MATH. SCAND. 22 (1968), 241–256

THE ALGEBRA OF POLYHEDRA AND THE DEHN-SYDLER THEOREM

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1.

The algebra of polyhedra has its origin in Euclid's treatment of the volume of polyhedra. Through the recent result of J.-P. Sydler [8], that the necessary conditions for the equivalence of polyhedra due to M. Dehn [1] are also sufficient, this subject has reached a certain completion. In the present paper we shall give a simplified proof of Sydler's theorem. The proof is closely related to the simple proof of Dehn's theorem due to H. Hadwiger [2]. It depends on certain algebraical theorems; the proofs of these theorems have been obtained in collaboration with J. Karpf and A. Thorup and will appear in a separate paper [4]. We also find certain additional results. In order to make the exposition self-contained we have included brief proofs of the known results which we need. For a detailed treatment of the basic results we refer to the book of Hadwiger [3].

2.

By the *polyhedron group* $\mathcal{D}$ in three-dimensional euclidean space we mean the free abelian group generated by the (closed, non-degenerate) polyhedra.

A polyhedron P is said to be composed of the polyhedra $P_1,\ldots,P_n$, or to be decomposed into $P_1,\ldots,P_n$, if $P=P_1\cup\ldots\cup P_n$ and the interiors of $P_1,\ldots,P_n$ are disjoint. A polyhedron Q is said to be congruent to the polyhedron P if there exists a motion t in the space such that $tP=Q$.

By $\mathcal{E}$ we denote the subgroup of $\mathcal{D}$ generated by all elements $P-P_1-\ldots-P_n$, where P is composed of $P_1,\ldots,P_n$, and all elements $P-Q$, where Q is congruent to P. If X and Y are elements of $\mathcal{D}$, we say that X is *equivalent* to Y if $X-Y$ belongs to $\mathcal{E}$. Thus the equivalence classes are the elements of the factor group $\mathcal{D}/\mathcal{E}$.

A polyhedron Q is said to be symmetric to the polyhedron P if there exists a symmetry t in the space (i.e. a transformation composed of a

Received December 27, 1966, and, in revised form, December 2, 1968.

motion and a reflexion in a plane) such that $tP = Q$. A classical result states:

If P and Q are symmetric polyhedra, then P and Q are equivalent.

Let us first consider the case where P is a tetrahedron $A_1A_2A_3A_4$. Let O denote the centre of the inscribed sphere and O_i its projection on the face $A_jA_kA_l$ (where i, j, k, l are distinct). Then P is composed of the six polyhedra $OO_iO_jA_kA_l$ (where again i, j, k, l are distinct). Since these polyhedra are self-symmetric, they are congruent to the corresponding polyhedra into which Q can be decomposed. Hence P and Q are equivalent.

In case of an arbitrary polyhedron P the theorem follows by decomposing P into tetrahedra.

3.

By $\mathrm{Vol}\colon \mathcal{D} \to \mathsf{R}$ we denote the homomorphism of $\mathcal{D}$ into the additive group of real numbers R, whose value for a polyhedron P is its volume. Clearly, Vol is surjective, and it vanishes on $\mathcal{E}$. Hence Vol is constant on each equivalence class, and therefore defines a surjective homomorphism $\mathrm{vol}\colon \mathcal{D}/\mathcal{E} \to \mathsf{R}$ whose value for each equivalence class is the value of Vol for its elements. Thus the condition $\mathrm{Vol}\,X = \mathrm{Vol}\,Y$ is necessary for the equivalence of X and Y. A classical result states that for prisms this condition is also sufficient:

If P and Q are prisms and $\mathrm{Vol}P = \mathrm{Vol}Q$, then P and Q are equivalent.

Let $R(a,b,c)$ denote any rectangular parallelepipedon with edges a, b, c. In order to prove that prisms with the same volume are equivalent, it

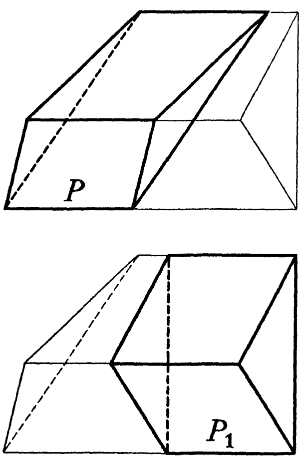

Fig. 1

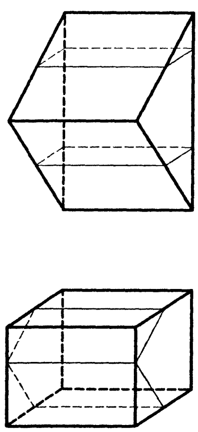

Fig. 2

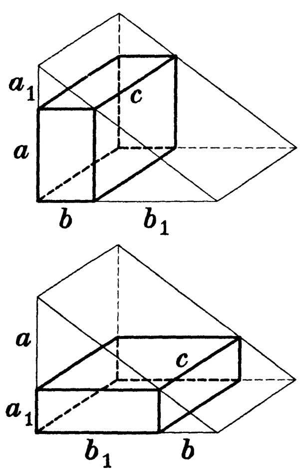

Fig. 3

is sufficient to prove that every prism P is equivalent to a rectangular parallelepipedon $R(1,1,v)$. For then we must have $\operatorname{Vol} P = v$, and consequently, if P and Q are prisms with the same volume, they are equivalent to the same rectangular parallelepipedon. Since every prism can be decomposed into triangular prisms, it is sufficient to prove the statement for triangular prisms.

From fig. 1 it is seen that any skew triangular prism P is equivalent to a right triangular prism P_1, and from fig. 2 it is seen that any right triangular prism is equivalent to a rectangular parallelepipedon. From fig. 3 it is seen that two rectangular parallelepipeda $R(a,b,c)$ and $R(a_1,b_1,c)$ with $ab = a_1 b_1$ are equivalent. Hence any rectangular parallelepipedon $R(a,b,c)$ is equivalent to a rectangular parallelepipedon $R(1,ab,c)$, and hence to a rectangular parallelepipedon $R(1,1,abc)$.

4.

By $\mathcal{J}$ we denote the subgroup of $\mathcal{D}$ generated by $\mathcal{E}$ and the prisms. If X and Y are elements of $\mathcal{D}$, we say that X is equivalent to Y *modulo prisms* if $X - Y$ belongs to $\mathcal{J}$. Thus the equivalence classes modulo prisms are the elements of the factor group $\mathcal{D}/\mathcal{J}$.

By $\mathcal{G}$ we denote the kernel of Vol, i.e., the subgroup consisting of those elements X of $\mathcal{D}$ for which $\operatorname{Vol} X = 0$.

From the theorem proved above, that prisms with the same volume are equivalent, it follows easily that $\mathcal{J} \cap \mathcal{G} = \mathcal{E}$. Thus the group $\mathcal{D}/\mathcal{E}$ is the direct sum of the subgroups $\mathcal{J}/\mathcal{E}$ and $\mathcal{G}/\mathcal{E}$, the restriction of vol: $\mathcal{D}/\mathcal{E} \to \mathsf{R}$ to $\mathcal{J}/\mathcal{E}$ is an isomorphism, and the map of $\mathcal{D}/\mathcal{J}$ into $\mathcal{G}/\mathcal{E}$ which takes every element of $\mathcal{D}/\mathcal{J}$ into its intersection with $\mathcal{G}$ is an isomorphism. In particular we have:

THEOREM 1. *Two elements X and Y of the polyhedron group $\mathcal{D}$ are equivalent if and only if* $\operatorname{Vol} X = \operatorname{Vol} Y$ *and X and Y are equivalent modulo prisms.*

5.

A polyhedron Q is said to be similar to the polyhedron P in the ratio $\lambda \in \mathsf{R}_+$ if there exists a similarity t with ratio λ in the space (i.e. a transformation composed of a motion and a dilation with ratio λ) such that $tP = Q$.

If P is a polyhedron and $\lambda, \mu \in \mathsf{R}_+$, and if Q, R, S are polyhedra similar to P in the ratios λ, μ, $\lambda + \mu$, then S is equivalent modulo prisms to $Q + R$. If P is a tetrahedron T, this follows from fig. 4, which shows that a tetrahedron similar to T in the ratio $\lambda+\mu$ can be decomposed into two tetrahedra similar to T in the ratios λ and μ, and two prisms. In the general case it follows by decomposing P into tetrahedra.

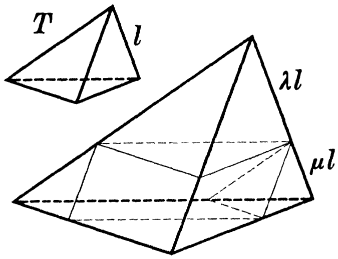

Fig. 4

Using this remark one easily proves that there exists in $\mathfrak{D}/\mathfrak{J}$ a unique multiplication with scalars $\lambda \in \mathsf{R}$ which makes the group $\mathfrak{D}/\mathfrak{J}$ into a vector space over R, such that if x is the class containing a polyhedron P, and $\lambda \in \mathsf{R}_+$, then λx is the class containing the polyhedra similar to P in the ratio λ.

6.

Let $\mathfrak{U}$ denote the free abelian group generated by all pairs (l,α) of real numbers and let $\mathfrak{S}$ denote the subgroup of $\mathfrak{U}$ generated by all elements $(l+m,\alpha)-(l,\alpha)-(m,\alpha)$, $(l,\alpha+\beta)-(l,\alpha)-(l,\beta)$, (l,π). The factor group $\mathfrak{U}/\mathfrak{S}$ is then the tensor product $\mathsf{R}\otimes\mathsf{R}_\pi$ of R and $\mathsf{R}_\pi=\mathsf{R}$ modπ. The equivalence class containing the pair (l,α) is denoted by $l\otimes\alpha$. In $\mathsf{R}\otimes\mathsf{R}_\pi$ there exists a unique multiplication with scalars $\lambda\in\mathsf{R}$ which makes the group $\mathsf{R}\otimes\mathsf{R}_\pi$ into a vector space over R, such that $\lambda(l\otimes\alpha)=(\lambda l)\otimes\alpha$.

The classical argument of M. Dehn [1] can now be presented as follows. By Δ we denote the group homomorphism of $\mathfrak{D}$ into $\mathsf{R}\otimes\mathsf{R}_\pi$ whose value for a polyhedron P is

$$\Delta(P) = \sum_{\nu=1}^{n} l_\nu \otimes \alpha_\nu ,$$

where $l_1,\ldots,l_n$ are the edges and $\alpha_1,\ldots,\alpha_n$ the corresponding (interior) dihedral angles of P. One easily proves that Δ vanishes on $\mathfrak{J}$. Hence Δ is constant on each equivalence class modulo prisms, and therefore defines a group homomorphism $\delta:\mathfrak{D}/\mathfrak{J}\to\mathsf{R}\otimes\mathsf{R}_\pi$ whose value for each equivalence class modulo prisms is the value of Δ for its elements. One immediately sees that δ is actually a vector space homomorphism, i.e. δ is a linear map. Denoting by $\varkappa:\mathfrak{D}\to\mathfrak{D}/\mathfrak{J}$ the canonical map which takes every element of $\mathcal{D}$ into the equivalence class modulo prisms to which it belongs, we have $\Delta = \delta \circ \varkappa$. Thus the condition $\Delta(X) = \Delta(Y)$ is necessary for the equivalence modulo prisms of X and Y.

Recently it was proved by J.-P. Sydler [8] that this condition is also sufficient. Thus we have the Dehn–Sydler theorem:

THEOREM 2. *Two elements X and Y of the polyhedron group $\mathcal{D}$ are equivalent modulo prisms if and only if $\Delta(X) = \Delta(Y)$.*

7.

Let X and Y belong to the subgroup of $\mathcal{D}$ generated by the polyhedra $P_1, \ldots, P_n$ and let $\{\pi, \beta_1, \ldots, \beta_s\}$ be a finite set of real numbers which is linearly independent over Q, such that all dihedral angles ξ of $P_1, \ldots, P_n$ are expressible in the form $\xi = \varrho\pi + \varrho_1\beta_1 + \ldots + \varrho_s\beta_s$ with rational coefficients. Then we find for $\Delta(X)$ and $\Delta(Y)$ expressions of the form $\sum_{\sigma=1}^{s} p_\sigma \otimes \beta_\sigma$ and $\sum_{\sigma=1}^{s} q_\sigma \otimes \beta_\sigma$, in which the numbers p_σ and q_σ are linear combinations with rational coefficients of the edges of $P_1, \ldots, P_n$. Since $1 \otimes \beta_1, \ldots, 1 \otimes \beta_s$ are linearly independent elements of the vector space $\mathsf{R} \otimes \mathsf{R}_\pi$, the condition $\Delta(X) = \Delta(Y)$ is equivalent to the s conditions $p_1 = q_1, \ldots, p_s = q_s$. For other ways of expressing Dehn's condition see M. Dehn [1], O. Nicoletti [6], H. Lebesgue [5], and H. Hadwiger [2].

Classical examples are provided by the regular polyedra. Let T, O, D, I denote a regular tetrahedron, octahedron, dodecahedron, icosahedron with edge 1, and let $\alpha_T, \alpha_O, \alpha_D, \alpha_I$ denote their dihedral angles. We have $\cos\alpha_T = \frac{1}{3}$, $\cos\alpha_O = -\frac{1}{3}$, $\cos 2\alpha_D = -\frac{3}{5}$, $\cos 2\alpha_I = \frac{1}{9}$. From the well-known result, that the only angles α for which both $\cos\alpha$ and α/π are rational are those for which $2\cos\alpha$ is an integer, it follows that $\Delta(T) = 6 \otimes \alpha_T$, $\Delta(O) = 12 \otimes \alpha_O$, $\Delta(D) = 30 \otimes \alpha_D$, $\Delta(I) = 30 \otimes \alpha_I$ are $\neq 0$, so that Dehn's result shows that T, O, D, I are not equivalent to prisms. Since $\alpha_T + \alpha_O = \pi$, we have $\Delta(2T + O) = 0$. Hence, by Sydler's result, $2T + O$ is equivalent to a prism. This is also evident from the fact that by placing regular tetrahedra on two opposite faces of a regular octahedron one obtains a prism. It was noticed by Lebesgue [5] that the set $\{\pi, \alpha_T, \alpha_D, \alpha_I\}$ is linearly independent over Q. Consequently, $\Delta(T), \Delta(D), \Delta(I)$ are linearly independent elements of $\mathsf{R} \otimes \mathsf{R}_\pi$. By Dehn's result, the elements of $\mathcal{D}/\mathcal{J}$ containing T, D, I are therefore linearly independent. The dimension of the vector space $\mathcal{D}/\mathcal{J}$ is therefore $\geqq 3$.

8.

Sydler's result means that the map δ is injective, and is therefore equivalent to the following theorem:

THEOREM 3. *For every linear map* $\tau : \mathcal{D}/\mathcal{J} \to \mathcal{V}$ *of* $\mathcal{D}/\mathcal{J}$ *into an arbitrary vector space* $\mathcal{V}$ *over* R *there exists a linear map* $\Phi : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathcal{V}$ *such that* $\tau = \Phi \circ \delta$, *or, equivalently, such that* $\tau \circ \varkappa = \Phi \circ \Delta$.

The situation is illustrated by the diagram

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\varkappa"'] \arrow[dr, "\Delta"] & \\ \mathcal{D}/\mathcal{J} \arrow[rr, "\delta"] \arrow[dr, "\tau"'] & & \mathsf{R} \otimes \mathsf{R}_\pi \arrow[dl, "\Phi"] \\ & \mathcal{V} & \end{tikzcd}$$

Actually, Sydler's result is equivalent to the existence of Φ in the special case where $\mathcal{V} = \mathcal{D}/\mathcal{J}$ and τ is the identical map; however, for the further discussion it will be convenient to work with an arbitrary $\mathcal{V}$.

9.

In the proof of Theorem 3 we shall need three geometrical lemmas; the remainder of the proof will be algebraical. The first lemma is the fundamental lemma of Sydler [7]; the two other lemmas were also used by Sydler.

Let a, b be two numbers in the interval $]0,1[$, and let α, β be the angles in the interval $]0, \frac{1}{2}\pi[$ determined by $\sin^2\alpha = a$, $\sin^2\beta = b$. We denote by $\alpha * \beta$ the angle in the interval $]0, \frac{1}{2}\pi[$ determined by $\sin^2(\alpha * \beta) = ab$. The composition $*$ is clearly commutative and associative. We denote by $T(a,b)$ any tetrahedron $ABCD$ in which the edges $AB = \cot\alpha$, $BC = \cot\alpha \cot\beta$, $CD = \cot\beta$ are orthogonal. Thus two tetrahedra $T(a,b)$ are either congruent or symmetric and are therefore equivalent. The dihedral angles corresponding to AB and CD are α and β. A simple calculation shows that the dihedral angle corresponding to AD is $\frac{1}{2}\pi - \alpha * \beta$, and that $AD = \cot(\alpha * \beta)$. The three remaining dihedral angles are right angles. Hence

$$\Delta(T(a,b)) = \cot\alpha \otimes \alpha + \cot\beta \otimes \beta - \cot(\alpha * \beta) \otimes (\alpha * \beta) .$$

For the volume of $T(a,b)$ we find the expression

$$\operatorname{Vol} T(a,b) = v(a) + v(b) - v(ab) ,$$

where v is the function defined by $v(u) = (u-1)/6u$.

Now, let a, b, c be numbers in $]0,1[$, and let α, β, γ be the angles in $]0, \frac{1}{2}\pi[$ determined by $\sin^2\alpha = a$, $\sin^2\beta = b$, $\sin^2\gamma = c$, and put

$$X = T(a,b) + T(ab,c) \quad \text{and} \quad Y = T(a,c) + T(ac,b) .$$

We then find

$$\operatorname{Vol} X = v(a)+v(b)+v(c)-v(abc) = \operatorname{Vol} Y,$$

$$\Delta(X) = \cot\alpha \otimes \alpha + \cot\beta \otimes \beta + \cot\gamma \otimes \gamma - \cot(\alpha*\beta*\gamma) \otimes (\alpha*\beta*\gamma) = \Delta(Y).$$

Thus X and Y satisfy the conditions of Theorems 1 and 2. The fundamental lemma states that they are equivalent:

LEMMA 1. *For arbitrary* $a,b,c \in\,]0,1[$ *the elements*

$$T(a,b)+T(ab,c) \quad \textit{and} \quad T(a,c)+T(ac,b)$$

of $\mathfrak{P}$ *are equivalent.*

Since the volumes are equal, it is enough to prove that the elements are equivalent modulo prisms.

From fig. 5 it is seen that a polyhedron $OPQRS$, where $OP=PQ$, $QS=2PR$, and PR and QS are orthogonal to the face OPQ, is equivalent to a prism (the point U is the mid-point of OS).

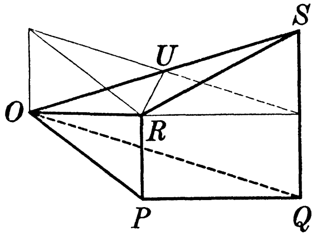

Fig. 5

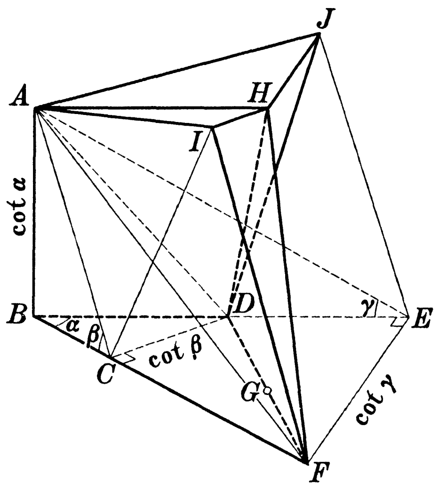

Fig. 6

On fig. 6 the tetrahedron $ABCD$ is $T(a,b)$ and the tetrahedron $ABEF$ is $T(a,c)$. In the quadrangle $CDEF$ the angles at C and E are right angles. Hence it is inscribable. The centre of its circumscribed circle is the mid-point G of the diagonal DF. The points $ACDEF$ therefore lie on a sphere, whose centre H lies on the normal to the plane $BCDEF$ in G. The points I and J are the projections of H on the planes $ABCF$ and $ABDE$. Since G and H have the same distances from these planes, we have $HI = \frac{1}{2}\cot\beta$, $HJ = \frac{1}{2}\cot\gamma$. The point I is the centre of the circumscribed circle for the triangle ACF, and the point J is the centre of the circumscribed circle for the triangle ADE. Hence the angle AIF is 2β and the angle AJD is 2γ. We consider the polyhedron $ABDFHIJ$.

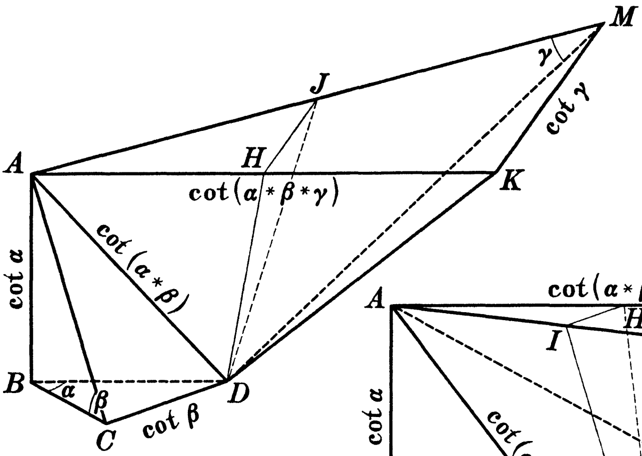

Fig. 7

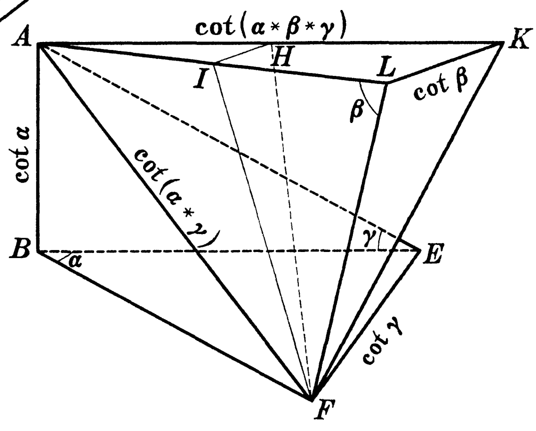

Fig. 8

On fig. 7 the points $ABCDHJ$ are as in fig. 6, so that the tetrahedron $ABCD$ is $T(a,b)$. We have $AD = \cot(\alpha * \beta)$. The points H and J are the mid-points of AK and AM. We see that AD, DM, MK are orthogonal, that $MK = \cot\gamma$, and that the angle AMD is γ. Thus the tetrahedron $ADMK$ is $T(ab,c)$.

Similarly, on fig. 8 the points $ABEFHI$ are as in fig. 6, so that the tetrahedron $ABEF$ is $T(a,c)$. We have $AF = \cot(\alpha * \gamma)$. The points H and I are the mid-points of AK and AL. We see that AF, FL, LK are orthogonal, that $LK = \cot\beta$, and that the angle ALF is β. Thus the tetrahedron $AFLK$ is $T(ac,b)$.

The edge AK, which appears in $T(ab,c)$ and $T(ac,b)$, is $\cot(\alpha * \beta * \gamma)$.

Since the polyhedra $AICHD$, $FICHD$ on fig. 6 and the polyhedron $DJMHK$ on fig. 7 are the same type as the polyhedron $OPQRS$ on fig. 5, we see that the polyhedron $ABDFHIJ$ on fig. 6 is equivalent modulo prisms to $T(a,b) + T(ab,c)$.

Similarly, since the polyhedra $AJEHF$, $DJEHF$ on fig. 6 and the polyhedron $FILHK$ on fig. 8 are the same type as the polyhedron $OPQRS$ on fig. 5, we see that the polyhedron $ABDFHIJ$ on fig. 6 is equivalent modulo prisms to $T(a,c) + T(ac,b)$.

Hence $T(a,b) + T(ab,c)$ and $T(a,c) + T(ac,b)$ are equivalent modulo prisms.

By $\lambda T(a,b)$ we denote any tetrahedron similar to a tetrahedron $T(a,b)$ in the ratio $\lambda \in \mathsf{R}_+$.

LEMMA 2. *For arbitrary $a,b,c \in \mathsf{R}_+$ the elements*

$$aT\left(\frac{a+b}{a+b+c}, \frac{a}{a+b}\right) + bT\left(\frac{a+b}{a+b+c}, \frac{b}{a+b}\right)$$

and

$$aT\left(\frac{a+c}{a+b+c}, \frac{a}{a+c}\right) + cT\left(\frac{a+c}{a+b+c}, \frac{c}{a+c}\right)$$

of $\mathfrak{P}$ are equivalent.

Let $OABC$ be a tetrahedron in which the edges $OA = (bc)^{\frac{1}{2}}$, $OB = (ca)^{\frac{1}{2}}$, $OC = (ab)^{\frac{1}{2}}$ are orthogonal. By the plane through OC orthogonal to AB the tetrahedron is decomposed into two tetrahedra. A simple calculation shows that these are the tetrahedra appearing in the first sum. Similarly, by the plane through OB orthogonal to AC the tetrahedron is decomposed into two tetrahedra, which are the tetrahedra appearing in the second sum.

LEMMA 3. *For three angles $\xi, \eta, \zeta \in \,]0, \tfrac{1}{2}\pi[$ with sum π there exists a rectangular parallelepipedon R with diagonals AB, CD, EF, GH such that the dihedral angles at the edge AB of the six pairwise symmetric tetrahedra of the type $\lambda T(a,b)$ into which R is decomposed by the planes $ABCD$, $ABEF$, $ABGH$ are ξ, η, ζ.*

The hexagon on fig. 9 with pairwise parallel sides, whose diagonals intersect each other at angles ξ, η, ζ, is, by a well-known theorem of axonometry, the orthogonal projection of a rectangular parallelepipedon R in the direction of a diagonal AB, whose image is the centre of the

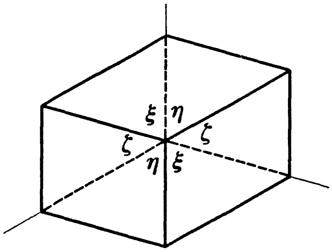

Fig. 9

hexagon; the images of the three other diagonals CD, EF, GH are the diagonals of the hexagon. The images of the six tetrahedra into which R is decomposed by the planes $ABCD$, $ABEF$, $ABGH$ are therefore the six triangles into which the hexagon is decomposed by its diagonals, which shows that the dihedral angles at the edge AB are ξ, η, ζ.

10.

We now turn to the proof of Theorem 3.

Let $\mathfrak{V}$ be an arbitrary vector space over R, and let $\tau : \mathfrak{D}/\mathfrak{J} \to \mathfrak{V}$ be a linear map. We must prove the existence of a linear map $\Phi \colon \mathsf{R} \otimes \mathsf{R}_\pi \to \mathfrak{V}$ such that $\tau \circ \varkappa = \Phi \circ \Delta$.

If Φ is a linear map of $\mathsf{R} \otimes \mathsf{R}_\pi$ into $\mathfrak{V}$, the function $\varphi : \mathsf{R} \to \mathfrak{V}$ defined by $\varphi(\xi) = \Phi(1 \otimes \xi)$ satisfies the conditions $\varphi(\xi + \eta) = \varphi(\xi) + \varphi(\eta)$, $\varphi(\pi) = 0$. Conversely, if $\varphi : \mathsf{R} \to \mathfrak{V}$ is a function satisfying these conditions, there exists a unique linear map $\Phi : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathfrak{V}$ such that $\Phi(1 \otimes \xi) = \varphi(\xi)$ for all $\xi \in \mathsf{R}$. Thus, in order to prove Theorem 3 we must prove:

There exists a function $\varphi : \mathsf{R} \to \mathfrak{V}$ satisfying the conditions $\varphi(\xi + \eta) = \varphi(\xi) + \varphi(\eta)$, $\varphi(\pi) = 0$, such that for every polyhedron P we have

$$\tau \circ \varkappa(P) = \sum_{\nu=1}^{n} l_\nu \varphi(\alpha_\nu)\,, \tag{1}$$

where $l_1, \ldots, l_n$ are the edges and $\alpha_1, \ldots, \alpha_n$ the corresponding dihedral angles of P.

(i) Let $F \colon]0,1[^2 \to \mathfrak{V}$ be the function defined by $F(a,b) = \tau \circ \varkappa(T(a,b))$. The function F satisfies the equations

$$\begin{aligned} F(a,b) &= F(b,a)\ , \\ F(a,b)+F(ab,c) &= F(a,c)+F(ac,b)\ . \end{aligned}$$

The first is obvious and the other follows from Lemma 1. The functions $F\colon]0,1[^2 \to \mathfrak{V}$ which satisfy these equations, are by [4] precisely the functions which can be represented by means of a function $f\colon]0,1[\to \mathfrak{V}$ through the formula

$$F(a,b) = f(a)+f(b)-f(ab)\ . \tag{2}$$

In the sequel, f denotes such a function.

(ii) From Lemma 2 it follows that for arbitrary $a,b,c \in \mathsf{R}_+$ we have

$$\begin{aligned} &aF\left(\frac{a+b}{a+b+c}, \frac{a}{a+b}\right) + bF\left(\frac{a+b}{a+b+c}, \frac{b}{a+b}\right) \\ &\qquad = aF\left(\frac{a+c}{a+b+c}, \frac{a}{a+c}\right) + cF\left(\frac{a+c}{a+b+c}, \frac{c}{a+c}\right). \end{aligned}$$

Using the formula (2) one obtains the relation

$$\begin{aligned} &\left[af\left(\frac{a}{a+b}\right)+bf\left(\frac{b}{a+b}\right)\right] + \left[(a+b)f\left(\frac{a+b}{a+b+c}\right)+cf\left(\frac{c}{a+b+c}\right)\right] \\ &\qquad = \left[af\left(\frac{a}{a+c}\right)+cf\left(\frac{c}{a+c}\right)\right] + \left[(a+c)f\left(\frac{a+c}{a+b+c}\right)+bf\left(\frac{b}{a+b+c}\right)\right]. \end{aligned}$$

Let $G\colon \mathsf{R}_+{}^2 \to \mathfrak{V}$ be the function defined by

$$G(a,b) = af\left(\frac{a}{a+b}\right) + bf\left(\frac{b}{a+b}\right).$$

The function G satisfies the equations

$$\begin{aligned} G(a,b) &= G(b,a)\ , \\ G(a,b)+G(a+b,c) &= G(a,c)+G(a+c,b)\ , \\ G(\lambda a,\lambda b) &= \lambda G(a,b)\ . \end{aligned}$$

The first and last are obvious and the second is precisely the relation above. The functions $G\colon \mathsf{R}_+{}^2 \to \mathfrak{V}$ which satisfy these equations, are by [4] precisely the functions which can be represented through the formula

$$G(a,b) = ag(a)+bg(b)-(a+b)g(a+b)$$

by means of a function $g\colon \mathsf{R}_+ \to \mathfrak{V}$ satisfying the equation

$$g(a)+g(b)-g(ab) = 0\ .$$

In the sequel, g denotes such a function. From the last equation we see that $g(1)=0$. For numbers $a,b \in]0,1[$, such that $a+b=1$, we therefore have

$$af(a)+bf(b) = ag(a)+bg(b)\,.$$

Introducing the function $h\colon]0,1[\to \mathfrak{V}$ defined by $h(a)=f(a)-g(a)$ we obtain for F the representation

$$F(a,b) = h(a)+h(b)-h(ab)\,, \tag{3}$$

where h satisfies the equation

$$ah(a)+bh(b) = 0$$

when $a+b=1$.

We now introduce the function $\varphi\colon \mathsf{R} \to \mathfrak{V}$ defined by

$$\varphi(\xi) = \begin{cases} \tan\xi\, h(\sin^2\xi) & \text{when } \xi \not\equiv 0 \bmod \tfrac{1}{2}\pi\,, \\ 0 & \text{when } \xi \equiv 0 \bmod \tfrac{1}{2}\pi\,. \end{cases} \tag{4}$$

We then have $\varphi(\xi)+\varphi(\eta)=0$ when $\xi+\eta\equiv 0 \bmod \frac{1}{2}\pi$, and the formula (3) takes the form

$$\tau\circ\varkappa\big(T(a,b)\big) = \cot\alpha\,\varphi(\alpha)+\cot\beta\,\varphi(\beta)+\cot(\alpha*\beta)\,\varphi(\tfrac{1}{2}\pi-\alpha*\beta)\,.$$

The formula (1) is therefore valid for every tetrahedron $T(a,b)$ and hence for every tetrahedron $\lambda T(a,b)$.

(iii) We know already that $\varphi(\pi)=0$, and that $\varphi(\xi+\eta)=\varphi(\xi)+\varphi(\eta)$ when $\xi+\eta\equiv 0 \bmod \frac{1}{2}\pi$.

For three angles $\xi,\eta,\zeta \in]0,\frac{1}{2}\pi[$ with sum π we now apply Lemma 3. Using formula (1) for the six tetrahedra, and the fact that the value of $\varkappa$ for the rectangular parallelepipedon is 0, we find the relation $\varphi(\xi)+\varphi(\eta)+\varphi(\zeta)=0$. (The contributions from all other angles cancel out.) Thus we have $\varphi(\xi+\eta)=\varphi(\xi)+\varphi(\eta)$, when $\xi,\eta \in]0,\frac{1}{2}\pi[$ and $\xi+\eta \in]\frac{1}{2}\pi,\pi[$, and hence also when $\xi,\eta \in [0,\frac{1}{2}\pi]$ and $\xi+\eta \in [\frac{1}{2}\pi,\pi]$.

If $\xi,\eta \in [0,\frac{1}{2}\pi]$ and $\xi+\eta \in [0,\frac{1}{2}\pi]$, we have $\frac{1}{2}\pi-\xi, \frac{1}{2}\pi-\eta \in [0,\frac{1}{2}\pi]$ and $\pi-(\xi+\eta) \in [\frac{1}{2}\pi,\pi]$. Hence

$$\varphi(\xi+\eta) = -\varphi\big(\pi-(\xi+\eta)\big) = -\varphi(\tfrac{1}{2}\pi-\xi)-\varphi(\tfrac{1}{2}\pi-\eta) = \varphi(\xi)+\varphi(\eta)\,.$$

For arbitrary $\xi,\eta \in \mathsf{R}$ we have $\xi=m\frac{1}{2}\pi-\xi_0$, $\eta=n\frac{1}{2}\pi-\eta_0$, where $\xi_0,\eta_0 \in [0,\frac{1}{2}\pi]$ and $m,n \in \mathsf{Z}$. Hence

$$\varphi(\xi+\eta) = -\varphi(\xi_0+\eta_0) = -\varphi(\xi_0)-\varphi(\eta_0) = \varphi(\xi)+\varphi(\eta)\,.$$

Thus φ satisfies the conditions $\varphi(\xi+\eta)=\varphi(\xi)+\varphi(\eta)$, $\varphi(\pi)=0$. Since the formula (1) holds for every tetrahedron $\lambda T(a,b)$, and since every polyhedron can be decomposed into tetrahedra $\lambda T(a,b)$, we conclude that the formula (1) holds for all polyhedra.

11.

Since the vector space $\mathfrak{D}/\mathfrak{J}$ is generated by the elements $\varkappa(T(a,b))$, its dimension is $\leqq \aleph$.

Let $a_1,\ldots,a_n$ be numbers in the interval $]0,1[$ which are algebraically independent over Q, and let $\alpha_1,\ldots,\alpha_n$ and $\delta_1,\ldots,\delta_n$ be the angles in the interval $]0,\frac{1}{2}\pi[$ determined by $\sin^2\alpha_\nu = a_\nu$, $\sin^2\delta_\nu = a_\nu{}^2$ [so that $\delta_\nu = \alpha_\nu * \alpha_\nu$]. Then the set $\{\pi,\alpha_1,\ldots,\alpha_n,\delta_1,\ldots,\delta_n\}$ is linearly independent over Q. Indeed, for arbitrary $p_1,q_1,\ldots,p_n,q_n \in \mathsf{Z}$ we have

$$\exp i \sum_{\nu=1}^{n} (p_\nu\alpha_\nu + q_\nu\delta_\nu) = \prod_{\nu=1}^{n} \left((1-a_\nu)^{\frac{1}{2}} + i a_\nu{}^{\frac{1}{2}}\right)^{p_\nu}\left((1-a_\nu{}^2)^{\frac{1}{2}} + i a_\nu\right)^{q_\nu},$$

and one easily sees that the right hand side is 1 only when all the numbers p_ν, q_ν are 0. Since

$$\Delta(T(a_\nu,a_\nu)) = 2\cot\alpha_\nu \otimes \alpha_\nu - \cot\delta_\nu \otimes \delta_\nu ,$$

we see that $\Delta(T(a_1,a_1)),\ldots,\Delta(T(a_n,a_n))$ are linearly independent elements of $\mathsf{R}\otimes\mathsf{R}_\pi$. Hence, by Dehn's result, if $\Omega \subset]0,1[$ is a transcendence basis of R over Q, the elements $\varkappa(T(a,a))$, $a\in\Omega$, are linearly independent elements of $\mathfrak{D}/\mathfrak{J}$. Since a transcendence basis of R over Q has the cardinal number $\aleph$, we conclude that the dimension of the vector space $\mathfrak{D}/\mathfrak{J}$ is $\geqq \aleph$.

Thus we have proved:

THEOREM 4. *The dimension of the vector space* $\mathfrak{D}/\mathfrak{J}$ *is* $\aleph$.

12.

The image $\Delta(\mathfrak{D}) = \delta(\mathfrak{D}/\mathfrak{J})$ is a linear subspace of $\mathsf{R}\otimes\mathsf{R}_\pi$. For every vector space $\mathfrak{V}$ over R, the linear maps $\Phi : \mathsf{R}\otimes\mathsf{R}_\pi \to \mathfrak{V}$, which take $\Delta(\mathfrak{D})$ into 0, are the maps satisfying the condition of Theorem 3 when $\tau = 0$. From the proof of Theorem 3 it is clear that these maps Φ are the maps for which $\varphi(\xi) = \Phi(1\otimes\xi)$ has the form (4), where $h :]0,1[\to \mathfrak{V}$ is a function such that

$$h(a) + h(b) - h(ab) = 0$$

for all $a,b \in]0,1[$, and

$$ah(a) + bh(b) = 0$$

when $a+b=1$.

A function d: $\mathsf{R} \to \mathfrak{V}$ is called a *derivation* if it satisfies the equations

$$\text{(5a)} \qquad d(a+b) = d(a)+d(b)\,,$$

$$\text{(5b)} \qquad d(ab) = b\,d(a)+a\,d(b)\,.$$

For a derivation $d\colon \mathsf{R} \to \mathfrak{V}$ we have by (5b) the relation $d(1)=0$, and (5a) therefore shows that d has the period 1. The function $h\colon]0,1[\to \mathfrak{V}$ defined by $h(a)=d(a)/a$ evidently satisfies the conditions mentioned above.

Conversely, if $h\colon]0,1[\to \mathfrak{V}$ satisfies these conditions, the function $d\colon \mathsf{R} \to \mathfrak{V}$ with period 1 for which $d(a)=ah(a)$ when $a \in]0,1[$, and $d(1)=0$, will be a derivation.

To see this, we first observe that (5a) holds for $a,b \in [0,1]$ and $a+b=1$, and that (5b) holds for $a,b \in [0,1]$. Suppose now that $a,b \in [0,1]$ and $a+b \in]0,1]$. We then find

$$d(a) = d\left((a+b)\frac{a}{a+b}\right) = \frac{a}{a+b}d(a+b) + (a+b)d\left(\frac{a}{a+b}\right),$$

$$d(b) = d\left((a+b)\frac{b}{a+b}\right) = \frac{b}{a+b}d(a+b) + (a+b)d\left(\frac{b}{a+b}\right).$$

Adding these relations we see that (5a) holds for $a,b \in [0,1]$, $a+b \in]0,1]$. Hence (5a) holds for $a,b,a+b \in [0,1]$. If $a,b \in [0,1]$ and $a+b \in [1,2]$, we have $1-a$, $1-b \in [0,1]$ and $(1-a)+(1-b) \in [0,1]$. Hence

$$\begin{aligned} d(a)+d(b) &= -d(1-a)-d(1-b) = -d(2-a-b) \\ &= d(a+b-1) = d(a+b)\,. \end{aligned}$$

Thus (5a) holds for $a,b \in [0,1]$.

For arbitrary $a,b \in \mathsf{R}$ we have $a=a_0+m$, $b=b_0+n$, where $a_0,b_0 \in [0,1]$ and $m,n \in \mathsf{Z}$. Hence

$$d(a+b) = d(a_0+b_0) = d(a_0)+d(b_0) = d(a)+d(b)\,.$$

Having thus established (5a), we find that

$$\begin{aligned} d(ab) &= d(a_0b_0+na_0+mb_0) = d(a_0b_0)+d(na_0)+d(mb_0) \\ &= b_0d(a_0)+a_0d(b_0)+nd(a_0)+md(b_0) = bd(a)+ad(b)\,, \end{aligned}$$

so that (5b) is established.

When $h\colon]0,1[\to \mathfrak{V}$ is defined by $h(a)=d(a)/a$, where $d\colon \mathsf{R} \to \mathfrak{V}$ is a derivation, the expression $\tan\xi\, h(\sin^2\xi)$ in (4) takes the form $2d(\sin\xi)/\cos\xi$. Thus, defining $0/0$ as 0, we have the following theorem:

Theorem 5. *For every vector space $\mathfrak{V}$ over R, the linear maps $\Phi\colon \mathsf{R}\otimes \mathsf{R}_\pi \to \mathfrak{V}$*

which take the subspace $\Delta(\mathfrak{P}) = \delta(\mathfrak{P}/\mathfrak{J})$ *into* 0 *are determined by means of the derivations* $d : \mathsf{R} \to \mathfrak{V}$ *through the formula*

$$\Phi(1 \otimes \xi) = d(\sin\xi)/\cos\xi .$$

13.

A derivation $d : \mathsf{R} \to \mathfrak{V}$ vanishes on the set of real algebraic numbers. If Ω is a transcendence basis of R over Q, every function from Ω into $\mathfrak{V}$ is the restriction of a unique derivation $d : \mathsf{R} \to \mathfrak{V}$. Proofs of these known results are contained in [4].

Considering now the special case of Theorem 3 where $\mathfrak{V} = \mathfrak{P}/\mathfrak{J}$ and τ is the identical map, we see that all linear maps $\Phi : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathfrak{P}/\mathfrak{J}$, for which $\varkappa = \Phi \circ \Delta$, coincide on the set of elements $1 \otimes \xi$ for which $\sin\xi$ is algebraic, and that if $\Omega \subset]0,1[$ is a transcendence basis of R over Q, and Γ is the set of those angles $\gamma \in]0, \frac{1}{2}\pi[$ for which $\sin\gamma \in \Omega$, there exists just one linear map $\Phi : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathfrak{P}/\mathfrak{J}$ for which $\varkappa = \Phi \circ \Delta$ and $\Phi(1 \otimes \xi) = 0$ for all $\xi \in \Gamma$. The set $\{\pi, \Gamma\}$ is linearly independent over Q. Let $\{\pi, \Xi\}$ be an extension of $\{\pi, \Gamma\}$ to a basis of R over Q. Then the set $\{1 \otimes \xi \mid \xi \in \Xi\}$ is a basis of $\mathsf{R} \otimes \mathsf{R}_\pi$, and it is easily seen that the set $\{\Phi(1 \otimes \xi) \mid \xi \in \Xi \setminus \Gamma\}$ is a basis of $\mathfrak{P}/\mathfrak{J}$.

14.

In Theorem 5 we take $\mathfrak{V} = \mathsf{R}$. The subspace $\Delta(\mathfrak{P})$ of $\mathsf{R} \otimes \mathsf{R}_\pi$ is the intersection of the kernels of the linear maps $\Phi : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathsf{R}$ which take $\Delta(\mathfrak{P})$ into 0. Thus we have the following theorem:

THEOREM 6. *The subspace* $\Delta(\mathfrak{P}) = \delta(\mathfrak{P}/\mathfrak{J})$ *of* $\mathsf{R} \otimes \mathsf{R}_\pi$ *consists of those elements* $\sum_{\nu=1}^{n} k_\nu \otimes \xi_\nu$ *for which*

$$\sum_{\nu=1}^{n} k_\nu d(\sin\xi_\nu)/\cos\xi_\nu = 0$$

for all derivations $d : \mathsf{R} \to \mathsf{R}$.

In particular, an element of the form $k \otimes \xi$ with $k \neq 0$ will belong to $\Delta(\mathfrak{P})$ if and only if $\sin\xi$ is algebraic.

For arbitrary angles $\xi_1, \ldots, \xi_n$, let $t_1, \ldots, t_s$ be real numbers which are algebraically independent over Q, such that the numbers $\sin\xi_\nu$ are algebraic over the field $\mathsf{Q}(t_1, \ldots, t_s)$. Assuming that the characteristic polynomials of the numbers $\sin\xi_\nu$ over the field $\mathsf{Q}(t_1, \ldots, t_s)$ are known, we find for the numbers $d(\sin\xi_\nu)/\cos\xi_\nu$, where $d : \mathsf{R} \to \mathsf{R}$ is a derivation, expressions of the form $\sum_{\sigma=1}^{s} c_{\nu\sigma} d(t_\sigma)$, in which the numbers $c_{\nu\sigma}$ are independent of d. Since the values $d(t_\sigma)$ can be chosen arbitrarily, we find that the sets $k_1,\ldots,k_n$ for which $\sum_{\nu=1}^{n} k_\nu \otimes \xi_\nu$ belongs to $\Delta(\mathfrak{D})$, are determined as the solutions of the s linear equations $\sum_{\nu=1}^{n} k_\nu c_{\nu\sigma} = 0$, $\sigma = 1,\ldots,s$.

Let $\mathcal{D}$ denote the set of all derivations $d : \mathsf{R} \to \mathsf{R}$, and consider the vector space $\mathsf{R}^{\mathcal{D}}$ of all real functions on $\mathcal{D}$. Let $\varepsilon : \mathsf{R} \otimes \mathsf{R}_\pi \to \mathsf{R}^{\mathcal{D}}$ denote the linear map which takes $1 \otimes \xi$ into the function $d \mapsto d(\sin\xi)/\cos\xi$. Then Theorem 6 states that $\Delta(\mathfrak{D})$ is the kernel of ε.

Theorems 2 and 6 are illustrated by the diagram

$$\begin{tikzcd} & & \mathfrak{D} \arrow[dl, "\varkappa"'] \arrow[dr, "\Delta"] & & \\ 0 \arrow[r] & \mathfrak{D}/\mathfrak{J} \arrow[rr, "\delta"] & & \mathsf{R} \otimes \mathsf{R}_\pi \arrow[r, "\varepsilon"] & \mathsf{R}^{\mathcal{D}} \end{tikzcd}$$

The polyhedron group $\mathfrak{D}$ is mapped canonically by $\varkappa$ onto the factor group $\mathfrak{D}/\mathfrak{J}$ and by Δ into the group $\mathsf{R} \otimes \mathsf{R}_\pi$. Dehn's result states the existence of the map $\delta : \mathfrak{D}/\mathfrak{J} \to \mathsf{R} \otimes \mathsf{R}_\pi$ such that $\Delta = \delta \circ \varkappa$. This map is linear. Sydler's result means that the first part of the horizontal sequence is exact, and our characterization of the image $\Delta(\mathfrak{D}) = \delta(\mathfrak{D}/\mathfrak{J})$ means that the second part of the horizontal sequence is exact.

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