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ON THE OSCILLATION FUNCTIONS OF GAUSSIAN PROCESSES

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1. Introduction and results obtained.

By a *Gaussian process* we shall understand a separable, measurable, jointly Gauss distributed process with the time parameter on $[0,1]$, continuous in the (second order) mean.

As to the regularity of the sample path of a stationary Gaussian process, we have Yu. K. Belayev's theorem of alternatives [1] which reads as follows: the sample function (path) of a stationary Gaussian process is either continuous with probability one or unbounded on every interval with probability one.

What will happen for a non-stationary Gaussian process? Our purpose is to answer this question.

Given a Gaussian process $x=x(t,\omega)$, $0\leqq t\leqq 1$, $\omega\in\Omega(\mathscr{B},P)$, we shall define the *oscillation function* of x by

$$W_x(t,\omega) = \lim_{\varepsilon\downarrow 0}\ \sup_{u,v\in(t-\varepsilon,\,t+\varepsilon)\cap[0,1]} |x(v,\omega)-x(u,\omega)|\,,$$

where we apply the usual convention $(+\infty)-(+\infty)=(-\infty)-(-\infty)=0$, etc. Because of the separability of our process x, the supremum can be taken only for the u and v in the separant Q of x which is countable, so that $W_x(t,\omega)$ is measurable in ω for each $t\in[0,1]$.

An interesting fact is that the oscillation function of a Gaussian process is a deterministic function. Precisely speaking, we have the following theorem which will be proved in Section 2.

THEOREM 1. *There exists a function* $\alpha=\alpha_x(t)$, $0\leqq t\leqq 1$, *which does not depend on* ω, *such that*

$$P[W_x(t,\omega)=\alpha(t) \text{ for every } t\in[0,1]] = 1\ .$$

In view of this theorem we call α the *oscillation function* of the Gaussian process x.

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Using the fact that the probability law of the process $x(t)-E(x(t))$, $0 \leqq t \leqq 1$, is invariant under reflection, we shall prove the following theorem in Section 3:

THEOREM 2. *For each* $t \in [0,1]$, *we have*

$$P\left[\overline{\lim_{s\to t}}\, x(s) = x(t) + \tfrac{1}{2}\alpha(t),\ \underline{\lim_{s\to t}}\, x(s) = x(t) - \tfrac{1}{2}\alpha(t)\right] = 1\,.$$

It is clear that, with probability one, the sample function of $x(t)$ is continuous at every point t where $\alpha(t)$ vanishes.

We have also the following theorem which will be proved in Section 4.

THEOREM 3. *If, for some constant* a, $\alpha(t) \geqq a > 0$ *on a dense subset* D *of an open interval* $I \subset [0,1]$, *then*

$$P\left[\overline{\lim_{s\to t}}\, x(s) = \infty,\ \underline{\lim_{s\to t}}\, x(s) = -\infty \text{ for every } t \in I\right] = 1\,.$$

In Section 5 and 6 we shall prove the following properties that characterize oscillation functions.

THEOREM 4. (a) *The oscillation function of a Gaussian process satisfies*

$(\alpha, 1)$ $\quad\alpha(t)$ *is upper semi-continuous,*

$(\alpha, 2)$ $\quad\{t : a \leqq \alpha(t) < \infty\}$ *is nowhere dense for every* $a > 0$.

(b) *Conversely, given a function* $\alpha: [0,1] \to [0,\infty]$ *satisfying* $(\alpha,1)$ *and* $(\alpha,2)$, *we can construct a (not necessarily unique) Gaussian process whose oscillation function is* α.

Let us derive Belayev's theorem of alternatives from Theorems 1 and 3. Suppose $x(t)$ is a stationary Gaussian process. The oscillation function $\alpha(t)$ of the restriction of x to $0 \leqq t \leqq 1$ is constant because of the stationarity. If the constant is 0, then almost all sample functions of x are continuous; if it is positive, then almost all sample functions are, by Theorem 3, unbounded both below and above on every interval.

2. Proof of Theorem 1.

We can assume that $Ex(t) = 0$, because $Ex(t)$ is continuous in t. Let $R(s,t)$, $t,s \in [0,1]$, be the covariance function of x. Then R is real, symmetric, positive-definite and continuous on $I \times I$. Using Mercer's theorem, we can expand R as follows:

$$R(t,s) = \sum_n \varphi_n(t)\,\varphi_n(s)/\lambda_n\,, \tag{2.1}$$

where $\{\lambda_n\}$ and $\{\varphi_n\}$ are, respectively, the positive eigenvalues and the corresponding real (normalized) eigenfunctions for the integral operator with the kernel $R(t,s)$, that is,

$$\varphi_n(t) = \lambda_n \int_0^1 R(t,s)\,\varphi_n(s)\,ds, \qquad 0 \leqq \lambda_1 \leqq \lambda_2 \leqq \dots\,, \tag{2.2}$$

and the sum in (2.1) converges absolutely and uniformly on $I \times I$.

We shall define a sequence of random variables x_n, $n=1,2,\dots$, as the Fourier coefficients of the sample function of $x(t)$ with respect to $\{\varphi_n\}$:

$$x_n = \int_0^1 x(t)\,\varphi_n(t)\,dt\,. \tag{2.3}$$

Observing that

$$E\left[\int_0^1 x(s)^2\,ds\right] = \int_0^1 R(s,s)\,ds < \infty\,,$$

we have

$$P\left[\int_0^1 x(s)^2\,ds < \infty\right] = 1\,,$$

so that x_n is well-defined.

A simple computation shows that

$$E(x_n x_m) = 0, \ n \neq m, \qquad E(x_n^2) = \lambda_n^{-1}\,,$$

which implies that x_n, $n=1,2,\dots$, are independent, each having the Gauss distribution with mean 0 and variance λ_n^{-1}. Observing that

$$E\left[\left|x(t) - \sum_1^N \varphi_n(t) x_n\right|^2\right] = R(t,t) - \sum_1^N \lambda_n^{-1}\varphi_n(t)^2 \to 0$$

as $n \to \infty$, we have

$$P\left[x(t) = \sum_{n=1}^{\infty} \varphi_n(t) x_n\right] = 1 \quad \text{for each } t; \tag{2.4}$$

the infinite series converges to $x(t)$ in the mean for each t and so it converges with probability one for each t because of the independence of x_n, $n=1,2,\dots$.

Let us define the *maximum oscillation* $W_y(s,t,\omega)$ of a separable process on the interval $[s,t]$ by

$$(2.5)\qquad W_y(s,t,\omega) = \lim_{n\uparrow\infty}\lim_{p\uparrow\infty}\sup_{\substack{u,v\in(s-n^{-1},t+n^{-1})\cap[0,1]\\ |u-v|<p^{-1}}} |y(u,\omega)-y(v,\omega)|\ ,$$

where the interval $[0,1]$ can be replaced by the separant Q of y.

We shall now prove that $W_x(s,t,\omega)$ is a constant; more precisely, that there exists a function $\alpha(s,t)$ independent of ω such that

$$(2.6)\qquad P[W_x(s,t,\omega)=\alpha(s,t)] = 1$$

for each pair $s\leqq t$.

Since $\varphi_j(t)$ is continuous in t, we have

$$(2.7)\qquad P[W_x(s,t,\omega)=W_{y_n}(s,t,\omega)] = 1\ ,$$

where

$$y_n(t) = x(t) - \sum_{j=1}^{n}\varphi_j(t)x_j = \sum_{j=n+1}^{\infty}\varphi_j(t)x_j\ .$$

The separant Q of x is also that of y_n because of the continuity of $\varphi_j(t)$. By virtue of (2.4), $y_n(t)$ is measurable with respect to $\mathscr{B}(x_k, k\geqq n)$ for each t. But $W_{y_n}(s,t,\omega)$ is measurable with respect to $\mathscr{B}(y_n(t), t\in Q)$. Since Q is countable, $W_{y_n}(s,t,\omega)$ is measurable with respect to $\mathscr{B}(x_k, k\geqq n)$ and so $W_x(s,t,\omega)$ is also measurable with respect to $\mathscr{B}(x_k,\ k\geqq n)$ for every n, by (2.7). Since the x_n, $n=1,2,\ldots$, are independent, Kolmogorov's zero-one law shows that $W_x(s,t,\omega)$ is a constant with probability one.

We shall now strengthen (2.6) to get

$$(2.6')\qquad P[W_x(s,t,\omega)=\alpha(s,t) \text{ for every pair } s\leqq t] = 1\ .$$

It follows from (2.6) that

$$(2.6'')\qquad P[W_x(s,t,\omega)=\alpha(s,t) \text{ for every rational pair } s\leqq t] = 1\ .$$

Since $W_x(s,t,\omega)$ is left-continuous in s and right-continuous in t, (2.6″) implies (2.6′). Writing $W_x(t,\omega)$ and $\alpha(t)$ for $W_x(t,t,\omega)$ and $\alpha(t,t)$, respectively, we get from (2.6′)

$$(2.7)\qquad P[W_x(t,\omega)=\alpha(t) \text{ for every } t] = 1\ .$$

3. Proof of Theorem 2.

We shall use the same notation as in Section 2. Using Kolmogorov's zero-one law in the same way as before, we can see that $\overline{\lim}_{s\to t}(x(s,\omega)-x(t,\omega))$ is a constant, say $\beta(t)$, with probability one. Since the process $y(t)\equiv -x(t)$ has the same Gaussian probability law as the process x, we have

$$\overline{\lim_{s\to t}}(-x(s)+x(t)) = \beta(t)\ ,$$

that is,

$$\underline{\lim}_{s\to t}(x(s)-x(t)) = -\beta(t)$$

with probability one. By definition we have

$$\begin{aligned} W_x(t,\omega) &= \overline{\lim}_{s\to t} x(s) - \underline{\lim}_{s\to t} x(s) \\ &= \overline{\lim}_{s\to t}(x(s)-x(t)) - \underline{\lim}_{s\to t}(x(s)-x(t)) . \end{aligned}$$

Therefore, $\alpha(t)$ must be equal to $2\beta(t)$. This completes the proof of Theorem 2.

4. Proof of Theorem 3.

In the following lemma and throughout this paper an open subset of $[0,1]$ means a subset open in $[0,1]$. For example, $[0,u)$ is open and 0 is an interior point of this interval.

LEMMA 4.1. *Let $x(t)$, $0 \leqq t \leqq 1$, be a separable process continuous in probability, and D a dense subset of an open subinterval I of $[0,1]$. For each $t \in I$, we can then find a sequence $s_n \in D$ such that $s_n \to t$ and that*

$$P\left[\overline{\lim}_{n\to\infty} x(s_n) = \overline{\lim}_{s\to t} x(s)\right] = 1 , \tag{4.1}$$

and hence a fortiori

$$P\left[\overline{\lim}_{\substack{s\to t\\ s\in D}} x(s) = \overline{\lim}_{s\to t} x(s)\right] = 1 . \tag{4.1'}$$

PROOF. Let $Q=\{t_n\}$ be a separant of $x(t)$ and let $\{U_n\}$ be a sequence of neighborhoods of t converging to t. Let $\{y_n(\omega)\}$ be a sequence converging to $\bar{x}(t)=\overline{\lim}_{s\to t}x(t)$ strictly from below, for example

$$y_n(\omega) = \min(\bar{x}(t),n) - 1/n .$$

Since Q is a separant of $x(t)$, we can find $u_{n1},u_{n2},\ldots,u_{np_n} \in U_n \cap Q$ such that

$$P[\max_k x(u_{nk}) > y_n(\omega)] > 1 - 2^{-n} .$$

By Borel–Cantelli's lemma we have

$$P[\lim_n \max_k x(u_{nk}) \geqq \bar{x}(t)] = 1 .$$

Writing $\{u_n\}$ for $\{u_{11},\ldots,u_{1p_1},u_{21},\ldots,u_{2p_2},\ldots\}$, we have

$$P\left[\overline{\lim_{n\to\infty}}\, x(u_n) \geqq \bar{x}(t)\right] = 1 . \tag{4.2}$$

Since $x(t)$ is continuous in probability, we can find

$$s_n \in (u_n - 1/n, u_n + 1/n) \cap D$$

such that

$$P[|x(u_n) - x(s_n)| > 2^{-n}] < 2^{-n}, \qquad n = 1, 2, \ldots .$$

Using Borel–Cantelli's lemma again, we have

$$P[\overline{\lim}_n x(u_n) = \overline{\lim}_n x(s_n)] = 1 , \tag{4.3}$$

which, combined with (4.2), implies (4.1) and so (4.1′).

We shall now prove Theorem 3. Using (4.1) and Theorem 2, we can see that the event

$$\Omega_1 = \left\{\omega : \overline{\lim_{\substack{s\to t\\ s\in D}}}\, x(s,\omega) = x(t,\omega) + \tfrac{1}{2}\alpha(t) \text{ for every } t \in D\right\}$$

has probability one, since D is a countable dense subset of I. For every $\omega \in \Omega_1$ and every $t \in D$, we have

$$\begin{aligned}\overline{\lim_{\substack{s\to t\\ s\in D}}}\, x(s,\omega) &= \overline{\lim_{\substack{s\to t\\ s\in D}}}\ \overline{\lim_{\substack{u\to s\\ u\in D}}}\, x(u,\omega)\\ &= \overline{\lim_{\substack{s\to t\\ s\in D}}}\left(x(s,\omega) + \tfrac{1}{2}\alpha(s)\right) \geqq \overline{\lim_{\substack{s\to t\\ s\in D}}}\, x(s,\omega) + \tfrac{1}{2}a .\end{aligned}$$

But this is impossible, unless

$$\overline{\lim_{\substack{s\to t\\ s\in D}}}\, x(s,\omega) = \infty ,$$

that is, unless $\overline{\lim}_{s\to t}\, x(s,\omega) = \infty$.

Therefore

$$P\left[\overline{\lim_{s\to t}}\, x(s,\omega) = \infty \text{ for every } t \in D\right] = 1 .$$

Similarly we have

$$P\left[\underline{\lim_{s\to t}}\, x(s,\omega) = -\infty \text{ for every } t \in D\right] = 1 .$$

This completes our proof, since D is dense in I.

5. Proof of Theorem 4(a).

Let $\alpha(t)$ be the oscillation function of a Gaussian process $x(t)$, $0<t<1$. Then we have

$$P[W_x(t,\omega)=\alpha(t) \text{ for every } t] = 1$$

by the definition of $\alpha(t)$ in Section 2. By the definition of $W_x(t,\omega)$, it holds that

$$\overline{\lim_{s\to t}} W_x(s,\omega) \leqq W_x(t,\omega) ,$$

so that we have

$$\overline{\lim_{s\to t}} \alpha(s) \leqq \alpha(t) ,$$

which shows that α is upper semi-continuous. Hence it follows that $T_a=\{t: \alpha(t)\geqq a\}$ is a closed subset of $[0,1]$. If T_a-T_∞ contains a dense subset D of an open interval for some $a>0$, then $D\subset T_\infty$ by Theorem 3, in contradiction with $D\subset T_a-T_\infty$. Therefore, T_a-T_∞ is nowhere dense for $a>0$. This completes the proof of Theorem 4(a).

6. Proof of Theorem 4(*b*).

We shall denote the mean square norm of a random variable x by $\|x\|$,

$$\|x\|^2 = E(x^2) .$$

Consider a Brownian motion $B(t)$, $0\leqq t\leqq 1$, and a stationary Gaussian process $S(t)$, $-\infty<t<\infty$, with $ES(t)=0$, $ES(t)^2=1$ and $\alpha(t,S)\equiv\infty$. The existence of the latter process was proved by Belayev [1]. We can assume that these two processes are independent. Let L be the $\|\ \|$-closure of all finite linear combinations of $B(t)$, $0\leqq t\leqq 1$, and $S(t)$, $0\leqq t\leqq 1$. It is clear that any process $x(t)$, $0\leqq t\leqq 1$, such that $x(t)\in L$ for each t is jointly Gauss distributed.

We shall prove Theorem 4(b) by constructing a Gaussian process $x(t)\in L$, $0\leqq t\leqq 1$, with $\alpha(t,x)=\alpha(t)$ for any given function

$$\alpha : [0,1] \to [0,\infty] \text{ satisfying } (\alpha,1) \text{ and } (\alpha,2) .$$

Let us start with some lemmas.

LEMMA 6.1. *Given* $I=[u,v]\subset[0,1]$ *and* $\varepsilon>0$, *we can construct a Gaussian process* $x(t)\in L$, $0\leqq t\leqq 1$, *satisfying*

(a) $x(t,\omega)=0$ *for* $t\in[0,1]-I^\circ$ ($I^\circ=$the interior of I),
(b) $\alpha(t,x)=\infty$ *for* $t\in I$,
(c) $\|x(t)\|\leqq\varepsilon$ *for* $t\in[0,1]$.

PROOF. Take a continuous function $f(t)$ such that $0<f(t)<\varepsilon$ in I° and $f(t)=0$ elsewhere. Then $x(t,\omega)\equiv f(t)S(t,\omega)$ is a Gaussian process satisfying our conditions.

LEMMA 6.2. *Given* $0<a<\infty$, $\varepsilon>0$, *and* $I=[u,v]\subset[0,1]$, *we can construct a Gaussian process* $x(t)\in L$, $0\leqq t\leqq 1$, *satisfying the following conditions*:

(a) $x(t,\omega)$ *has continuous paths*,
(b) $x(t,\omega)=0$ *for* $t\in[0,1]-I^\circ$,
(c) $Ex(t)=0$, $\|x(t)\|<\varepsilon$ *for every* t,
(d) $P(|\sup_I x(t)-a|>\varepsilon)<\varepsilon$.

Such a process will be denoted by $x(t;I,a,\varepsilon)$.

PROOF. Let $B(t)$ be a Brownian motion and define a process $y(t)$ by

$$\begin{aligned} y(t) &= 0, && 0\leqq t\leqq u\,, \\ &= \frac{a(B(t)-B(u))}{(2(t-u)\log\log(t-u)^{-1})^{\frac{1}{2}}}, && u\leqq t\leqq v'\equiv\min(v,u+9)\,, \\ &= \frac{a(B(v')-B(u))}{(2(v'-u)\log\log(v'-u)^{-1})^{\frac{1}{2}}}, && v'\leqq t\leqq 1\,. \end{aligned}$$

Then $y(t)$ is jointly Gauss distributed with $Ey(t)=0$ and the sample path of $y(t)$ is continuous except at $t=u$. The continuity in the mean follows from

$$E[y(t)^2] = \frac{a^2}{2\log\log(t-u)^{-1}} \to 0 \quad \text{as } t\downarrow u\,.$$

By the law of the iterated logarithm we have

$$P\left[\overline{\lim_{t\downarrow u}}\, y(t)=a\right] = 1\,.$$

We now determine $u<s_1<s_2<s_3<s_4<v$ as follows. By taking s_4 sufficiently close to u, we have

$$E[y(t)^2] < \varepsilon^2\,,$$

$$P\left[\sup_{u<t<s_4} y(t)<a+\varepsilon\right] > 1-\tfrac{1}{2}\varepsilon\,,$$

$$P\left[\sup_{u<t<s_4} y(t)>a-\varepsilon\right] > 1-\tfrac{1}{2}\varepsilon\,.$$

By taking s_2 sufficiently close to u and s_3 sufficiently close to s_4, we have

$$P\left[\sup_{s_2 \leqq t \leqq s_3} y(t) > a - \varepsilon\right] > 1 - \tfrac{1}{2}\varepsilon .$$

We shall take s_1 in (u, s_2) arbitrarily.

Let $f(t)$ be a polygonal function of t vanishing on $[0, s_1] \cup [s_4, 1]$, equal to 1 on $[s_2, s_3]$ and linear in each of $[s_1, s_2]$ and $[s_3, s_4]$. Then $x(t) = f(t)\, y(t)$ is a Gaussian process satisfying our conditions.

LEMMA 6.3. *Suppose that $\alpha_1(t)$ and $\alpha_2(t)$ satisfy $(\alpha, 1)$ and the following condition (stronger than $(\alpha, 2)$):*

$(\alpha, 2')$ $\qquad\qquad \{t : \alpha_2(t) > 0\}$ is nowhere dense .

For any Gaussian process $x_1(t)$ with $\alpha(t, x_1) = \alpha_1(t)$ and any $\varepsilon > 0$, we can construct a Gaussian process $x_2(t) \in L$, $0 \leqq t \leqq 1$, satisfying the conditions:

(a) $\alpha(t, x_1 + x_2) = \alpha_1(t) + \alpha_2(t)$,
(b) $\|x_2(t)\| < \varepsilon$,
(c) $P[\sup_t |x_2(t)| > \sup_t \alpha_2(t)] < \varepsilon$.

PROOF. We can assume that $c = \sup \alpha_2(t) > 0$. If otherwise, $x_2(t) \equiv 0$ will satisfy our conditions trivially.

Write $\alpha(t)$ for $\alpha_1(t) + \alpha_2(t)$. The set $\{(t, \alpha(t)) : \alpha(t) > 0\}$ is a subset of $[0, 1] \times [0, \infty]$. Let $\{(t_n, \alpha(t_n))\}_n$ be a countable dense subset of $\{(t, \alpha(t)) : \alpha(t) > 0\}$. Then the sets $\{t_n\}_n$ are dense in $\{t : \alpha(t) > 0\}$ and so dense in its closure F. By our assumptions $(\alpha, 2')$, F is nowhere dense and its complement G is a dense open subset of $[0, 1]$. Since $\alpha(t) \geqq \alpha(t, x_1)$, we have $\alpha(t, x_1) = 0$ for $t \in G$, so that the sample path of $x_1(t)$ is continuous in $t \in G$ with probability one.

Using Theorem 2 and Lemma 4.1, we can find $\{t_{1n}\}_n$ in G tending to t_1 as $n \to \infty$ such that

$$P\left[\overline{\lim_{n\to\infty}}\, x_1(t_{1n}) = x(t_1) + \tfrac{1}{2}\alpha_1(t_n)\right] = 1 .$$

Since the path of $x_1(t)$ is continuous in $t \in G$, we can find, for each t_{1n}, a closed interval $I_{1n} \subset G$ containing t_{1n} in its interior such that

$$P\left[x_1(t_{1n}) \geqq \inf_{I_{1n}} x_1(s) \geqq x_1(t_{1n}) - 2^{-n}\right] < 2^{-n} .$$

This implies

$$P\left[\overline{\lim_{n}}\, \inf_{I_{1n}} x_1(s) = \overline{\lim_{n\to\infty}}\, x_1(t_{1n})\right] = 1$$

by Borel–Cantelli's lemma. Thus we have

$$P\left[\overline{\lim_{n\to\infty}}\ \inf_{I_{1n}} x_1(s) = x(t_1) + \tfrac{1}{2}\alpha_1(t_1)\right] = 1\,.$$

By taking I_{1n} sufficiently small, we can achieve that the $\{I_{1n}\}_n$ are disjoint. Then $\sum_n |I_{1n}| \leqq 1$ where $|\cdot|$ denotes length. Therefore $|I_{1n}| \to 0$ and so I_{1n} tends to t_1 by $t_{1n} \to t_1$.

It is clear that $\bigcup_n I_{1n} \cup \{t_1\}$ is a closed set which does not contain t_2. Therefore we can find a neighborhood U of t_2 which does not intersect this closed set. In the same way as above, we can find a sequence of disjoint intervals $I_{2n} \subset G \cap U$, $n = 1, 2, \ldots$, tending to t_2 such that

$$P\left[\overline{\lim_{n\to\infty}}\ \inf_{I_{2n}} x_1(s) = x_1(t_2) + \tfrac{1}{2}\alpha_1(t_2)\right] = 1\,.$$

Continuing this procedure we can get a double sequence of disjoint closed intervals $I_{kn} \subset G$, $k, n = 1, 2, \ldots$, such that I_{kn} tends to t_k as $n \to \infty$ for each k and that

$$\text{(6.1)} \qquad P\left[\overline{\lim_{n\uparrow\infty}}\ \inf_{I_{kn}} x_1(s) = x_1(t_k) + \tfrac{1}{2}\alpha_1(t_k)\right] = 1, \quad k = 1, 2, \ldots\,.$$

By removing a finite number of intervals from $\{I_{kn}\}_n$ we can achieve that $I_{kn} \subset (t_k - 1/k, t_k + 1/k)$ for each k.

Take $\varepsilon_{kn} > 0$ such that

$$\sum_{kn} \varepsilon_{kn} < \min\left(\tfrac{1}{2}\varepsilon, \tfrac{1}{2}c\right)$$

and set

$$x_{kn}(t) = x(t; I_{kn}, a_{kn}, \varepsilon_{kn}) \quad \text{(see Lemma 6.2)},$$

where $a_{kn} = \frac{1}{2}\alpha_2(t_{kn})$ if this is finite and $= n$ otherwise, so that $a_{kn} \uparrow \frac{1}{2}\alpha_2(t_{kn})$ as $n \to \infty$ for each k. We shall prove that

$$x_2(t) = \sum_{kn} x_{kn}(t)$$

is a Gaussian process satisfying our conditions.

Since this implies

$$\begin{aligned} x_2(t) &= x_{kn}(t) && \text{on} \quad I_{kn}, \quad k, n = 1, 2, \ldots\,, \\ &= 0 && \text{elsewhere}\,, \end{aligned}$$

$x_2(t)$ is well-defined and $x_2(t) \in L$. Therefore $x_2(t)$ is jointly Gauss distributed, separable and measurable.

By $\|x_{kn}(t)\| < \varepsilon_{kn}$ and $\sum \varepsilon_{kn} < \infty$, the series $\sum_{kn} x_{kn}(t)$ converges in the mean uniformly in t and so $x_2(t)$ is continuous in the mean and we have

$$\|x_2(t)\| < \sum_{kn} \|x_{kn}(t)\| < \varepsilon ,$$

which proves (b).

Observing that

$$\begin{aligned} P\left[\sup_t x_2(t) > c\right] &\leqq \sum_{kn} P\left[\sup_{t\in I_{kn}} x_2(t) > c\right] \\ &\leqq \sum_{kn} P\left(\sup_t x_{kn}(t) > a_{kn} + \varepsilon_{kn}\right) \\ &\leqq \sum_{kn} \varepsilon_{kn} < \tfrac{1}{2}\varepsilon , \end{aligned}$$

we have

$$P(\sup_t |x_2(t)| > c) < \varepsilon ,$$

since the probability law of the sample path of $x_2(t)$ is symmetric by $Ex_2(t) = 0$. Thus (c) is proved.

Now we shall prove that $\alpha(t, x_2) \leqq \alpha_2(t)$.

If $t_0 \in G$, then t_0 has a positive distance from F. Since I_{kn} is in the $1/k$-neighborhood of $t_k \in F$, a small neighborhood U ($\subset G$) of t_0 does not intersect I_{kn} with $k \geqq k_0$, $n = 1, 2, \ldots$, for some k_0. Since $I_{kn} \to t_k \in F$ for each k, U can intersect only a finite number of intervals among $\{I_{kn}\}_{kn}$. Let us denote these intervals by $I_{k(i), n(i)}$, $i = 1, 2, \ldots, m$. Then $x_2(t)$ is the sum of $x_{k(i), n(i)}$, $i = 1, 2, \ldots, m$, as far as t lies in U. Therefore the path of $x_2(t)$ is continuous in U and so $\alpha(t_0, x) = 0 \leqq \alpha_2(t_0)$.

If $t_0 \in F$, then $x_2(t_0) = 0$ by our construction. Take an arbitrary $\delta > 0$. Then there exists a neighborhood U_1 of t_0 such that

$$\sup_{s\in U_1} \alpha_2(s) < \alpha_2(t_0) + \delta .$$

Take a neighborhood U_2 of t_0 such that $\overline{U}_2 \subset U_1$. Then the distance $\varrho(U_2, U_1{}^c)$ is positive. Since the $\{I_{kn}\}_{kn}$ are disjoint and $\sum_{kn} |I_{kn}| \leqq 1$, we have only a finite number of intervals among $\{I_{kn}\}_{kn}$ with the length $\geqq \varrho(U_2, U_1{}^c)$ and only such intervals can intersect both U_2 and $U_1{}^c$. Since each I_{kn} has positive distance from t_0, we have a neighborhood U_3 ($\subset U_2$) of t_0 such that $I_{kn} \subset U_1$ as far as I_{kn} intersects U_3.

Write $\sum'$ for the summation over those indices (k,n) for which I_{kn} intersects U_3. Then any interval I_{kn} with the index (k,n) appearing in $\sum'$ is contained in U_1. By taking U_3 small enough, we can achieve that

$$\sum_{kn}{}' \varepsilon_{kn} < \delta .$$

We then have

$$P\left[\sup_{s\in U_3} x_2(s) > \tfrac{1}{2}\alpha_2(t_0)+2\delta\right]$$

$$\leqq \sum{}' P\left[\sup x_{kn}(s) > \sup_{s\in U_1}\alpha_2(s)+\delta\right]$$

$$\leqq \sum{}' P(\sup x_{kn}(s) > a_{kn}+\varepsilon_{kn}) < \sum{}'\varepsilon_{kn} < \delta ,$$

by the construction in Lemma 6.2. Therefore we get

$$P\left[\overline{\lim_{s\to t_0}} x_2(s) > \tfrac{1}{2}\alpha_2(t_0)+2\delta\right] < \delta$$

for every $\delta > 0$. Letting $\delta\downarrow 0$, we have

$$P\left[\overline{\lim_{s\to t_0}} x_2(s) \geqq x_2(t_0)+\tfrac{1}{2}\alpha_2(t_0)\right] = 0. \qquad (\text{Note } x_2(t_0)=0.)$$

This implies that $\alpha(t_0,x_2)\leqq\alpha_2(t_0)$ by Theorem 2.

Thus $\alpha(t,x_2)\leqq\alpha_2(t)$ is proved for every t. By the definition of the oscillation function we have

$$\alpha(t,x_1+x_2) \leqq \alpha(t,x_1)+\alpha(t,x_2) \leqq \alpha_1(t)+\alpha_2(t) = \alpha(t) .$$

We shall now prove that

$$\alpha(t,x_1+x_2) \geqq \alpha(t) .$$

Consider first the case $t=t_k$. It holds that

$$\text{(6.2)}\quad P\left[\overline{\lim_{s\to t_k}}\,(x_1(s)+x_2(s))\geqq x_1(t_k)+x_2(t_k)+\tfrac{1}{2}\alpha(t_k)\right]$$

$$\geqq P\left[\overline{\lim_{n\to\infty}}\,\sup_{I_{kn}}[x_1(s)+x_2(s)]\geqq x_1(t_k)+x_2(t_k)+\tfrac{1}{2}\alpha(t_k)\right]$$

$$\geqq P\left[\overline{\lim_{n\to\infty}}\left(\inf_{I_{kn}} x_1(s)+\sup_{I_{kn}} x_2(s)\right)\geqq x_1(t_k)+x_2(t_k)+\tfrac{1}{2}\alpha(t_k)\right]$$

$$\geqq P\left[\overline{\lim_{n\to\infty}}\,\inf_{I_{kn}} x_1(s)+\underline{\lim_{n\to\infty}}\,\sup_{I_{kn}} x_2(s)\geqq x_1(t_k)+x_2(t_k)+\tfrac{1}{2}\alpha(t_k)\right].$$

Since we have

$$P[\sup_{I_{kn}} x_2(s) < a_{kn}-\varepsilon_{kn}] < \varepsilon_{kn}$$

by virtue of $x_2(s)=x_{kn}(s)$ on I_{kn}, we get

$$\text{(6.3)}\qquad P\left[\underline{\lim_{n\to\infty}}\,\sup_{I_{kn}} x_2(s)\geqq x_2(t_k)+\tfrac{1}{2}\alpha_2(t_k)\right] = 1$$

by $x_2(t_k)=0$ and $\sum_n \varepsilon_{kn}<\infty$. By (6.1), (6.2) and (6.3) we get

$$P\left[\overline{\lim_{s\to t_k}}\,(x_1(s)+x_2(s)) \geqq x_1(t_k)+x_2(t_k)+\tfrac{1}{2}(t_k)\right] = 1 ,$$

that is, by Theorem 2,

$$\alpha(t_k, x_1+x_2) \geqq \alpha(t_k) .$$

For $t \neq t_k$, $k=1,2,\ldots$, the point $(t,\alpha(t))$ is an accumulation point of $(t_n,\alpha(t_n))$, $n=1,2,\ldots$. Therefore we have a subsequence $\{s_n\}$ of $\{t_n\}$ such that

$$\alpha(t) = \lim_n \alpha(s_n) .$$

By Theorem 4 (a) $(\alpha, 1)$ we have

$$\alpha(t, x_1+x_2) \geqq \overline{\lim}_n \alpha(s_n, x_1+x_2) = \overline{\lim}_n \alpha(s_n) = \alpha(t) ,$$

which completes the proof of Lemma 6.3.

Now we shall come back to the proof of Theorem 4 (b). We shall first assume

$(\alpha, 2'')$ $\{t : \alpha(t) \geqq c\}$ is nowhere dense for every $c>0$;

this is stronger than $(\alpha, 2)$ but weaker than $(\alpha, 2')$. Set

$$\begin{aligned} \alpha_0(t) &= 0 , \\ \alpha_1(t) &= \max\left(\alpha(t), \tfrac{1}{2}\right) - \tfrac{1}{2} , \\ \alpha_n(t) &= \max\left(\min\left(\alpha(t), 2^{-n+1}\right), 2^{-n}\right) - 2^{-n}, \qquad n=2,3,\ldots . \end{aligned}$$

It is then easy to verify the following properties of $\alpha_n(t)$:

$$\begin{aligned} &0 \leqq \alpha_n(t) \leqq 2^{-n-1}, \qquad n=2,3,\ldots , \\ &\textstyle\sum_0^n \alpha_i(t) \uparrow \alpha(t) \\ &\{t : \alpha_n(t)>0\} \text{ is nowhere dense .} \end{aligned}$$

Starting with the Gaussian process $x_0(t)\equiv 0$ whose oscillation function is $\alpha_0(t)\equiv 0$, we can use Lemma 6.3 to define a sequence of Gaussian processes $x_n(t)\in L$, $0\leqq t\leqq 1$, $n=1,2,\ldots$, satisfying

(a) $\alpha(t, \sum_0^n x_i) = \sum_0^n \alpha_i(t)$,
(b) $\|x_n(t)\| < 2^{-n}$,
(c) $P[\sup_t |x_n(t)| > \sup_t \alpha_n(t)] < 2^{-n}$.

Now set

$$x(t) = \textstyle\sum_n x_n(t) .$$

By Borel–Cantelli's lemma it follows from (c) and $\alpha_n(t) \leqq 2^{-n-1}$, $n\geqq 2$, that this infinite series converges uniformly in t with probability one.

Therefore $x(t)$ is well-defined, separable, measurable and jointly Gauss distributed, and we have

$$\alpha(t,x) = \lim_n \alpha(t, \textstyle\sum_0^n x_i) = \lim_n \textstyle\sum_0^n \alpha_i(t) = \alpha(t) .$$

It follows from (b) that this infinite series converges also in the mean, uniformly in t, so that $x(t)$ is continuous in the mean.

We shall now remove the assumption that $T_c = \{t: \alpha(t) \geqq c\}$ is nowhere dense.

Let $I_1, I_2, \ldots$ be the maximal intervals contained in the set $T_\infty = \{t: \alpha(t) = \infty\}$. Since, by $(\alpha, 2)$, $\{t: c \leqq \alpha(t) < \infty\}$ is nowhere dense, $I_1, I_2, \ldots$ are also the maximal intervals contained in the set T_c $(c > 0)$.

We now define

$$\begin{aligned} \beta(t) &= 0, \qquad t \in I_n{}^\circ, \quad n = 1, 2, \ldots , \\ &= \alpha(t) \quad \text{elsewhere} . \end{aligned}$$

Then $\beta(t)$ satisfies $(\alpha, 1)$ and $(\alpha, 2'')$. Therefore we can construct a Gaussian process $y(t) \in L$, $0 \leqq t \leqq 1$, such that $\alpha(t, y) = \beta(t)$, as we proved above.

By Lemma 6.1, we can construct a sequence of Gaussian processes $y_n(t) \in L$, $0 \leqq t \leqq 1$, $n = 1, 2, \ldots$, such that $\alpha(t, y_n) = \infty$ on I_n, $y_n(t) = 0$ on $[0,1] - I_n{}^\circ$ and $\|y_n(t)\| < 2^{-n}$. Now consider

$$x(t) = y(t) + \textstyle\sum_n y_n(t) .$$

Then

$$\begin{aligned} x(t) &= y_n(t) + y(t), \qquad t \in I_n{}^\circ, \quad n = 1, 2, \ldots , \\ &= y(t), \qquad \text{elsewhere}, \end{aligned}$$

and $x(t) \in L$, $0 \leqq t \leqq 1$. Therefore $x(t)$ is jointly Gaussian, measurable and separable. Its continuity in the mean follows from $\|y_n(t)\| < 2^{-n}$.

To complete our proof, we need only to show that $\alpha(t, x) = \alpha(t)$. Since $x(t) = y(t)$ on the set $G = [0,1] - \overline{\bigcup_n I_n}$ which is open in $[0,1]$, we have

$$\alpha(t,x) = \alpha(t,y) = \beta(t) = \alpha(t), \quad t \in G .$$

Since $x(t) = y_n(t) + y(t)$ and $y(t)$ is continuous in $I_n{}^\circ$, we have

$$\alpha(t,x) = \alpha(t, y_n) = \infty = \alpha(t), \quad t \in \textstyle\bigcup_n I_n{}^\circ .$$

If $t \in \overline{\bigcup_n I_n} - \bigcup_n I_n{}^\circ$, then t is an accumulation point of $\bigcup_n I_n{}^\circ$ and so we get

$$\alpha(t,x) \geqq \varlimsup_{\substack{s \to t \\ s \in \bigcup I_n{}^\circ}} \alpha(s,x) = \infty$$

by Theorem 4 (a) $(\alpha, 1)$.

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