a compact Hausdorff space, we denote by $\mathscr{M}_1(S)$ the set of all Radon probability measures on S endowed with the w^*-topology as a subset of $C^*(S)$. It is well known that $\mathscr{M}_1(S)$ is an r-simplex and conversely, every r-simplex admits such a representation (cf. [5]).

COROLLARY 3.3. *If $\{S_i : i \in I\}$ is a family of compact Hausdorff spaces, then $\mathscr{M}_1(\times S_i)=\prod \mathscr{M}_1(S_i)$. The affine product of a family of simplexes is an r-simplex if and only if each factor is an r-simplex.*

PROOF. By the preceding theorem $\partial \prod \mathscr{M}_1(S_i)=\times \partial \mathscr{M}_1(S_i)=\times S_i$ and the result follows from the fact that an r-simplex is completely determined by its set of extreme points. The second assertion is a trivial consequence of Theorem 3.2, the Tychonoff theorem and its converse.

REMARK. It is easy to check that even if K_1, K_2 are one-dimensional simplexes, the image of $K_1 \times K_2$ does not fill up the affine product $K_1 \prod K_2$.

If $f \in H$ then we denote by $\tilde{f}$ the continuous and affine function on $\prod\{K_i : i \in I\}$ corresponding to f, that is, $\tilde{f}(k)=f(k)$, $k \in \times\{K_i : i \in I\}$. So far only real valued functions have been extended to $\prod\{K_i : i \in I\}$. The next theorem shows that continuous and multi-affine maps into a locally convex linear topological space can be extended to the affine product.

THEOREM 3.4. *Let C be a compact convex subset of a locally convex linear topological space E and $T : \times\{K_i : i \in I\} \to C$ a continuous and multi-affine map. Then there exists a unique continuous and affine map $\tilde{T} : \prod\{K_i : i \in I\} \to C$ such that $\tilde{T}(\tilde{k})=T(k)$ for every $k \in \times\{K_i : i \in I\}$. If E is a Banach space, then*

$$\sup_{l \in \Pi K_i} \|\tilde{T}(l)\| = \sup_{k \in \times K_i} \|T(k)\| .$$

PROOF. Put $K = \prod\{K_i : i \in I\}$ and let $T' : \partial K \to C$ be the map defined by $T'(\tilde{k}) = T(k)$, $k \in \times\{\partial K_i : i \in I\}$. For any $x^* \in E^*$ we have

$$\widehat{x^* \circ T'} = \widecheck{x^* \circ T'} = \overline{x^* \circ T}$$

since $\overline{x^* \circ T}$ is continuous and affine on K, and on ∂K coincides with $x^* \circ T'$. By Theorem 2.2, T' has a continuous and affine extension $\tilde{T} : K \to C$. If $k \in \times\{K_i : i \in I\}$ and $x^* \in E^*$ then

$$x^*(\tilde{T}(\tilde{k})) = \overline{x^* \circ T}(\tilde{k}) = x^*(T(k)) ,$$

which means that $\tilde{T}(\tilde{k}) = T(k)$. The second assertion of the theorem follows easily from the fact that $\tilde{T}(K) = \overline{\operatorname{conv}}\, T(\times K_i)$.

REMARK. In case C is a simplex the same result was obtained by Davies and Vincent-Smith [7].

COROLLARY 3.5. *Let $\{K_i : i \in I_1\}$, $\{K_i : i \in I_2\}$ be two families of complexes. If $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \emptyset$, then $\prod\{K_i : i \in I\}$ is affinely homeomorphic to $(\prod_{i \in I_1} K_i) \prod (\prod_{i \in I_2} K_i)$.*

PROOF. Denote $K_1 = \prod\{K_i : i \in I_1\}$, $K_2 = \prod\{K_i : i \in I_2\}$. Let H be, as usual, the space of all continuous and multi-affine functions on $\times\{K_i : i \in I\}$. Let H' be the similarly defined space for $\times\{K_i : i \in I_2\}$. By Theorem 2.3 it suffices to show that H is isometrically isomorphic to $A(K_1 \prod K_2) = A(K_1, A(K_2)) = A(K_1, H')$. Pick $f \in H$. Define

$$f' : \times\{K_i : i \in I_1\} \to H'$$

by

$$f'(k^1)(k^2) = f(k), \qquad k^1 \in \times\{K_i : i \in I_1\},\ k^2 \in \times\{K_i : i \in I_2\} ,$$

where

$$p_i(k) = \begin{cases} p_i(k^1), & i \in I_1 , \\ p_i(k^2), & i \in I_2 . \end{cases}$$

From the previous theorem we infer that there is a continuous and affine function $\tilde{f}' : K_1 \to H'$ such that $\tilde{f}'(\tilde{k}^1) = f'(k^1)$. Clearly, $\|\tilde{f}'\| = \|f\|$ and the map $f \to \tilde{f}'$ from H to $A(K_1, H')$ is linear. It is easily seen that the image of H by this map is all of $A(K_1, H')$. Thus we have found the desired isometry.

4. Projections and closed faces.

Now we turn to investigate some properties of the faces of the affine product $\prod\{K_i : i \in I\}$.

For any index $i \in I$ one can embed $A(K_i)$ isometrically into $A(\prod K_i)$ by the map

$$T_i f = \tilde{f}', \quad f \in A(K_i)\,,$$

where

$$f'(k) = f(p_i(k)), \quad k \in \times\{K_i : i \in I\}\,.$$

$T_i{}^*$ maps $A^*(\prod K_i)$ onto $A^*(K_i)$ and the image of $\prod\{K_i : i \in I\}$ is K_i. We denote by P_i the restriction of $T_i{}^*$ to $\prod\{K_i : i \in I\}$ and call it the projection of the affine product onto K_i. Obviously P_i is continuous, affine, and if $k \in \times\{K_i : i \in I\}$, then $P_i(k) = p_i(k)$. Actually, one may define P_i as the extension of p_i given by Theorem 3.4.

PROPOSITION 4.1. *If F is a closed face of $\prod\{K_i : i \in I\}$, then $P_i(F)$ is a closed face of K_i.*

PROOF. Since $P_i(\partial F) \subset \partial K_i$ by Theorem 3.2, this is a particular case of a result of Davies and Vincent-Smith [7, Lemma 6, Corollary 6].

THEOREM 4.2. *If P_i is a closed face of K_i for every $i \in I$, then $F = \bigcap_{i\in I} P_i{}^{-1}(F_i)$ is a closed face of $\prod\{K_i : i \in I\}$ affinely homeomorphic to $\prod\{F_i : i \in I\}$.*

PROOF. Clearly, F is a closed convex subset of $\prod\{K_i : i \in I\}$. Suppose that $l_1, l_2 \in \prod\{K_i : i \in I\}$, $0 < \lambda < 1$, and $\lambda l_1 + (1-\lambda)l_2 \in F$. Then $\lambda P_i(l_1) + (1-\lambda)P_i(l_2) \in F_i$ and this implies $P_i(l_1) \in F_i$, $P_i(l_2) \in F_i$. Therefore, $l_1 \in F$, $l_2 \in F$, which means that F is a face of K.

Now we turn to show that $A(F)$ is isometrically isomorphic with H, the space of all continuous and multi-affine functions on $F' = \times\{F_i : i \in I\}$; according to Theorem 2.3, from this it will result that F and $\prod\{F_i : i \in I\}$ are affinely homeomorphic. Consider F' canonically embedded in F. The restrictions to F' of the functions of $A(F)$ form (by [4]) a space isometrically isomorphic with $A(F)$, which we denote by $A(F)_{|F'}$. We have $A(F)_{|F'} \subset H \subset C(F')$, and it is easy to see that the Choquet boundary of both $A(F)_{|F'}$ and H is ∂F. Since $A(F)_{|F'}$ has the Riesz separation property, it follows from the generalization of the Stone–Weierstrass theorem by Edwards and Vincent-Smith [10] that $A(F)_{|F'} = H$.

ADDED IN PROOF: Some of our results were obtained independently by A. Hulanicki and R. R. Phelps in a forthcoming paper about tensor products of ordered linear spaces.

REFERENCES

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. 15 (1964), 97–110.
2. E. M. Alfsen, *A note on affine functions on compact sets*, Math. Seminar Reports, Oslo Univ., Vol. 3, No. 13 (1965).
3. E. M. Alfsen, *Boundary values for homomorphisms of compact convex sets*, Math. Scand. 19 (1966), 113–121.
4. H. Bauer, *Minimalstellen von Funktionen und Extremalpunkte*, Archiv der Math. 9 (1958), 389–393.
5. H. Bauer, *Šilovscher Rand und Dirichletsches Problem*, Ann. Inst. Fourier (Grenoble) 11 (1961), 89–136.
6. G. Choquet et P. A. Meyer, *Existence et unicité des représentations intégrales dans les convexes compacts quelconques*, Ann. Inst. Fourier (Grenoble) 13 (1963), 139–154.
7. E. B. Davies and G. F. Vincent-Smith, *Tensor products and projective limits of Choquet simplexes*, Math. Scand. 22 (1968), 145–164.
8. M. M. Day, *Normed linear spaces*, (Ergebn. der Math., Neue Folge 21) Berlin · Göttingen · Heidelberg, 1958.
9. D. A. Edwards, *Séparation des fonctions réelles définies sur un simplexe de Choquet*, C. R. Acad. Sci. Paris 261 (1965), 2798–2800.
10. D. A. Edwards and G. F. Vincent-Smith, *A Weierstrass–Stone theorem for Choquet simplexes* (to appear).
11. E. G. Effros, *Structure in simplexes* II, J. Functional Analysis 1 (1967), 379–391.
12. A. J. Lazar, *Spaces of affine continuous functions on simplexes* (to appear).
13. A. J. Lazar, *Affine functions in simplexes and extreme operators*, Israel J. Math. 5 (1967), 31–43.
14. J. Lindenstrauss, *Extension of compact operators*, (Memoirs Amer. Math. Soc. 48), Providence, R. I., 1964.
15. R. R. Phelps, *Lectures on Choquet's theorem* (Van Nostrand Mathematical Studies 7), Princeton, N.J., 1966.
16. Z. Semadeni, *Free compact convex sets*, Bull. Acad. Polon. Sci. 13 (1964), 141–146.

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