MATH. SCAND. (1968), 227–240

SOME REMARKS ON GENERAL COHOMOLOGY THEORIES

A. DELEANU and P. J. HILTON

0. Introduction.

The starting point of this note is the observation that, in the Atiyah–Hirzebruch spectral sequence passing from ordinary cohomology H to a (generalized) cohomology theory h, defined on the category of CW-complexes X, the limit term E_∞ fails to capture precisely those classes in $h(X)$ which vanish on every skeleton X^q of X. The set of such classes yields a subgroup $\tilde{h}(X)$ of $h(X)$ which we call the group of *extraordinary* cohomology classes of X in the theory h. (This terminology seems to us very natural; on the other hand we find it unnatural to refer to h as an "extraordinary" cohomology theory, since it is very curious to stigmatize all theories with one single exception as extraordinary! We prefer, and adopt, the terminology "generalized" (see [8]) or "general" to describe a cohomology theory which is not required to satisfy the dimension axiom.) Moreover $\tilde{h}$ comes very close to being itself a cohomology theory. It satisfies the axioms (as stated in § 1 for a single-space theory) with the exception of the exactness axiom: given

$$A \overset{i}{\subseteq} X \xrightarrow{p} X/A$$

we can only guarantee that $\tilde{h}(i)\,\tilde{h}(p)=0$,

$$\tilde{h}(X/A) \xrightarrow{\tilde{h}(p)} \tilde{h}(X) \xrightarrow{\tilde{h}(i)} \tilde{h}(A)\,. \tag{0.1}$$

However, it turns out (0.1) almost always fails to be exact! Precisely we prove that if h is an additive (i.e., representable) theory and if $\tilde{h}$ is also a cohomology theory, then $\tilde{h}=0$ (Corollary 1.8). If h is not additive, $\tilde{h}$ can be a cohomology theory without being trivial: we cite as an example the James-Whitehead [5] zero-coefficient theory. Further we can have, even where h is additive, $\tilde{h}\neq 0$ (so that $\tilde{h}$ is not a cohomology theory). For the computations of L. Hodgkin show that, if h is complex K-theory, then $h(K(Q,n))=\tilde{h}(K(Q,n))\neq 0$, where n is even and Q is the group of

Received May 9, 1967.

rationals. (We are grateful to Mr. D. Burghelea for drawing our attention to this example.)

In § 2 we consider first a construction which in some sense generalizes that of $\tilde{h}$. The idea of this construction is certainly not new. We take a full subcategory $\mathscr{C}_0$ of the category $\mathscr{C}$ of CW-complexes and let $\mathscr{C}_{0h}, \mathscr{C}_h$ stand for the categories obtained by collecting the morphisms of $\mathscr{C}_0, \mathscr{C}$ into homotopy classes. For any cohomology theory h on $\mathscr{C}$ we set

$$h_0(X) = \varprojlim_f h(Y), \quad Y \in \mathscr{C}_0 ,$$

where $f\colon Y \to X$ in $\mathscr{C}_h$. There is then a natural transformation $r_0\colon h \to h_0$ and it turns out that $\tilde{h}$ is the kernel of r_0 if $\mathscr{C}_0$ is the category of finite-dimensional complexes. Moreover h_0 inherits all the "cohomology" properties of h (including additivity) with the possible exception of the exactness condition. Of course, Brown made very effective use of this construction (with $\mathscr{C}_0$ the category of finite complexes) in proving his representability theorem (see [1], [1a]). It then follows immediately from Brown's theorem that if h is representable and $\mathscr{C}_0$ contains all spheres, then h_0 is a cohomology theory if and only if $h_0 = h$. This result yields an immediate generalization of Corollary 1.8.

The process of obtaining h_0 from h is dualizable in an obvious way: we replace maps $f\colon Y \to X$ in $\mathscr{C}_h$, with $Y \in \mathscr{C}_0$, by maps $f\colon X \to Y$ in $\mathscr{C}_h$, with $Y \in \mathscr{C}_0$. We prove a theorem giving sufficient conditions on $\mathscr{C}_0$ for the resulting functor

$${}_0h(X) = \varinjlim_f h(Y)$$

to be a cohomology theory. Essentially we require that we can pull back in $\mathscr{C}_0$ a diagram

$$Y_0 \to Y \leftarrow Y_1$$

of fibrations in $\mathscr{C}_0$; and that mapping cones of maps in $\mathscr{C}_0$ lie in $\mathscr{C}_0$. This leads to several open questions as to whether given subcategories of $\mathscr{C}$ satisfy these conditions: for example, we do not know whether the category of complexes with finite homotopy groups does so. We can however say that we do obtain examples by imposing a bound on the cardinality of the set of cells of $Y \in \mathscr{C}_0$. There is also the question of elucidating the circumstances under which ${}_0h$, whether or not it is a cohomology theory, inherits additivity from h.

1. Extraordinary cohomology classes.

Let $\mathscr{C}$ be the category of CW-complexes with base-point and of base-point-preserving continuous maps, and let $\mathscr{A}b$ be the category of abelian groups. We recall [8] that a (generalized or extraordinary) cohomology theory h on $\mathscr{C}$ is a sequence of contravariant functors

$$h^n\colon \mathscr{C} \to \mathscr{A}\ell, \qquad -\infty < n < +\infty ,$$

together with a sequence of natural transformations

$$\sigma^n\colon h^n \to h^{n+1}\Sigma, \qquad -\infty < n < +\infty ,$$

where Σ is the suspension functor, satisfying the following conditions:

(1) If $f_0 \simeq f_1$, then $h^n(f_0) = h^n(f_1)$.

(2) For any $X \in \mathscr{C}$

$$\sigma^n(X)\colon h^n(X) \approx h^{n+1}(\Sigma X) .$$

(3) If A is a subcomplex of X, $i\colon A \to X$ the inclusion map and $p\colon X \to X/A$ the identification map, then the sequence

$$h^n(X/A) \xrightarrow{h^n(p)} h^n(X) \xrightarrow{h^n(i)} h^n(A)$$

is exact.

A cohomology theory h on $\mathscr{C}$ is said to be *additive* if it satisfies the following general wedge axiom:

(4) If $(X_\alpha)_\alpha$ is a family of objects of $\mathscr{C}$ and $j_\beta\colon X_\beta \to \bigvee_\alpha X_\alpha$ is the inclusion map, then

$$\prod_\alpha h^n(j_\alpha)\colon h^n(\bigvee_\alpha X_\alpha) \approx \prod_\alpha h^n(X_\alpha) \quad \text{for all } n .$$

We shall sometimes drop the superscript n on h^n, when there is no danger of confusion.

Let now $X \in \mathscr{C}$ and let $i_q(X)\colon X^q \to X$ be the inclusion of the q-skeleton. We define a sequence of contravariant functors

$$\tilde{h}^n\colon \mathscr{C} \to \mathscr{A}\ell, \qquad -\infty < n < +\infty ,$$

as follows:

$$\tilde{h}^n(X) = \bigcap_{q \geqq 0} \operatorname{Ker} h^n(i_q(X)) .$$

Let $f\colon X \to Y$ be a map in $\mathscr{C}$. Select a *cellular* map $g\colon X \to Y$ such that $g \simeq f$. Then we have for each $q \geqq 0$ and for each n a commutative diagram

$$\begin{tikzcd}
h^n(X) \arrow[r, "h^n(i_q(X))"] & h^n(X^q) \\
h^n(Y) \arrow[u, "h^n(f)"] \arrow[r, "h^n(i_q(Y))"'] & h^n(Y^q) \arrow[u, "h^n(g|X^q)"']
\end{tikzcd}$$

which shows that $h^n(f)$ maps $\tilde{h}^n(Y)$ into $\tilde{h}^n(X)$ and thus defines $\tilde{h}^n$ by restriction as a functor. It also shows that the values of $\tilde{h}^n$ depend only on the topological structure (indeed, the homotopy structure) and not on the cellular structure of X. Moreover we have a natural transformation $s: \tilde{h} \to h$ with $s(X)$ monic.

We now define a sequence of natural transformations

$$\tilde{\sigma}^n:\ \tilde{h}^n \to \tilde{h}^{n+1} \circ \Sigma, \qquad -\infty < n < +\infty\ ,$$

as follows: For each $q \geqq 0$ and each n we have a commutative diagram:

$$\begin{tikzcd}
h^n(X) \arrow[r, "h^n(i_q(X))"] \arrow[d, "\sigma^n(X)"', "\approx"] & h^n(X^q) \arrow[d, "\approx"', "\sigma^n(X^q)"] & \\
h^{n+1}(\Sigma X) \arrow[r, "h^{n+1}(i_{q+1}(\Sigma X))"'] & h^{n+1}((\Sigma X)^{q+1}) \arrow[r, phantom, "="] & h^{n+1}(\Sigma X^q)
\end{tikzcd}$$

which enables us to set by definition

$$\tilde{\sigma}^n(X) = \sigma^n(X) \mid \bigcap_{q \geqq 0} \operatorname{Ker} h^n(i_q(X))\ .$$

It is straightforward that $s: \tilde{h} \to h$ is compatible with $\sigma, \tilde{\sigma}$ and that the functors $\tilde{h}^n$ together with the natural transformations $\tilde{\sigma}^n$ satisfy conditions (1) and (2) in the definition of a generalized cohomology theory. As to condition (3), we can only prove in general that $\tilde{h}^n(i) \circ \tilde{h}^n(p) = 0$. It is also easy to check that, if h^n satisfies the general wedge axiom (4), then so does $\tilde{h}^n$.

Proposition 1.1. *Let h be an additive cohomology theory on $\mathscr{C}$. Define for any integer n and any $X \in \mathscr{C}$ the homomorphism*

$$\varphi_X^n:\ \prod_{q \geqq 0} h^n(X^q) \to \prod_{q \geqq 0} h^n(X^q)$$

by

$$\varphi_X^n(x_0, x_1, x_2, \ldots) = \left(x_0 - h^n(i_0{}^1)(x_1), x_1 - h^n(i_1{}^2)(x_2), x_2 - h^n(i_2{}^3)(x_3), \ldots\right),$$

where $x_q \in h^n(X^q)$ and $i_q{}^{q+1} = i_q{}^{q+1}(X): X^q \to X^{q+1}$ is the inclusion map, $q \geqq 0$.

The necessary and sufficient condition for $\tilde{h}^n(X) = 0$ is that $\varphi_X{}^{n-1}$ be an epimorphism.

Proof. Consider the natural homomorphism

$$r = \varprojlim_q h^n(i_q):\ h^n(X) \to \varprojlim_q h^n(X^q)\ .$$

According to a result of Milnor (Lemma 2 of [7]) whose proof—as he himself observes—does not require the dimension axiom, we have

$$\operatorname{Ker} r = \operatorname{Ker}(\varprojlim_{q} h^n(i_q)) = \operatorname{Coker} \varphi_X{}^{n-1} .$$

On the other hand, it is obvious that

$$\tilde{h}^n(X) = \bigcap_{q \geqq 0} \operatorname{Ker} h^n(i_q) = \operatorname{Ker}(\varprojlim_{q} h^n(i_q)) = \operatorname{Ker} r .$$

REMARK 1.2. The result of Milnor mentioned above also states that r is an epimorphism for any additive cohomology theory.

COROLLARY 1.3. *Let h be an additive cohomology theory on $\mathscr{C}$ and let n be an integer and $X \in \mathscr{C}$. If the inverse system of abelian groups $\{h^{n-1}(X^q), h^{n-1}(i_q{}^{q+1}(X))\}$ satisfies the Mittag-Leffler condition, i.e. if for any integer $q \geqq 0$ there exists m such that*

$$\operatorname{Im}[h^{n-1}(X^{q+m}) \to h^{n-1}(X^q)] = \operatorname{Im}[h^{n-1}(X^{q+k}) \to h^{n-1}(X^q)]$$

for any $k \geqq m$, then $\tilde{h}^n(X) = 0$.

PROOF. The assertion follows immediately from Proposition 1.1 in view of the fact that $\operatorname{Coker} \varphi_X{}^{n-1}$ is isomorphic to $\varprojlim{}^{(1)} h^{n-1}(X^q)$, where $\varprojlim{}^{(1)}$ is the first derived functor of $\varprojlim$ [7], and the fact that this derived functor is zero for any inverse system of abelian groups which satisfies the Mittag-Leffler condition (see, for instance, [6, p. 17]).

REMARK 1.4. If the abelian groups $h^{n-1}(X^q)$ are countable for all q, the converse statement holds, namely that $\tilde{h}^n(X) = 0$ implies that the inverse system $\{h^{n-1}(X^q), h^{n-1}(i_q{}^{q+1}(X))\}$ satisfies the Mittag-Leffler condition. This follows at once from a result of B. I. Gray [4].

THEOREM 1.5. *Let h be an additive cohomology theory on $\mathscr{C}$, n an integer and $X \in \mathscr{C}$. If there exists an integer N such that*

$$h^{n-1}(i_q{}^{q+1}) \colon h^{n-1}(X^{q+1}) \to h^{n-1}(X^q)$$

is an epimorphism for $q \geqq N$, then $\tilde{h}^n(X) = 0$.

PROOF. The hypothesis implies that the system $\{h^{n-1}(X^q), h^{n-1}(i_q{}^{q+1})\}$ satisfies the Mittag-Leffler condition. Apply Corollary 1.3.

REMARK 1.6. An easy consequence of Theorem 1.5 is the statement that, if h is an additive cohomology theory on $\mathscr{C}$ such that there exists an integer N with $h^n(S^0) = 0$ for any $n \leqq N$, then $\tilde{h}^n(X) = 0$ for any n and any X. Now this statement actually follows from the following much stronger result, whose proof relies on E. H. Brown's theorem [1] to the effect that any additive cohomology theory on $\mathscr{C}$ is representable by an Ω-spectrum and on simple considerations of obstruction theory:

PROPOSITION 1.7. *Let h be an additive cohomology theory on $\mathscr{C}$ such that there exists an integer N with $h^n(S^0)=0$ for any $n \leqq N$. Then, for any n and X,*

$$h^n(i_q)\colon\ h^n(X) \to h^n(X^q)$$

is an isomorphism for sufficiently large q.

COROLLARY 1.8. *Let h be an additive cohomology theory on $\mathscr{C}$. If $\tilde{h}$ is a cohomology theory, then $\tilde{h}^n(X)=0$ for any n and any X.*

PROOF. It is straightforward that $\tilde{\tilde{h}}^n(X)=\tilde{h}^n(X)$ for any n and any X. Now $\tilde{h}^n(X)=0$ for every finite-dimensional CW-complex X, so we may apply Proposition 1.7 with any N.

REMARK 1.9. The James–Whitehead theory with zero coefficients [5] provides an example of a theory with $h^n(S^0)=0$ for any integer n, but such that $\tilde{h}=h \neq 0$. Of course, this theory is not representable.

2. Limits over subcategories.

It will be convenient in this section to enlarge the category of application of our cohomology theories to include all spaces of the (based) homotopy type of CW-complexes. Our reason for doing this is simply to permit ourselves to apply path-space functors; any other device achieving this effect (e.g. working in a category of simplicial sets satisfying the Kan condition) would have been equally acceptable.

Thus $\mathscr{C}$ will denote the category of based spaces of the based homotopy type of CW-complexes and base-point-preserving continuous maps and we consider cohomology theories on $\mathscr{C}$ to $\mathscr{A}b$. The arguments of the previous section carry over in this more general context. If $X \in \mathscr{C}$ then we will consider, in formulating condition (3) for a cohomology theory, those subspaces $A \subseteq X$ such that the pair (X,A) is homotopically equivalent to a CW-pair. Again let $X = \mathscr{C}$ and let $u\colon X \to \bar{X}$ be a homotopy equivalence of X with a CW-complex $\bar{X}$; we extend $\tilde{h}$ to the whole of $\mathscr{C}$ by setting

$$\tilde{h}(X) = h(u)\,\tilde{h}(\bar{X})\,.$$

If $v\colon X \to \bar{\bar{X}}$ is another homotopy equivalence of X with a CW-complex $\bar{\bar{X}}$, then there is a homotopy equivalence $w\colon \bar{X} \to \bar{\bar{X}}$ with $wu \simeq v$; moreover $\tilde{h}(w)$ is an isomorphism so

$$h(v)\,\tilde{h}(\bar{\bar{X}}) = h(u)\,h(w)\,\tilde{h}(\bar{\bar{X}}) = h(u)\,\tilde{h}(\bar{X})\,.$$

Thus $\tilde{h}(X)$ is well-defined; similarly we define $\tilde{h}(f)$ for any $f\colon X \to Y$ in $\mathscr{C}$

and thus define the functor $\tilde{h}$. Of course $\tilde{h}$ is by definition homotopy invariant; and $\tilde{h}(f)$ is defined by restricting $h(f)$.

We now extend the sequence $\tilde{\sigma}^n\colon \tilde{h}^n \to \tilde{h}^{n+1}\Sigma$, of natural transformations, to the whole of $\mathscr{C}$. Given X and $u\colon X \to \bar{X}$ as above, we define $\tilde{\sigma}^n(X)$ by imposing commutativity on the diagram

$$\begin{tikzcd} \tilde{h}^n(X) \arrow[d, "\tilde{\sigma}^n(X)"'] & \tilde{h}^n(\bar{X}) \arrow[l, "\tilde{h}^n(u)"'] \arrow[d, "\tilde{\sigma}^n(\bar{X})"] \\ \tilde{h}^{n+1}(\Sigma X) & \tilde{h}^{n+1}(\Sigma \bar{X}) \arrow[l, "\tilde{h}^{n+1}(\Sigma u)"'] \end{tikzcd} .$$

Of course, all arrows in this diagram are isomorphisms. Again one readily shows that $\tilde{\sigma}^n(X)$ is independent of the choice of u, and $\tilde{\sigma}^n$ constitutes a sequence of natural transformations.

The results of the previous section are now readily interpretable in our enlarged category; where, in the enunciation of some result explicit mention is made of the skeleton filtration of a complex X, it is now necessary merely to replace the given hypothesis by the same hypothesis with respect to *any* complex of the homotopy type of X. Thus we may assume all results of the previous section, modified in this evident sense, to be available.

Let $\mathscr{C}_h$ be the homotopy category of $\mathscr{C}$, i.e. the category whose objects are those of $\mathscr{C}$ and whose morphisms are the based homotopy classes of maps in $\mathscr{C}$.

Let $\mathscr{C}_0$ be a full subcategory of $\mathscr{C}$, and $\mathscr{C}_{0h}$ the corresponding full subcategory of $\mathscr{C}_h$. We may then form, for any object X in $\mathscr{C}_h$, the $\mathscr{C}_{0h}$-category over X, $(\mathscr{C}_{0h}, X)$. An object of $(\mathscr{C}_{0h}, X)$ is a map $f\colon Y \to X$ in $\mathscr{C}_h$ with $Y \in \mathscr{C}_0$, and a morphism $u\colon f_0 \to f_1$ is a map $u\colon Y_0 \to Y_1$ in $\mathscr{C}_{0h}$ such that

$$\begin{tikzcd} Y_0 \arrow[d, "u"'] \arrow[dr, "f_0"] & \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

commutes in $\mathscr{C}_h$.

Now, if h is a cohomology theory on $\mathscr{C}$, we may construct the inverse limit $\varprojlim_f (h(Y), h(u))$; call this $h_0(X)$. Clearly h_0 is a contravariant functor and there is a natural transformation $r_0\colon h \to h_0$, such that $r_0(Z)$ is an isomorphism for all $Z \in \mathscr{C}_0$.

We mention that, in the case when $\mathscr{C}_0$ is the category of finite CW-complexes, this process of taking inverse limits has been used by E. H.

Brown [1, p. 478] and by A. Dold for half-exact functors [3, p. 9.2]. We also mention that h_0 satisfies condition (4) if h is additive.

REMARK 2.1. Assume $\mathscr{C}_0$ contains all finite CW-complexes. Then, if h and h_0 are both additive cohomology theories (we understand by this that h_0 is provided with a sequence of natural equivalences $\sigma_0{}^n$ satisfying condition (2), and compatible with r_0), it follows that r_0 is a natural equivalence: $r_0: h \approx h_0$. For, according to E. H. Brown's representability theorem [1], there exist CW-complexes Y, Y_0 and natural equivalences

$$h(X) = [X, Y], \quad h_0(X) = [X, Y_0], \quad X \in \mathscr{C}.$$

Now the natural transformation r_0 is induced by a map $f \colon Y \to Y_0$ and we clearly have for each sphere S^n

$$r_0(S_n) = f^*\colon [S_n, Y] \approx [S_n, Y_0].$$

Thus, by J. H. C. Whitehead's theorem, f is a homotopy equivalence, so that r_0 is a natural equivalence.

THEOREM 2.2. *If $\mathscr{C}_0$ is the category of finite-dimensional complexes, then r factors as $r = r_1 r_0$, where*

$$r_1 = r_1(X)\colon\ h_0(X) = \varprojlim_f\, (h(Y), h(u)) \to \varprojlim h(X^q)$$

is an isomorphism.

The natural transformation $r \colon h(X) \to \varprojlim h(X^q)$ was defined in the proof of Proposition 1.1.

PROOF. An element of $\varprojlim_f (h(Y), h(u))$ is a family of elements $\alpha_f \in h(Y)$ for each f, such that $h(u)\alpha_{f_1} = \alpha_{f_0}$ for any u with $f_1 u = f_0$. By restricting f to be the homotopy class of an inclusion $X^k \subset X$ and u to be the homotopy class of an inclusion $X^l \subset X^m$, for any k, l, m, we define r_1, and plainly $r = r_1 r_0$. We wish to show that r_1 is an isomorphism. (As explained at the beginning of this section we may confine attention to CW-complexes X.)

Let $\xi \in \varprojlim_f (h(Y), h(u))$ with $r_1(\xi) = 0$. Let α_f be a component of ξ, where $f \colon Y \to X$. By cellular approximation $f = i_q f'$ for some q, where $f' \colon Y \to X^q$. But the i_q-component of ξ is zero and is mapped by $h(f')$ to the f-component of ξ, namely α_f. Thus $\alpha_f = 0$, so that $\xi = 0$.

Now let $\eta \in \varprojlim h(X^q)$. For each f choose $k = \dim Y + 1$ and a homotopy class $f' \colon Y \to X^k$ with $i_k f' = f$. Then f' is uniquely determined (by the requirement that it contains a cellular map).

Let $\alpha_f = h(f')\eta_k$, where η_k is the i_k-component of η. We assert that the α_f determine an element $\xi \in \varprojlim_f (h(Y), h(u))$ such that $r_1(\xi) = \eta$. Plainly the i_q-component of ξ is η_q for it is $h(j)\eta_{q+1}$, where $j: X^q \subset X^{q+1}$. Thus it remains to show that ξ belongs to the limit. Consider the commutative diagram in $\mathscr{C}_h$

$$\begin{tikzcd} Y_0 \arrow[d, "u"'] \arrow[dr, "f_0"] & \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd}$$

where $Y_0, Y_1 \in \mathscr{C}_0$. Choose $m = \max(\dim Y_0, \dim Y_1) + 1$,

$$f_0'': \; Y_0 \to X^m, \qquad f_1'': \; Y_1 \to X^m$$

being uniquely determined "cellular classes" as above, with $f_0 = i_m f_0''$, $f_1 = i_m f_1''$. Then if $\alpha_{f_0} = h(f_0')\eta_k$, as above, we claim that $\alpha_{f_0} = h(f_0'')\eta_m$. For if $j: X^k \subset X^m$, then $f_0'' = jf_0'$, $\eta_k = h(j)\eta_m$, so that

$$h(f_0')\,\eta_k \;=\; h(f_0')\,h(j)\,\eta_m \;=\; h(jf_0')\,\eta_m \;=\; h(f_0'')\,\eta_m\,.$$

Similarly $\alpha_{f_1} = h(f_1'')\eta_m$.

It follows immediately that $\alpha_{f_1} = h(u)\alpha_{f_1}$. For $f_1''u = f_0''$, so that

$$\alpha_{f_0} \;=\; h(f_0'')\,\eta_m \;=\; h(u)\,h(f_1'')\,\eta_m \;=\; h(u)\,\alpha_{f_1}\,.$$

Thus $\xi \in \varprojlim_f (h(Y), h(u))$ and $r_1(\xi) = \eta$, establishing the claim that r_1 is an isomorphism.

Corollary 2.3. *If $\mathscr{C}_0$ is the category of finite-dimensional complexes and h is additive, then r_0 is an epimorphism. Indeed, there is then a short exact sequence*

$$\tilde{h} \overset{s}{\rightarrowtail} h \overset{r_0}{\twoheadrightarrow} h_0\,.$$

If, moreover, h_0 or $\tilde{h}$ is a cohomology theory, then $\tilde{h} = 0$ and $r_0: h \approx h_0$.

Proof. Now $r(X): h(X) \twoheadrightarrow \varprojlim h(X^q)$ if h is additive and hence representable. Thus r_0 is an epimorphism. Moreover $\ker r_0 = \ker r = \operatorname{Im} \tilde{h}$ (see the proof of Proposition 1.1) so we obtain the short exact sequence

$$\tilde{h} \overset{s}{\rightarrowtail} h \overset{r_0}{\twoheadrightarrow} h_0\,.$$

We use this sequence to induce natural isomorphisms $\sigma_0{}^n: h_0{}^n \approx h_0{}^{n+1}\Sigma$. Plainly then $\tilde{h}$ and h_0 are both cohomology theories if either is; but we have observed that $\tilde{h}$ satisfies the wedge axiom (4) if h is additive. Thus if h_0 or $\tilde{h}$ is a cohomology theory then $\tilde{h}$—and hence also h_0—is an additive cohomology theory. We now complete the argument by applying Remark 2.1. Alternatively, we could have completed the argument by appeal to Corollary 1.8.

Naturally, in Theorem 2.2 and Corollary 2.3 we may throw into $\mathscr{C}_0$ any object of $\mathscr{C}$ homotopy equivalent to a finite-dimensional complex.

We now consider the dual process. We again let $\mathscr{C}_0$ be a full subcategory of $\mathscr{C}$ and consider, for an object X of $\mathscr{C}_h$, the $\mathscr{C}_{0h}$-category *under* X, $(X,\mathscr{C}_{0h})$. An object of $(X,\mathscr{C}_{0h})$ is a map $f\colon X \to Y$ in $\mathscr{C}_h$ with $Y \in \mathscr{C}_0$, and a morphism $u\colon f_0 \to f_1$ is a map $u\colon Y_0 \to Y_1$ in $\mathscr{C}_{0h}$ such that $uf_0 = f_1$. We may then construct the direct limit $\varinjlim_f(h(Y),h(u))$; call this ${}_0h(X)$. Clearly ${}_0h$ is a contravariant functor and there is a natural transformation $t\colon {}_0h \to h$. We will be concerned to give reasonable conditions under which ${}_0h$ is a cohomology theory.

THEOREM 2.4. *Assume that $\mathscr{C}_{0h}$ contains the one-point space and, with any space, all spaces equivalent to it. Assume also that $\mathscr{C}_0$ contains pull-backs of fibrations and mapping cones. Under these conditions, ${}_0h$ is a cohomology theory.*

PROOF. First of all, note that the conditions imposed on $\mathscr{C}_{0h}$ and $\mathscr{C}_0$ guarantee that $\mathscr{C}_{0h}$ and $\mathscr{C}_0$ contain finite direct products and that $\mathscr{C}_0$ contains, with any space X, the loop space ΩX and the suspension ΣX. They also imply that $\mathscr{C}_{0h}$ has *weak* pull-backs with respect to $\mathscr{C}_h$, that is, given $f_i\colon X_i \to Y$, $i=1,2$, in $\mathscr{C}_{0h}$ there exist $g_i\colon Z \to X_i$, $1=1,2$, in $\mathscr{C}_{0h}$, such that $f_1g_1 = f_2g_2$ and, for any $l_i\colon W \to X_i$ in $\mathscr{C}_h$ with $f_1l_1 = f_2l_2$, there exists a (not necessarily unique) morphism $c\colon W \to Z$ in $\mathscr{C}_h$ such that $l_i = g_ic$, $i=1,2$. (To see this one can apply an argument similar to that of [2, p. 298], where weak pull-backs are referred to as "generators of intersection ideals".) It is just these conditions on $\mathscr{C}_{0h}$, together with the existence of cones, that the hypotheses were designed to ensure.

We now go ahead with the actual proof. Under the given hypotheses, we may represent $\varinjlim_f(h(Y),h(u))$ as follows. We consider, for each $f\colon X \to Y$, the group $h(Y)$; an element of the set $\bigcup_f h(Y)$ will be denoted by (α,f) or α_f, $\alpha \in h(Y)$. We then set up an equivalence relation in $\bigcup_f h(Y)$ by declaring $\alpha_{f_0} \sim \alpha_{f_1}$ if there is a commutative diagram in $\mathscr{C}_h$

$$(2.5)\qquad \begin{tikzcd} & Y_0 \\ X \arrow[ur] \arrow[r, "f"] \arrow[dr] & Y \arrow[u, "u_0"'] \arrow[d, "u_1"] \\ & Y_1 \end{tikzcd}, \qquad u_0, u_1 \in \mathscr{C}_{0h},$$

with Y_0, Y_1, Y in $\mathscr{C}_0$, $h(u_0)\alpha_{f_0} = h(u_1)\alpha_{f_1}$. The proof that this is an equivalence relation depends on the fact that $\mathscr{C}_{0h}$ has weak pullbacks with respect to $\mathscr{C}_h$. We introduce a group structure into the set of equivalence classes as follows: given $\alpha_{f_0} \in h(Y_0)$, $\alpha_{f_1} \in h(Y_1)$, consider the commutative diagram

$$\begin{tikzcd} & Y_0 \\ X \arrow[ur] \arrow[r, "f"] \arrow[dr] & Y_0 \times Y_1, \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & Y_1 \end{tikzcd} \qquad f = \{f_0, f_1\},$$

where p_0 and p_1 are the canonical projections. Then the sum of the equivalence classes represented by α_{f_0} and α_{f_1} is by definition the equivalence class represented by $h(p_0)\alpha_{f_0} + h(p_1)\alpha_{f_1} \in h(Y_0 \times Y_1)$. Plainly the resulting group is the direct limit $\varinjlim_f (h(Y), h(u)) = {}_0h(X)$. Moreover $t\colon {}_0h \to h$ simply associates $h(f)\alpha$ with the classe of α_f. We now establish the exactness axiom (3).

We observe from (2.5) that if α_f represents the zero of ${}_0h(X)$, then there is a commutative diagram in $\mathscr{C}_h$,

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[dr, "f_0"'] & \\ & Y_0 \arrow[uu, "u"'] \end{tikzcd} \qquad Y_0, Y \in \mathscr{C}_0,$$

such that $h(u)\alpha_f = 0$. Consider now a cofibre sequence

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

in $\mathscr{C}$ and the resulting sequence

$${}_0h(X/A) \xrightarrow{{}_0h(p)} {}_0h(X) \xrightarrow{{}_0h(i)} {}_0h(A)$$

in $\mathscr{A}\mathscr{b}$. Let $\xi \in {}_0h(X)$ go to zero in ${}_0h(A)$, and let ξ be represented by $\alpha = \alpha_f \in h(Y)$, $f\colon X \to Y$. Then $\alpha = \alpha_{fi} \in h(Y)$, $fi\colon A \to Y$, represents zero in ${}_0h(A)$; that is, there exists a commutative diagram in $\mathscr{C}_h$,

$$\text{(2.6)} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "i"] \arrow[r, "f_0"'] & Y_0 \arrow[u, "u"'] \end{tikzcd}$$

with $h(u)\alpha = 0$. Then (2.6) induces a diagram

$$
\begin{tikzcd}
X/A \arrow[r, "f_1"] & C_u \\
X \arrow[u, "p"] \arrow[r, "f"] & Y \arrow[u, "v"'] \\
A \arrow[u, "i"] \arrow[r, "f_0"] & Y_0 \arrow[u, "u"']
\end{tikzcd}
\tag{2.7}
$$

such that the top square is also (homotopy) commutative, where C_u is the mapping cone of u. Since $h(u)\alpha=0$, we have $\alpha=h(v)\beta$ for some $\beta \in h(C_u)$. Then $\beta=\beta_{f_1}$ represents an element η of ${}_0h(X/A)$ such that ${}_0h(p)\eta$ is represented by $\beta=\beta_{f_1p}=\beta_{vf}$. But then ${}_0h(p)\eta$ is also represented by $\alpha=\alpha_f$, so that ${}_0h(p)\eta=\xi$.

It remains to verify the suspension axiom (2). We define

$$
{}_0\sigma^n\colon\ {}_0h^n(X) \to {}_0h^{n+1}(\Sigma X)
$$

by mapping a representative $\alpha_f=(\alpha,f)$, $f\colon X \to Y$, $\alpha \in h^n(Y)$, to $(\sigma\alpha,\Sigma f)$, where we denote by $\sigma\alpha$ the image of α under the isomorphism $\sigma^n(Y)\colon h^n(Y) \to h^{n+1}(\Sigma Y)$. This clearly induces a map ${}_0\sigma^n$. That ${}_0\sigma^n$ is a homomorphism may be seen from the commutative diagram

$$
\begin{tikzcd}
 & \Sigma(Y_1 \times Y_2) \arrow[d, "\{\Sigma p_1,\, \Sigma p_2\}"] \\
\Sigma X \arrow[ur, "\Sigma\{f_1,f_2\}"] \arrow[r, "\{\Sigma f_1,\, \Sigma f_2\}"'] & \Sigma Y_1 \times \Sigma Y_2
\end{tikzcd}
$$

together with the fact that $h\{\Sigma p_1,\Sigma p_2\}(\sigma\alpha_1,\sigma\alpha_2)=\sigma(\alpha_1,\alpha_2)$, where $(\alpha_1,\alpha_2) \in h(Y_1 \times Y_2)$ stands for $h(p_1)\alpha_1+h(p_2)\alpha_2$, with a similar convention for $(\sigma\alpha_1,\sigma\alpha_2)$. It is also clear that $t\colon {}_0h \to h$ is compatible with ${}_0\sigma,\sigma$.

We next show that ${}_0\sigma^n$ is onto. Recall that we may take loopspaces and suspensions in $\mathscr{C}_0$. Let $d\colon Y \to \Omega\Sigma Y$, $e\colon \Sigma\Omega Y \to Y$ be the natural maps. Given $f\colon \Sigma X \to Y$ and $\alpha \in h^{n+1}(Y)$ consider the commutative diagram

$$
\begin{tikzcd}
\Sigma X \arrow[r, "f"] \arrow[dr, "\Sigma f'"'] & Y \\
 & \Sigma\Omega Y \arrow[u, "e"']
\end{tikzcd}
$$

where $f'\colon X \to \Omega Y$ is adjoint to f. Then $(\alpha,f)\sim(h(e)\alpha,\Sigma f')$. Thus if $\beta \in h^n(\Omega Y)$ is such that $\sigma\beta=h(e)\alpha$, then (β,f') represents an element of ${}_0h^n(X)$ which suspends to the class represented by (α,f).

Finally we show that ${}_0\sigma^n$ is one-to-one. Suppose given (α,f), where $f\colon X \to Y$, $\alpha \in h^n(Y)$ and a commutative diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[dr, "g"'] & \Sigma Y \\ & Y_0 \arrow[u, "u_0"'] \end{tikzcd} \tag{2.8}$$

with $h(u_0)(\sigma\alpha)=0$. Now $e\Sigma g'=g$, where $g'\colon X \to \Omega Y_0$ is adjoint to g. Thus, by setting

$$Y_1 = \Omega Y_0, \quad f_1 = g', \quad u = u_0 e ,$$

we may replace (2.8) by the commutative diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[dr, "\Sigma f_1"'] & \Sigma Y \\ & \Sigma Y_1 \arrow[u, "u"'] \end{tikzcd} \tag{2.9}$$

with $h(u)(\sigma\alpha)=0$. Consider further the diagrams, where $(\Sigma f)'$ is the adjoint of Σf and u' the adjoint of u:

$$\begin{tikzcd} & Y \arrow[d, "d"] \\ X \arrow[ur, "f"] \arrow[r, "(\Sigma f)'"] \arrow[dr, "f_1"'] & \Omega\Sigma Y \\ & Y_1 \arrow[u, "u'"'] \end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd} \Sigma Y \arrow[d, shift right, "\Sigma d"'] \\ \Sigma\Omega\Sigma Y \arrow[u, shift right, "e"'] \\ \Sigma Y_1 \arrow[u, "\Sigma u'"] \end{tikzcd} \tag{2.11}$$

Then (2.10) commutes, and in (2.11) we have

$$e\Sigma d = 1, \quad e\Sigma u' = u .$$

Let $\beta \in h^n(\Omega\Sigma Y)$ be such that $\sigma\beta = h(e)\sigma\alpha$. Then $h(\Sigma d)(\sigma\beta)=\sigma\alpha$, so

$$h(d)\beta = \alpha .$$

Thus $(\alpha,f) \sim (\beta,(\Sigma f)') \sim (h(u')\beta,f_1)$. But

$$h(\Sigma u')(\sigma\beta) = h(\Sigma u')h(e)(\sigma\alpha) = h(u)(\sigma\alpha) = 0 ,$$

so

$$h(u')(\beta) = 0.$$

Thus $(\alpha,f) \sim (0,f_1)$, so (α,f) represents zero. This completes the proof of the theorem.

We remark that a suitable choice for the category $\mathscr{C}_0$ is that of countable CW-complexes. Plainly ${}_0h \neq h$ in general with this choice of $\mathscr{C}_0$.

We close this section with the following observation. We might be tempted to adopt the simple-minded device of seeking to construct a cohomology theory out of a given theory by restricting attention to those classes which are "represented" by complexes in $\mathscr{C}_0$. That is, we consider the subgroup of $h(X)$ consisting of those classes α such that there exists $f: X \to Y$, $Y \in \mathscr{C}_0$, with $\alpha \in h(f)h(Y)$. This is a subgroup provided $\mathscr{C}_0$ admits finite products and we again have a functor. However, exactness will, in general, fail to hold. Consider, for example, the case when $\mathscr{C}_0$ is the category of finite-dimensional complexes, and let h be ordinary cohomology. Consider the sequence

$$S^2 \xrightarrow{i} K(Z,2) \xrightarrow{p} K(Z,2)/S^2 .$$

Then the admissible subgroup of $h^2(K(Z,2))$ is obviously zero since elements in the subgroup must have finite multiplicative order, so we certainly do not get exactness.

REFERENCES

1. E. H. Brown, *Cohomology theories*, Ann. of Math. 75 (1962), 467–484.

1a. E. H. Brown: *Abstract homotopy theory*, Trans. Amer. Math. Soc. 115 (1965), 79–85.

2. B. Eckmann and P. J. Hilton, *Unions and intersections in homotopy theory*, Comment. Math. Helv. 38 (1964), 293–307.

3. A. Dold, *Halbexakte Homotopiefunktoren*, Lecture Notes in Math. nr. 12, Springer, 1966.

4. B. I. Gray: *Spaces of the same n-type, for all n*, Topology 5 (166), 241–243.

5. I. M. James and J. H. C. Whitehead, *Homology with zero coefficients*, Quart. J. Math. Oxford Ser. 9 (1958), 317–320.

6. O. A. Laudal, *Sur la limite projective et la théorie de la dimension* II, Séminaire de topologie et géométrie differentielle dirigé par Ch. Ehresmann, vol. III, Institut H. Poincaré, Paris (1962).

7. J. Milnor, *On axiomatic homology theory*, Pacific J. Math. 12 (1962), 337–341.

8. G. W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. 102 (1962), 227–283.

INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES, BUCHAREST, RUMANIA
EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE, ZÜRICH, SWITZERLAND,
AND
CORNELL UNIVERSITY, ITHACA, N.Y., U.S.A.