MATH. SCAND. 22 (1968), 187–196

EQUATIONAL CLASSES OF LATTICES

BJARNI JÓNSSON

Introduction.

The principal result proved in this note is

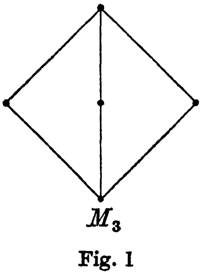
THEOREM 1. *For any equational class* V *of modular lattices the following conditions are equivalent*:

(i) $M_{3,3} \notin V$.

(ii) *Every member of* V *is a subdirect product of lattices of dimension two or less.*

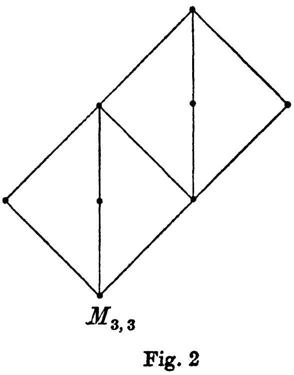
(iii) *The inclusion* $a(b+cd)(c+d) \leq b+ac+ad$ *holds in* V.

The lattice $M_{3,3}$ referred to in this theorem is the eight-element lattice pictured in Fig. 2. It is easy to determine all the equational classes V that satisfy (ii); we actually give axiom systems for all of these classes, and determine all the equational classes which cover them. Thus we obtain as a special case a solution to Problem 45 in Birkhoff [1, p. 157]. Our methods are in part borrowed from Grätzer [2], where a related but more special problem is considered.



M_3

Fig. 1



$M_{3,3}$

Fig. 2

1. Preliminaries.

Consider a modular lattice L. By a *diamond* in L we mean a five-termed sequence (v,x,y,z,u) of elements of L, whose terms are either all

Received June 12, 1967.

equal (in which case the diamond is said to be degenerate), or else form a non-distributive sublattice M_3 (Fig. 1), with u the largest element and v the smallest. The intervals $[x,u]$, $[y,u]$, $[z,u]$ are called the upper edges of the diamond, and $[v,x]$, $[v,y]$, $[v,z]$ the lower edges.

Two intervals $[a,b]$ and $[c,d]$ in L with $a+d=b$ and $ad=c$ are said to be *transposes* of each other. More specifically, we say that $[a,b]$ *transposes down* onto $[c,d]$, and that $[c,d]$ *transposes up* onto $[a,b]$, and we write

$$[a,b]\searrow[c,d], \qquad [c,d]\nearrow[a,b].$$

Two intervals $[a,b]$ and $[c,d]$ that are projective to each other are said to be *connected* by the *sequence of transposes* $[a_k,b_k]$, $k=0,1,\ldots,n$ if $[a_0,b_0]=[a,b]$ and $[a_n,b_n]=[c,d]$, and for $i=0,1,\ldots,n-1$ the i-th term transposes alternately up and down onto the next one. Two intervals are said to be *projective in n steps* if they are connected by an $n+1$-termed sequence of transposes. A sequence of transposes is said to be *normal* if, for $0<k<n$,

(1) either $[a_{k-1},b_{k-1}]\nearrow[a_k,b_k]\searrow[a_{k+1},b_{k+1}]$ and $b_k = b_{k-1}+b_{k+1}$,

(2) or $[a_{k-1},b_{k-1}]\searrow[a_k,b_k]\nearrow[a_{k+1},b_{k+1}]$ and $a_k = a_{k-1}a_{k+1}$.

If in addition, for each such k, $b_{k-1}b_{k+1}\leq a_k$ in the first case, but $a_{k-1}+a_{k+1}\geq b_k$ in the second, then the sequence is said to be *strongly normal.* It is shown in Grätzer [2] that two intervals that are projective in n steps are connected by a normal $n+1$-termed sequence of transposes.

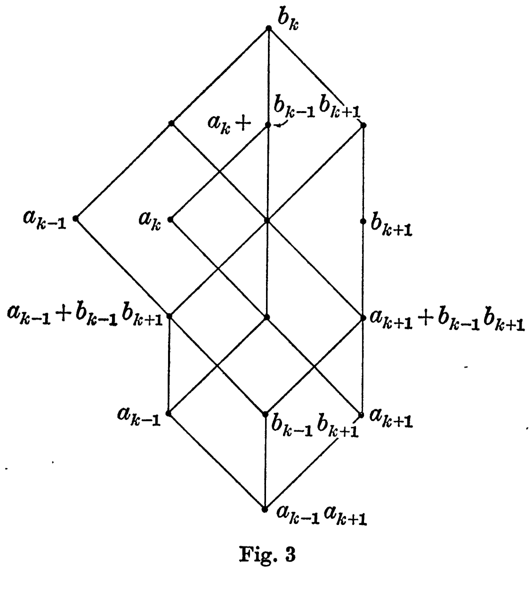


Fig. 3

On the other hand it is not always possible to take the sequence to be strongly normal.

In a normal sequence of transposes $[a_i, b_i]$, $i = 0, 1, \ldots, n$, the lattice generated by the six endpoints of three successive intervals, say by a_i, b_i with $i = k-1, k, k+1$, is in fact generated by three of these endpoints,

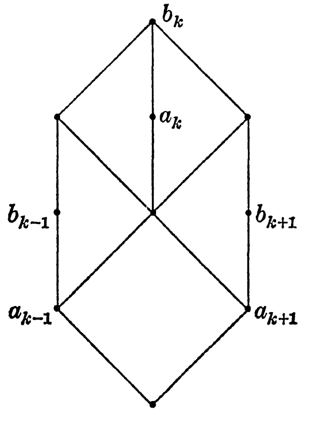


Fig. 4

and is therefore finite. If (1) holds, then this lattice is generated by b_{k-1}, a_k, b_{k+1}, and it is a homomorphic image of the lattice in Fig. 3. If (2) holds, then this lattice is of course a homomorphic image of the dual of the lattice in Fig. 3. If the sequence is strongly normal, then the lattice generated by the six endpoints is a homomorphic image of the lattice in Fig. 4, or of its dual. Each of these lattices contains a diamond, and we shall later have to investigate how two such diamonds fit together.

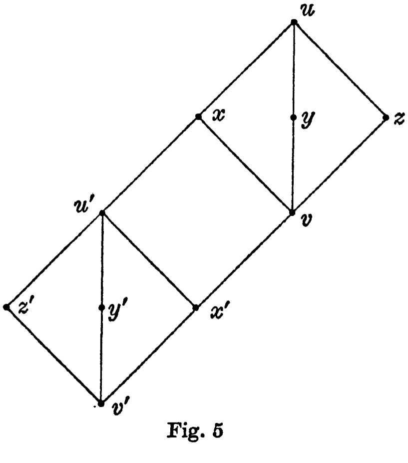


Fig. 5

Observe in this connection that if (v,x,y,z,u) and (v',x',y',z',u') are two diamonds, and if a lower edge in one transposes down onto an upper edge of the other, say $[v,x]\searrow[x',u']$, then the ten elements form a lattice which is a homomorphic image of the lattice in Fig. 5. Unless the two diamonds are degenerate, this lattice has $M_{3,3}$ as a homomorphic image.

Given a diamond (v,x,y,z,u) and an element w that belongs to one of its edges, these six elements generate a finite lattice whose isomorhism type is completely determined. In fact, if say $z<w<u$, then this lattice is given in Fig. 6. Observe that the adjunction of the new element yields two new diamonds.

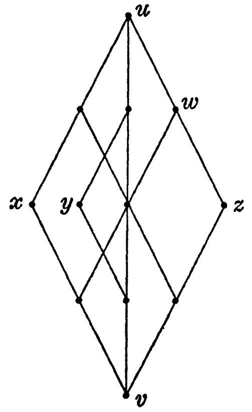


Fig. 6

2. Connecting sequences.

It is easy to see that in a subdirectly irreducible modular lattice any two non-trivial intervals have non-trivial sub-intervals that are projective to each other. Assuming that these subintervals have been so chosen that they are projective in as few steps as possible, we examine the sequences of transposes that connect them.

Lemma 2. *Suppose L is a modular lattice and $[a,b]$ and $[c,d]$ are non-trivial intervals in L that are projective in n steps. If no non-trivial subintervals of $[a,b]$ and $[c,d]$ are projective in fewer than n steps, then either $n\leq 2$, or else $[a,b]$ and $[c,d]$ are connected by a strongly normal $(n+1)$-termed sequence of transposes.*

Proof. Consider a normal sequence of transposes $[a_i,b_i]$, $i=0,1,\ldots,n$, that connects $[a,b]$ and $[c,d]$. If the sequence is not strongly normal, then for some k with $0<k<n$ we have

$$[a_{k-1}, b_{k-1}] \nearrow [a_k, b_k] \searrow [a_{k+1}, b_{k+1}] \quad \text{and} \quad b_{k-1}b_{k+1} \nleq a_k\,,$$

or else we have the dual situation. Let $c_{k-1} = a_{k-1} + b_{k-1}b_{k+1}$, and for $i \leq n$ with $i \neq k-1$ let c_i be the element of $[a_i, b_i]$ that corresponds to c_{k-1} under the given transpositions. Fig. 3 suggests, and it is easy to check arithmetically, that

$$[a_{k-1}, c_{k-1}] \searrow [a_{k-1}a_{k+1}, b_{k-1}b_{k+1}] \nearrow [a_{k+1}, c_{k+1}]\,.$$

Therefore, if $k > 1$, then

$$[a_{k-2}, c_{k-2}] \searrow [a_{k-1}a_{k+1}, b_{k-1}b_{k+1}] \nearrow [a_{k+1}, c_{k+1}]\,,$$

and if $k < n-1$, then

$$[a_{k-1}, c_{k-1}] \searrow [a_{k-1}a_k, b_{k-1}b_k] \nearrow [a_{k+2}, c_{k+2}]\,.$$

In either case it follows that the non-trivial subintervals $[a, c_0]$ of $[a, b]$ and $[c, c_n]$ of $[c, d]$ are projective in $n-1$ steps, contrary to our assumption. Thus we must have $k \leq 1$ and $k \geq n-1$, which implies that $n \leq 2$.

Lemma 3. *Suppose L is a modular lattice such that $M_{3,3}$ is not a homomorphic image of a sublattice of L. If (v,x,y,z,u) and (v',x',y',z',u') are diamonds in L such that $y' = yu'$, $z = z' + v$ and $[y,u] \searrow [v',z']$, then $[v,u] \searrow [v',u']$.*

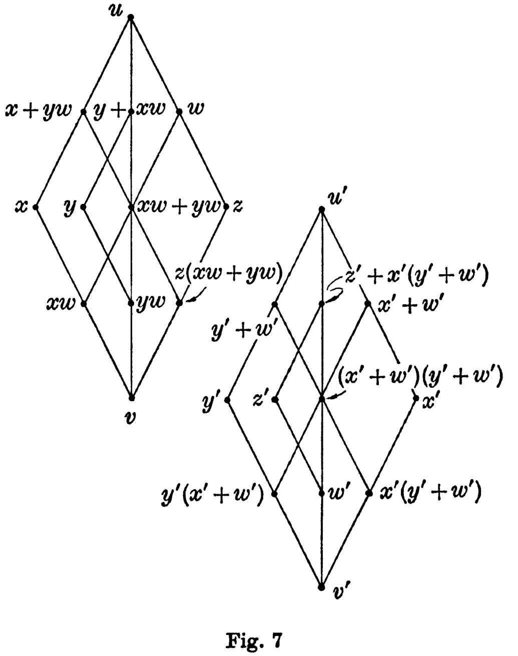


Fig. 7

PROOF. Our reasoning is motivated by Fig. 7. The element $w=v+u'$ belongs to the edge $[z,u]$ of the first diamond. If $z<w<u$, then u, x, y, z, v and w generate the lattice in the upper half of the figure. Under the transposition $[y,u]\searrow[v',z']$ the element $y+xw$ goes into $w'=z'(y+xw)$, and together with u', x', y', z' and v' this element generates the lattice in the lower half of the figure.

Without appealing to the figure one easily checks that the sequences

$$(xw+yw,x+yw,y+xw,w,u),$$
$$((x'+w')(y'+w'),y'+w',z'+x'(y'+w'),x'+w',u')$$

are diamonds. We claim that

$$[xw+yw,w]\searrow[y'+w',u'] .$$

In fact, since $u'\leq w\leq(x+u')(y+u')$, we have

$$\begin{aligned} xw+yw+u' &= (x+u')w+(y+u')w = w , \\ y'+w' &= y'+z'(y+xw) \\ &= (y'+z')(y+xw) \\ &= u'(y+xw) = u'w(y+xw) = u'(xw+yw) . \end{aligned}$$

The two diamonds must therefore be degenerate. In particular $w=u$, that is, $u=v+u'$. Dually $v'=u'v$, and the proof is complete.

LEMMA 4. *If L is a modular lattice such that $M_{3,3}$ is not a homomorphic image of a sublattice of L, then any two non-trivial intervals in L that are projective to each other have non-trivial subintervals that are projective to each other in three steps or less.*

PROOF. Assume that the given intervals, $[a,b]$ and $[c,d]$, are projective in four steps, and that no non-trivial subintervals are projective in fewer than four steps. It is clearly sufficient to show that these assumptions lead to a contradiction.

By Lemma 2 there exists a strongly normal sequence of transposes

$$[a,b] = [a_0,b_0]\nearrow[a_1,b_1]\searrow[a_2,b_2]\nearrow[a_3,b_3]\searrow[a_4,b_4] = [c,d] ,$$

or dually. We therefore obtain three diamonds

$$(a_0+a_2,b_0+a_2,a_1,a_0+b_2,b_1) ,$$
$$(a_2,a_1b_3,b_2,b_1a_3,b_1b_3) ,$$
$$(a_2+a_4,b_2+a_4,a_3,b_2+a_4,b_3) .$$

We claim that

(1) $$[a_0+a_2,b_1]\searrow[a_2,b_1b_3]\nearrow[a_2+a_4,b_3] .$$

In fact,

$$a_1b_3 = a_1(b_1b_3), \quad a_0+b_2 = b_2+(a_0+a_2), \quad [a_1,b_1]\searrow[a_2,b_2]\,,$$
$$b_1a_3 = a_3(b_1b_3), \quad b_2+a_4 = b_2+(a_2+a_4), \quad [a_3,b_3]\searrow[a_2,b_2]\,,$$

whence the assertion follows by Lemma 3. We now verify that

$$(2) \qquad [a_0,b_0]\nearrow[b_2+a_0+a_4,b_1+b_3]\searrow[a_4,b_4]\,.$$

Indeed,

$$b_0+(b_2+a_0+a_4) = b_1+a_4 = b_1+b_1b_3+a_2+a_4 = b_1+b_3\,,$$

$$\begin{aligned} b_0(b_2+a_0+a_4) &= a_0+b_0(b_2+a_4) \\ &= a_0+b_0b_1(b_2+a_4) \\ &= a_0+b_0(b_2+b_1a_4) \le a_0+b_0(b_2+a_2) \\ &\qquad\qquad\qquad\quad = a_0+b_0b_2 = a_0 \end{aligned}$$

Here we have made use of the fact that, by (1).

$$b_1b_3+(a_2+a_4) = b_3 \quad \text{and} \quad b_1a_4 = b_1b_3a_4 \le b_1b_3(a_2+a_4) = a_2\,.$$

Thus the first part of (2) holds, and the second part follows by symmetry.

By (2) the intervals $[a,b]=[a_0,b_0]$ and $[c,d]=[a_4,b_4]$ are projective in two steps, contrary to our assumption. This contradiction completes the proof.

3. Proof of Theorem 1.

LEMMA 5. *Suppose L is a modular lattice, $a, b, c, d \in L$ and $a<b\le c<d$. If $[a,b]$ and $[c,d]$ are projective in three steps, then $[a,b]$ transposes up onto a lower edge of a diamond, and $[c,d]$ transposes down onto an upper edge of a diamond.*

PROOF. The condition $b\le c$ implies that no non-trivial subintervals of $[a,b]$ and $[c,d]$ are projective to each other in fewer than three steps. It follows by Lemma 2 that $[a,b]$ and $[c,d]$ are connected by a strongly normal four-termed sequence of transposes

$$[a,b] = [a_0,b_0]\nearrow[a_1,b_1]\searrow[a_2,b_2]\nearrow[a_3,b_3] = [c,d]\,.$$

Then $[a,b]$ transposes up onto the lower edge $[a_0+a_2,b_0+a_2]$ of the diamond

$$(a_0+a_2,b_0+a_2,a_1,a_1,a_0+b_2,b_1)$$

(Fig. 4), and $[c,d]$ transposes down onto the upper edge $[b_1a_3,b_1b_3]$ of the diamond

$$(a_2,a_1b_3,b_2,b_1a_3,b_1b_3)\,.$$

PROOF OF THEOREM 1. Assume that the condition (i) of the theorem is satisfied, but (ii) fails. Then there exists a subdirectly irreducible lattice L in V whose dimension is larger than 2. Choose $a_0, a_1, a_2, a_3 \in L$ with $a_0 < a_1 < a_2 < a_3$. Then some non-trivial subintervals $[a,b]$ of $[a_0,a_1]$ and $[c,d]$ of $[a_1,a_2]$ are projective to each other, and by Lemma 4 these intervals can be so chosen that they are projective in three steps. Similarly, some non-trivial subintervals $[c',d']$ of $[c,d]$ and $[e,f]$ of $[a_2,a_3]$ are projective to each other, and again we can assume that they are projective in three steps. Finally $[c',d']$ is projective to a subinterval of $[a,b]$, also in three steps. We infer by Lemma 5 that $[c',d']$ transposes up onto a lower edge $[v,x]$ of a diamond (v,x,y,z,u), and transposes down onto an upper edge $[x',u']$ of a diamond (v',x',y',z',u'). Thus $[v,x]\searrow[x',u']$, whence it follows that the ten elements u, x, y, z, v, u', x', y', z', v' form a lattice which has $M_{3,3}$ as a homomorphic imate. This contradicts (i). Thus (i) implies (ii).

To prove that (ii) implies (iii) we need only observe that the inclusion

$$a(b+cd)(c+d) \leq b + ac + ad \tag{1}$$

holds in every lattice of dimension 2. Indeed, in such a lattice we always have $c \leq d$ or $d \leq c$ or $cd = 0$, and each of these conditions implies that the inclusion holds.

Finally, (1) fails in $M_{3,3}$ if we take for a, b, c and d the four elements that are both additively and multiplicatively irreducible, with a and b in the lower diamond. Therefore (iii) implies (i).

4. Applications.

For each cardinal $n \geq 3$ there exists one and, up to isomorphism, only one lattice M_n of dimension 2 and order $n+2$. All these lattices are simple, and therefore subdirectly irreducible. To complete the list, let M_1 be the one-element lattice and M_2 the two-element lattice. The lattices M_n with n infinite obviously all generate the same equational class U_ω, since they all have the same finitely generated sublattices. For n finite let U_n be the equational class generated by M_n. Also let $U_{3,3}$ be the equational class generated by $M_{3,3}$.

COROLLARY 6. *The equational classes U_n, $n = 1,2,\ldots,\omega$, form a strictly increasing sequence. They all have the property that $M_{3,3} \notin U_n$, and they are the only equational classes of modular lattices that have this property.*

PROOF. Clearly $U_n \subseteq U_m$ whenever $1 \leq n \leq m$. According to Jónsson [3], Corollary 3.4, if n is finite, then every subdirectly irreducible member of U_n is a homomorphic image of a sublattice of M_n. Therefore, if $n<m$, then $M_m \notin U_n$, so that $U_n \subset U_m$.

By Jónsson [3], Theorem 3.2, every subdirectly irreducible member of U_ω is a homomorphic image of a sublattice of an ultraproduct of lattices M_n, and is therefore isomorphic to one of the lattices M_n. Thus $M_{3,3} \notin U_\omega$, and hence $M_{3,3} \notin U_n$ for every n. Conversely, suppose V is an equational class of modular lattices such that $M_{3,3} \notin V$. By Theorem 1, every member of V is a subdirect product of lattices of the form M_n, and V is therefore completely determined by the class J of all cardinals n with $M_n \in V$. If J has a largest member k, then $V = U_k$, and k is finite. In the alternative case it is easy to see that J is the class of all positive cardinals, and therefore $V = U_\omega$.

We note that the two applications of Jónsson [3] in the above proof can be easily avoided. Regarding the first application this will be clear from the next corollary, and to eliminate the second application we need only examine the proof of Theorem 1. It is shown there that every two-dimensional lattice satisfies the inclusion in condition (iii), while $M_{3,3}$ does not satisfy this inclusion.

Corollary 7. *U_ω is the class of all modular lattices that satisfy the inclusion*

$$a(b+cd)(c+d) \leq b+ac+ad\,,$$

and for $1<n<\omega$, U_n is the class of all modular lattices that satisfy this inclusion and the inclusion

$$a \prod_{0 \leqq i<j<n} (x_i+x_j) \leq \sum_{0 \leqq i<n} ax_i$$

Proof. The first assertion is an immediate consequence of Theorem 1 and Corollary 6. To prove the second statement, observe that in a two-dimensional lattice the second inclusion is satisfied whenever two of the elements $a, x_0, x_1, \ldots, x_{n-1}$ are equal as well as when one of them is 0 or 1, but fails whenever all $n+1$ elements are distinct atoms. Hence this inclusion holds in U_n but fails in M_{n+1}.

Corollary 8. *U_3 is the class of all modular lattices that satisfy the inclusion*

$$a(b+c)(c+d)(d+b) \leq ab+ac+ad\,.$$

Proof. This is but a relettering of the second inclusion in Corollary 7 for $n=3$, and it clearly implies the first inclusion there.

This result is a solution to Problem 45 in Birkhoff [1], p. 157. In Schützenberger [4] another characterization of U_3 is offered without a proof. The axiom system proposed there consists of the lattice axioms and the modular law, together with the identity

$$a(b+c(d+e)) = a(b+cd) + a(b+ce) + ac(d+e) + ad(c+be) + ae(c+bd) .$$

This identity holds in M_3 but fails in M_4 and $M_{3,3}$, and Schützenberger's assertion therefore follows from our Theorem 1.

COROLLARY 9. *In the lattice of all equtional classes of modular lattices,* U_3 *is covered by precisely two classes,* U_4 *and* $U_{3,3}$, *and every class that properly contains* U_3 *contains either* U_4 *or* $U_{3,3}$.

PROOF. That U_4 and $U_{3,3}$ cover U_3 follows from Jónsson [3], Corollary 3.4. If V is an equational class that properly contains U_3, then either $U_{3,3} \subseteq V$, or else V is one of the classes U_n with $n > 3$. In the latter case we therefore have $U_4 \subseteq V$.

A weaker form of this result was proved in Grätzer [2]: If V is an equational class of modular lattices that properly contains U_3, and if V is generated by a finite lattice, then $U_4 \subseteq V$ or $U_{3,3} \subseteq V$.

COROLLARY 10. *In the lattice of all equational classes of modular lattices, if* $3 < n < \omega$, *then* U_n *is covered by precisely two classes,* U_{n+1} *and* $U_n + U_{3,3}$, *and every class that properly contains* U_n *contains either* U_{n+1} *or* $U_{3,3}$.

PROOF. The class $U_n \cap U_{3,3} = U_3$ is covered by $U_{3,3}$, whence it follows that U_n is covered by $U_n + U_{3,3}$. Other than this, the proof is exactly analogous to the one for Corollary 9.

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VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37203, U.S.A.