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ON ONE-SIDED INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS ON $\mathsf{R}^\varkappa$

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1. Introduction.

Consider distribution functions (d.f.'s) on a $\varkappa$-dimensional Euclidean space $\mathsf{R}^\varkappa$. Denote the points on $\mathsf{R}^\varkappa$ by letters $x,y,\ldots$, and their coordinates by the same letters with subscripts, $x=(x_1,x_2,\ldots x_\varkappa)$. A direction is a vector $\nu=(\nu_1,\nu_2,\ldots,\nu_\varkappa)$. A point x^0 is called a boundary point of the support of a distribution function G with respect to the direction ν if

$$\int\limits_{\nu\cdot(x-x^0)<\varepsilon} dG(x) \begin{cases} = 0 & \text{for } \varepsilon=0 \\ > 0 & \text{for } \varepsilon>0 \,. \end{cases}$$

In the 1-dimensional case there are only two directions. The boundaries of the support of an infinitely divisible d.f. G in R^1 have been investigated by G. Baxter and J. G. Shapiro [1]; G. G. Tucker [5], C. G. Esseen [3], and M. Jiřina [4] gave the exact boundaries in relation to the invariants in the Lévy representation of the characteristic function.

It may be of interest to investigate boundaries in some sense also for d.f.'s on $\mathsf{R}^\varkappa$ for $\varkappa>1$.[1] We shall show that it is possible to associate these boundaries with the divisibility property and with invariants directly related to this property, and also with the invariants in the Lévy representation. For this purpose we essentially use the limit theorem for sequences of convolutions of d.f.'s. Hence our method differs from the methods used for one-dimensional d.f.'s by the authors quoted above. The most laborious part of our investigation is the construction of a d.f. with given invariants, carried out in Section 2. This construction, however, may have an interest of its own.

From [2] we quote two theorems which we state here in special forms in accordance with the present applications. At first we shall make some remarks on concepts used in the book mentioned.

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[1] Dr. Richard Savage inspired me to this investigation by pointing out its importance for some applications. The work was prepared while I was a visiting professor in the Department of Statistics, Florida State University, Tallahassee, Florida.

Consider functions q which are nondecreasing in the sense used in [2, p. 238] at all points different from the origin. Let the function Q_j defined by

$$(1.1) \qquad Q_j(x_j) = \lim_{x_k \to +\infty} q(x), \quad k=1,2,\ldots,\varkappa,\ k \neq j$$

exist for $j=1,2,\ldots,\varkappa$. It is called the jth marginal function of q. A point x is called exceptional for q if it is a discontinuity point of some marginal function of q.

For a d.f. F_n on $\mathsf{R}^{\varkappa}$ we consider the sequence $\{n[F_n-e]\}$, where e denotes the unit d.f. and say that this sequence tends completely to a function q (which then is of the type just mentioned) if the sequence tends to q at all non-exceptional points for q and if, at any infinite limit point $\bar{x}$ (i.e., with at least one coordinate equal to $+\infty$ or $-\infty$), it tends to the corresponding limiting value $q(\bar{x})$. For this complete convergence we use the notation c-lim $n[F_n-e]$.

We say that a number η is proper for q if all points x with $x_j=+\eta$ or $x_j=-\eta$ for some j are non-exceptional for q. Of course the set of proper numbers η for q is dense on $(0,+\infty)$ since the discontinuity points of Q_j are countable. We use the notations

$$D(\eta) = \{x :\ |x_k| < \eta,\ k=1,2,\ldots,\varkappa\},$$
$$\bar{D}(\eta) = \{x :\ |x_k| \leqq \eta,\ k=1,2,\ldots,\varkappa\}.$$

The following theorem is a special form of Bergström [2, Theorem II 6.4, p. 321].

THEOREM 1.1. *A sequence F_n^{*n} of convolution powers of* d.f.'s *F_n on $\mathsf{R}^{\varkappa}$, $n=1,2,\ldots$, converges completely to a* d.f. *G if and only if the sequence $\{n[F_n-e]\}$ is Cauchy convergent in the Gaussian norm.*

We combine this theorem with Theorem II 5.3.3 in [2] which we use in the following special cases.

THEOREM 1.2. *A sequence $\{F_n\}$ of* d.f.'s *is Cauchy convergent in the Gaussian norm if and only if F_n tends completely to a* d.f. *F.*

THEOREM 1.3. *The sequence $\{n[F_n-e]\}$ defined in Theorem 1 is Cauchy convergent in the Gaussian norm if and only if the following conditions hold:*

$$\text{(i)} \qquad \text{c-lim}\, n[F_n(x)-e(x)] = q(x)$$

exists (q finite for $x \neq 0$),

$$\text{(ii)} \qquad \lim_{n\to+\infty} n \int_{\bar{D}(\eta)} x_j \, dF_n(x) = \alpha_j(\eta)$$

exists for numbers η proper for q,

(iii)
$$\lim_{n+\infty} n \int_{\bar{D}(\eta)} x_i x_j \, dF_n(x) = \alpha_{ij}(\eta)$$

exists for numbers η proper for q.

REMARK 1. The condition (i) implies the following properties of q:

a) q is nondecreasing and bounded on $\mathsf{R}^{\varkappa} - D(\eta)$ for any $\eta > 0$,
b) $\lim_{x_k \to -\infty} q(x) = 0$ for any k,
c) $\lim_{\text{all } x_k \to +\infty} q(x) = 0$.

Hence

$$\int_{|x_j| \geqq \eta_0} dq(x) < +\infty \tag{1.2}$$

for any $\eta_0 > 0$ and any j.

The condition (iii) implies the relation

$$\int_{\bar{D}(\eta) - D(0+)} x_j^2 \, dq(x) < +\infty , \tag{1.3}$$

the notation indicating the limit as $\eta' \to 0+$ of the corresponding integral over $\bar{D}(\eta) - D(\eta')$.

REMARK 2. The quantities q, α_i and α_{ij} are invariants of the infinitely divisible d.f. $G = \text{c-lim} F_n{}^{*n}$. We call q the first invariant, the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ the second invariant, and the matrix (α_{ij}) the third invariant. The second invariant is uniquely determined by q and $\alpha(\eta_0)$ for any number $\eta_0 > 0$ which is proper for q. The third invariant is uniquely determined by q and the limits $\alpha_{ij}(0+) = \lim_{\eta \to 0+} \alpha_{ij}(\eta)$ which exist when η tends to $0+$ through proper values.

REMARK 3. The conditions (ii) and (iii) are here given in a form slightly different from the conditions 2° in the quoted theorem, but it is easily seen that the conditions 1° and 2° in that theorem are equivalent to the conditions (i)–(iii) in the theorem above.

According to Theorems 1 and 2 an infinitely divisible d.f. can be constructed as a limit of a sequence of convolution powers. We shall show in Section 2 how this construction can be carried out when the invariants are given.

Now we state the main theorem of this paper, the proof of which will be given in Section 3.

THEOREM 1.4. *In order that x^0 be a boundary point of the support of an* *infinitely divisible* d.f. G *on* $R^\varkappa$ *with respect to the direction* ν *it is necessary and sufficient that the invariants* q, α *and* (α_{ij}) *of* G *satisfy the conditions*

1° $\displaystyle\int_{\nu\cdot x<0} dq(x) = 0,$

2° $\nu\cdot\alpha(\eta) \to \nu\cdot x^0$ *as* $\eta \to 0+$,

3° $\{\alpha_{ij}(0+)\}$ *is the covariance matrix of a singular normal* d.f. *which has its support on the hyperplane* $\nu\cdot x = 0$.

2. Construction of infinitely divisible d.f.'s.

LEMMA 2.1. *Let* F *and* G *be infinitely divisible* d.f.'s *on* $R^\varkappa$ *and* $F = F_n{}^{*n}$, $G = G_n{}^{*n}$ *for any positive integers* n. *Then*

$$F * G = \text{c-lim}\,[\tfrac{1}{2}(F_n + G_n)]^{*2n}\,.$$

REMARK. More generally we have the relation

$$F_1 * F_2 * \ldots * F_k = \lim_{n\to+\infty}\left(\frac{F_{1n} + F_{2n} + \ldots + F_{kn}}{k}\right)^{*kn}$$

if the F_i are infinitely divisible d.f.'s and $F_i = F_{in}^{*n}$ for all n. This can be proved by means of the general product-sum inequality in Bergström [2, p. 161]. However, one may use a special method for the case $k = 2$, as we show here.

PROOF. For any two d.f.'s F and G we have (cf. Bergström [2, Lemma 7.11, p. 82]).

$$\begin{aligned}(2.1)\qquad \|(F_n * G_n)^{*n} - \tfrac{1}{2}(F_n + G_n)^{*2n}\| &\leqq n\|F_n * G_n - \tfrac{1}{2}(F_n + G_n)^2\| \\ &= \tfrac{1}{4}n\,\|(F_n - G_n)^{*2}\| = \tfrac{1}{4}n\,\|[(F_n - e) - (G_n - e)]^{*2}\|\,,\end{aligned}$$

where $\|\cdot\|$ denotes the Gaussian norm. By [2, Theorem II 4.2.3, p. 301], we have

$$\|(F - e) * (G - e)\| \leqq \gamma_1\|F - e\|\;\|G - e\|$$

with a constant γ_1 for any d.f.'s F and G. Using this inequality we get from (2.1)

$$(2.2)\qquad \|(F_n * G_n)^{*n} - \tfrac{1}{2}(F_n + G_n)^{*2n}\| \leqq \tfrac{1}{4}n\gamma_1[\|F_n - e\| + \|G_n - e\|]^2\,.$$

Since $F_n{}^{*n}$ and $G_n{}^{*n}$ tend completely to F and G, respectively, we find by Theorem 1.1 and Theorem 1.3 that $\{n[F_n - e]\}$ and $\{n[G_n - e]\}$ are Cauchy convergent in the Gaussian norm and so they are bounded in this norm. Hence the right hand side of (2.2) tends to 0 as $n \to +\infty$ and by Theorem 1.2, $[\frac{1}{2}(F_n+G_n)]^{*2n}$ tends completely to $F*G$ which obviously is infinitely divisible.

It may be observed that we use the boundedness of $n[F_n-e]$ and $n[G_n-e]$ in the Gaussian norm only in order to prove that the right hand side of (2.2) tends to zero. By means of Lemma 1 the construction problem can be reduced. At first we observe that the normalized normal d.f. Φ with moment matrix $(\alpha_{ij}(0+))$ has first and second invariants equal to 0. Hence if F is an infinitely divisible d.f. with the invariants q, α, (α_{ij}'), with $\alpha_{ij}'(0+)=0$ for all i and j, then $\Phi*F$ has the invariants q, α, (α_{ij}).

Since q has the property (1.2), we can determine a sequence $\{\eta_n\}$ of values proper for q such that $\eta_n \to 0+$ as $n \to +\infty$, and

$$\frac{1}{n} \int_{R^{\varkappa}-D(\eta_n)} dq(x) = \beta_n < 1 .$$

Define the d.f. H_n by

$$H_n(x) = \frac{1}{n} \int_{[t:t\leqq x]-D(\eta_n)} dq(t) + (1-\beta_n)e(x) \tag{2.3}$$

and the moment vector $(c_1^{(n)},\ldots,c_{\varkappa}^{(n)})$ by

$$c_j^{(n)} = \int_{D(\eta_0)} x_j \, dH_n(x) \tag{2.4}$$

for some value η_0 proper for q. Then put

$$d^{(n)} = c^{(n)} - \alpha(\eta_0)/n \tag{2.5}$$

and

$$F_n(x) = \tfrac{1}{2}[H_n(x) + e(x+d^{(n)})] . \tag{2.6}$$

It easily follows that F_n is mean-continuous when q is mean-continuous and η_0 is proper for q. We state

THEOREM 1. *The* d.f. $F = \text{c-lim} F_n{}^{*2n}$, *where* F_n *is defined by* (2.6), *is infinitely divisible with the invariants* q, α, (α_{ij}'), $\alpha_{ij}'(0+)=0$ *for all* i *and* j.

For the proof of this theorem we need

LEMMA 2.2. *We have* $c_j^{(n)} = o(n^{-\frac{1}{2}})$ *as* $n \to +\infty$.

We first use this lemma to prove the theorem and then prove the lemma.

PROOF OF THEOREM 1. By the definition of F_n we get, choosing $x_j \leqq -\eta < -\eta_n$ for some j,

$$2n[F_n(x)-e(x)] = 2nF_n(x) = q(x)+ne(x+d^{(n)}) \to q(x) \quad \text{as } n \to \infty,$$

since $x_j+d_j^{(n)}<0$ for sufficiently large n. Choosing $x_j \geqq \eta > 0$ for all j we get

$$2n[F_n(x)-e(x)] \to q(x) \quad \text{as } n \to +\infty ,$$

since $x_j+c_j^{(n)}>0$ for sufficiently large n. Furthermore,

$$2n \int_{\overline{D}(\eta_0)} x_j\, dF_n(x) = n \int_{\overline{D}(\eta_0)} x_j\, dH_n(x) - nc_j^{(n)} + \alpha_j(\eta_0) = \alpha_j(\eta_0)$$

and

$$2n \int_{\overline{D}(\eta)} x_j^2\, dF_n(x) \leqq \int_{\overline{D}(\eta)-D(0+)} x_j^2\, dq(x) + 2n(d_j^{(n)})^2 \to \int_{\overline{D}(\eta)-D(0+)} x_j^2\, dq(x) ,$$

since $d_j^{(n)}=o(n^{-\frac{1}{2}})$ by Lemma 2 and the definition of $d^{(n)}$. Clearly the last integral tends to 0 as $\eta \to 0+$. Then also

$$2n \int_{\overline{D}(\eta)-\overline{D}(0+)} x_i x_j\, dF_n(x) \to 0$$

as first $n \to +\infty$ and then $\eta \to 0+$. The rest of the proof follows from Theorems 1.1 and 1.3.

PROOF OF LEMMA 2.2. We observe that

$$|c_j^{(n)}| = \left| \int_{\overline{D}(\eta_0)} x_j\, dH_n(x) \right| \leqq \frac{1}{n} \int_{|\eta_n| \leqq |x_j| \leqq \eta_0} |x_j|\, dq(x) + \frac{1}{n}\eta_n \int_{\overline{D}(\eta_0)-\overline{D}(\eta_n)} dq(x) .$$

The first term on the right hand side is equal to

$$\frac{1}{n}\left| \int_{-\eta_0}^{-\eta_n} x_j\, dQ_j(x_j) \right| + \frac{1}{n}\int_{\eta_n}^{\eta_0} x_j\, dQ_j(x_j) .$$

Both these terms are $o(n^{-\frac{1}{2}})$ according to Bergström [2, Lemma I, 7.11, p. 102], and

$$\frac{1}{n}\eta_n \int_{D(\eta_0)-D(\eta_n)} dq(x) \leqq \frac{1}{n}\sum_{j=1}^{\varkappa} \eta_n \int_{\eta_n \leqq |x_j| \leqq \eta_0} dQ_j(x_j)$$

$$\leqq \frac{1}{n}\sum_{j=1}^{\varkappa} \int_{\eta_n \leqq |x_j| \leqq \eta_0} |x_j|\, dQ_j(x_j) = o(n^{-\frac{1}{2}}) .$$

3. Proof of the main theorem.

LEMMA 3.1. *For any direction* ν *and* d.f.'s F *and* G *the following implications hold*:

$$1^\circ \quad \int_{\nu\cdot x<0} dF(x) = 0, \int_{\nu\cdot x<0} dG(x) = 0 \Rightarrow \int_{\nu\cdot x<0} dF*G(x) = 0\,,$$

$$2^\circ \quad \int_{\nu\cdot x<0} dF*G(x) > 0 \Leftarrow \int_{\nu\cdot x<0} dF(x) > 0, \int_{\nu\cdot x<0} dG(x) > 0\,,$$

$$3^\circ \quad \int_{\nu\cdot x<0} dF(x) = 0 \Leftrightarrow \int_{\nu\cdot x<0} dF^{*n}(x) = 0, \qquad n \text{ any positive integer.}$$

PROOF. The first two implications follow from the inequalities

$$\int_{\nu\cdot x<0} dF*G(x) \leqq \int_{\nu\cdot x<0} dF(x) + \int_{\nu\cdot x<0} dG(x)\,, \tag{3.1}$$

$$\int_{\nu\cdot x<0} dF*G(x) \geqq \int_{\nu\cdot x<0} dF(x) \int_{\nu\cdot x<0} dG(x)\,, \tag{3.2}$$

and 3° follows from 1° and 2°.

LEMMA 3.2. *In order that the zero point be a boundary point of the support of an infinitely divisible* d.f. G *on* $R^\varkappa$ *with respect to the direction* ν *it is necessary that its invariants* q *and* $\{\alpha_{ij}\}$ *satisfy the following conditions*:

$$1^\circ \quad \int_{\nu\cdot x<0} dq(x)=0,$$

$$2^\circ \quad \int_{\overline{D}(\eta)-D(0+)} \nu\cdot x\, dq(x) < +\infty \text{ for any } \eta>0,$$

3° $\{\alpha_{ij}(0+)\}$ *is the covariance matrix of a singular normal* d.f. *which has its support on the hyperplane* $\nu\cdot x=0$.

PROOF. Let $G=G_n{}^{*n}$ for any positive integer n where G_n is a d.f. If zero is a boundary point of G with respect to ν then, according to Lemma 3.1, 3°, it is also a boundary point of G_n with respect to ν. Hence applying Theorem 1.3 we get

$$\int_{\nu\cdot x<0} dq(x) = \lim_{n\to+\infty} n \int_{\nu\cdot x<0} dG_n(x) = 0\,,$$

and thus 1°.

According to the same theorem we also have

$$\nu\cdot\alpha(\eta) = \lim_{n\to+\infty} n \int_{\overline{D}(\eta)} \nu\cdot x \, dG_n(x)$$

and

$$\nu\cdot\alpha(\eta) \geqq \lim_{n\to+\infty} n \int_{\overline{D}(\eta)-D(\varepsilon)} \nu\cdot x \, dG_n(x) = \int_{\overline{D}(\eta)-D(\varepsilon)} \nu\cdot x \, dq(x) ,$$

if η and ε are proper for q. Letting $\varepsilon \to 0$ we get 2°.

Since G is uniquely determined by its invariants, we find, by the construction in Section 2, that G is a convolution of two infinitely divisible d.f.'s H and Φ with the invariants q', α', (α_{ij}') and q'', α'', (α_{ij}''), respectively, where $q'=q$, $\alpha'=\alpha$, $\alpha_{ij}'(0+)=0$, $q''=0$, $\alpha''=0$, $\alpha_{ij}''(0+)=\alpha_{ij}(0+)$. Then Φ is a normal d.f. with the covariance matrix $\{\alpha_{ij}(0+)\}$. We have to show that it is singular with its support on the hyperplane $\nu\cdot x=0$. Now if Φ has not its support on the hyperplane $\nu\cdot x=0$, we have

$$\int_{\nu\cdot x<-h} d\Phi(x) > 0$$

for any real number h and thus

$$\int_{\nu\cdot x<0} dG(x) \geqq \int_{\nu\cdot x<h} dH(x) \int_{\nu\cdot x<-h} d\Phi(x) > 0$$

for $h>0$ sufficiently large. This contradicts the fact that zero is a boundary point of the support of G with respect to ν.

The lemma is proved.

In order to prove Theorem 1.4 we first suppose that G is an infinitely divisible d.f. with the invariants q, α and (α_{ij}), and assume that x° is a boundary point of the support of G with respect to the direction ν. The conditions 1°–3° of Lemma 3.2 must be satisfied. Denote by $G(\cdot+x^0)$ the function which has the value $G(x+x^0)$ at the point x. Then the zero point is a boundary point of the support of $G(\cdot+x^0)$ with respect to ν and $G(\cdot+x^0)$ has the invariants q, $\alpha-x^0$, and (α_{ij}). In fact we have

$$G(\cdot+x^0) = G * e(\cdot+x^0),$$

and by Lemma 2.1 we get

$$G(\cdot+x^0) = \text{c-lim}\{\tfrac{1}{2}[G_n+e(\cdot+x^0/n)]\}^{*2n} .$$

Hence (cf. Theorem 1.3) $G(\cdot+x^0)$ has the invariants given above. We have thus reduced the proof to the case where $x^0=0$.

It is easily seen that $\nu\cdot\alpha(\eta)$ is non-negative and nondecreasing. Hence

$$\lim_{\eta\to 0+}\nu\cdot\alpha(\eta) = h \tag{3.3}$$

exists and $h \geqq 0$. Assume that h is not 0. If y is a point on the hyperplane $\nu\cdot x = h$, then $G(\cdot+y)$ has the invariants q, $\alpha - y$, (α_{ij}), and

$$\lim_{\eta\to 0+}\nu\cdot(\alpha(\eta)-y) = 0\,. \tag{3.4}$$

As in Section 2 we construct an infinitely divisible d.f. $\bar{G}$ with these invariants. Let

$$F = \text{c-lim}\, F_n{}^{*2n}, \qquad \bar{G} = F*\Phi\,,$$

where

$$F_n(x) = \tfrac{1}{2}[H_n(x) + e(x+d^{(n)})]\,, \tag{3.5}$$

$$d^{(n)} = c^{(n)} - [\alpha(\eta_0)-y]/n\,, \tag{3.6}$$

$$c_j^{(n)} = \int\limits_{\bar{D}(\eta_0)} x_j\, dH_n(x) = \frac{1}{n}\int\limits_{\bar{D}(\eta_0)-D(\eta_n)} x_j\, dq(x)\,.$$

The last relation follows from the definition of H_n (cf. (2.3) and (2.4)). Since $q(x)$ is constant on $\nu\cdot x<0$, we get

$$H_n(x) = \frac{1}{n}\int\limits_{\{[t:t\leqq x]-D(\eta_n)\}\cap[\nu\cdot x\geqq 0]} dq(t) + [1-\beta_n]e(x)\,.$$

By (3.3), (3.4), and Lemma 2 we get

$$\nu\cdot(\alpha(\eta_0)-y) = \nu\cdot\alpha(\eta_0) - \lim_{\eta\to 0+}\nu\cdot\alpha(\eta) = \int\limits_{D(\eta_0)-D(0+)} \nu\cdot x\, dq(x)\,.$$

Regarding (3.6) we thus get

$$n\nu\cdot d^{(n)} = n\nu\cdot c^{(n)} - \nu\cdot(\alpha(\eta_0)-y) = \int\limits_{\bar{D}(\eta_0)-D(\eta_n)} \nu\cdot x\, dq(x) - \int\limits_{\bar{D}(\eta_0)-D(0+)} \nu\cdot x\, dq(x)\,,$$

where the last difference is non-positive since $q(x)$ is constant on $\nu\cdot x<0$. Thus $\nu\cdot d^{(n)}\leqq 0$ and hence $e(x+d^{(n)})$ is constant on $\nu\cdot x<0$. Since $H_n(x)$ and $e(x+d^{(n)})$ are constant on $\nu\cdot x<0$, $F_n(x)$ is constant on $\nu\cdot x<0$, and thus $F(x)$ is constant on $\nu\cdot x<0$, according to Lemma 3.1.

Now $F*\Phi$ has the same invariants as $G(\cdot+y)$ and thus is identical with this d.f. But then $G(x)$ is constant on $\nu\cdot x<h$ in contradiction to the assumption that the zero point was a boundary point of the support of G with respect to the direction ν. Hence $h=0$. By the construction of F we have proved that there exist infinitely divisible d.f.'s to any given invariants satisfying the conditions in Theorem 1.4.

By the method that we have presented it is possible to construct infinitely divisible d.f's with a given support, since we may consider boundary points with respect to different points and directions.

At last we shall sketch an alternative formally simpler but less elementary proof of Theorem 1.4, still using the theorem about the existence of infinitely divisible d.f.'s G with given invariants and the factorization $G=F*\Phi$. We then give a procedure by which we reduce the proof in the general case to the special case $\varkappa=1$. However, it is easily seen that the simplification is only formal.

Let us study the measures μ_G, μ_F and μ_Φ which belong to G, F and Φ respectively. We introduce a new coordinate system by a linear transformation $x=By$, chosen so that $\nu\cdot x=y_1$. We require that the zero point be a boundary point of the support of G with respect to the direction ν. Let $\bar{G}$, $\bar{F}$ and Φ be the d.f.'s corresponding to μ_G, μ_F, and μ_Φ, respectively, in the new coordinate system. These d.f.'s are infinitely divisible and the zero-point is a boundary point of the support of $\bar{G}$ with respect to the direction orthogonal to the hyperplane $y_1=0$. Define $\bar{G}^{(1)}$ by

$$\bar{G}^{(1)}(y_1) = \lim_{y_2,\ldots,y_\varkappa\to+\infty}\bar{G}(y)\,,$$

and $\bar{F}^{(1)}$ and $\bar{\Phi}^{(1)}$ in the same way. Clearly $\bar{\Phi}^{(1)}$ is the unit d.f. on the line. Further, putting $F=\bar{F}_n{}^{*n}$ for every positive integer n and observing that F is infinitely divisible, we get $\bar{F}^{(1)}=(\bar{F}_n{}^{(1)})^{*n}$. If now Theorem 1.4 is proved in the case $\varkappa=1$ we find by this theorem that the necessary and sufficient conditions for the zero-point being a left boundary point of $\bar{F}^{(1)}$ are as follows:

$$1^\circ\ \lim_{n\to+\infty} n\int_{y_1<-\eta} d\bar{F}_n{}^{(1)}(y_1) = 0,\quad \eta>0,$$

$$2^\circ\ \lim_{\eta\to0+}\ \lim_{n\to+\infty} n\int_{|y_1|<\eta} y_1\,dF_n{}^{(1)}(y_1) = 0\,.$$

These conditions may be written

$$\lim_{n\to+\infty} n\int_{y_1<-\eta} d\bar{F}_n(y) = 0,\quad \eta>0\,,$$

$$\lim_{\eta\to0+}\ \lim_{n\to+\infty} n\int_{|y_1|\leqq\eta} y_1\,dF_n(y) = 0\,,$$

respectively, or, if the transformation $y=B^{-1}x$ is performed,

$$\lim_{n\to+\infty} n \int_{\nu\cdot x<-\eta} dF_n(x) = \int_{\nu\cdot x<-\eta} dq(x) = 0, \quad \eta > 0\,,$$

$$\lim_{\eta\to 0+} \lim_{n\to+\infty} n \int_{\overline{D}(\eta)} \nu\cdot x\, dF_n(x) = \lim_{\eta\to 0+} \nu\cdot\alpha(\eta) = 0\,.$$

By these conditions F is determined and Φ can be derived in the same way as before.

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