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# UHF ALGEBRAS ARE SINGLY GENERATED

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In this note, we show that every UHF algebra is generated, as a $C^*$-algebra, by one of its elements. These algebras were defined by Glimm and extensively studied by him in [1]. The richness of the representation theory for UHF algebras is ample evidence of their importance in the theory. Indeed, Powers [3] has recently exhibited a continuum of non-isomorphic type III factor representations of a UHF algebra.

It is easy to obtain factor representations of types $\mathrm{I}_\infty$ and $\mathrm{II}_1$ for any UHF algebra. The former may be produced from pure states and the latter from the trace state. Since two such representations are disjoint, their direct sum yields a faithful non-factor representation on a separable space. Just how general such non-factor representations can be seems to be unknown.

Let $\mathscr{A}$ be a UHF algebra. It is known that $\mathscr{A}$ has a "factorization" (see [3] for details) so that there is a sequence $\mathscr{M}_n$ of finite type I von Neumann factors having the following properties:

1) $I \in \mathscr{M}_n \subset \mathscr{A}$.
2) The $\mathscr{M}_n$'s commute in pairs.
3) The $C^*$-algebra generated by $\bigcup_{n=1}^\infty \mathscr{M}_n$ is $\mathscr{A}$.

Now each $\mathscr{M}_n$ has a single generator as a $C^*$-algebra [2], call it $G_n$. The real part of $G_n$ is a finite real linear combination of mutually commuting projections $\{E_i^{(n)}\}$ in $\mathscr{M}_n$, and the imaginary part of $G_n$ is a finite real linear combination of mutually commuting projections $\{F_i^{(n)}\}$ in $\mathscr{M}_n$. Let $\mathscr{E}$ be the collection of all $E_i^{(n)}$'s, and $\mathscr{F}$ the collection of all $F_i^{(n)}$'s for $n=1,2,\ldots$. Then by 2), $\mathscr{E}$ is a countable commuting family of projections, as is $\mathscr{F}$. Let $\mathscr{R}$ (resp. $\mathscr{S}$) be the abelian $C^*$-algebra generated by $\mathscr{E}$ (resp. $\mathscr{F}$). According to [4, pp. 293–294], $\mathscr{R}$ (resp. $\mathscr{S}$) has a single Hermitian $C^*$-generator $R$ (resp. $S$). Put $G=R+iS$. We assert that $G$ generates $\mathscr{A}$ as a $C^*$-algebra. To see this, let $\mathscr{G}$ be the

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$C^*$-algebra generated by $G$. Then $R, S \in \mathscr{G}$ so that $\mathscr{R}, \mathscr{S} \subset \mathscr{G}$. It follows that $G_n \in \mathscr{G}$, for each $n$. But then $\mathscr{M}_n \subset \mathscr{G}$, for each $n$, so that $\mathscr{A} \subset \mathscr{G}$. We have therefore proved the following

THEOREM. *Any UHF algebra is generated, as a $C^*$-algebra, by a single operator.*

The reader will note that our method of proof is similar to the von Neumann algebra techniques of [2] and [5].

COROLLARY 1. *There exist operators on a separable Hilbert space which generate irreducible $C^*$-algebras containing no non-zero compact operators.*

PROOF. The image of the generator $G$ in any irreducible representation of $\mathscr{A}$ is easily seen to have the desired property (any non-trivial representation of $\mathscr{A}$ will be faithful, since $\mathscr{A}$ is a simple algebra [1, Theorem 5.1, p. 338], and every cyclic representation of $\mathscr{A}$ is on a separable Hilbert space, since $\mathscr{A}$ is norm separable).

A von Neumann algebra is said to be *hyperfinite* if it is the weak closure of a UHF algebra.

COROLLARY 2. *Every hyperfinite von Neumann algebra is singly generated as a von Neumann algebra.*

In particular, each of the uncountably many non-isomorphic type III factors of Powers [3] is singly generated as a von Neumann algebra.

Erling Størmer has pointed out the following application of the above technique to another class of algebras. Let $(\mathscr{A}_i)$ be a countable family of pairwise commuting von Neumann algebras, each of which is singly generated as a von Neumann algebra. Then the weak closure, $(\otimes_{i=1}^{\infty} \mathscr{A}_i)^-$, of the $C^*$-tensor product is singly generated as a von Neumann algebra.

ADDED IN PROOF. Since this paper was written, W. Wogen has shown that *every* properly infinite von Neumann algebra acting separably is singly generated. His result will appear shortly in the Bulletin of the American Mathematical Society.

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