MATH. SCAND. 22 (1968), 266–268

A DECOMPOSITION THEOREM FOR C^* ALGEBRAS

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As pointed out to us by T. Bai Andersen the proof of [1, theorem 1.4] is incomplete. We give here a correct proof (corollary 6) as a consequence of a decomposition theorem, which may have independent interest. The observation, as well as the proof, that the decomposition theorem implies the result of E. Størmer [2] is due to F. Combes.

Let A be a C^* algebra represented as operators on some Hilbert space. For any operator x let $[x]$ denote the range projection of x.

Let $x_1, x_2, \ldots, x_k$ be a finite set of positive operators from A, and set $x = \sum x_i$.

We denote by B^+ the smallest closed order ideal in A^+ containing x. Then [1, Lemma 1.1] B^+ is the uniform closure of the order ideal

$$\{y \in A^+ \mid \exists \alpha \in R^+ \colon\ y \leqq \alpha x\},$$

also B^+ is the positive part of an order-related C^* subalgebra B, where $B = L^* \cap L$, L being the closed left ideal generated by x. Clearly B is also the order-related C^* subalgebra generated by $x_1, x_2, \ldots, x_k$.

Theorem 1. *For any $y \in B^+$ there exist $y_i \in B$, $y_i = [x_i] y_i$, $i = 1, 2, \ldots, k$, such that $y = \sum y_i^* y_i$.*

Proof. Define $u_n = (n^{-1} + x)^{-1} x$. Then $0 \leqq u_n \leqq u_m \leqq 1$ for $n \leqq m$. For any $z \in B$ there is a sequence $\{\alpha_m\}$ such that $z^* z \leqq \alpha_m x + m^{-1}$. Hence

$$\|z(1-u_n)\|^2 = \|(1-u_n) z^* z (1-u_n)\| \leqq m^{-1} + \alpha_m \sup_{\alpha} \frac{n^{-2}\alpha}{(n^{-1}+\alpha)^2} \leqq m^{-1} + \alpha_m (4n)^{-1}.$$

It follows that $\{u_n\}$ is an approximative unit for B. Now take any $y \in B^+$. Then

$$y = \lim y^{\frac12} u_n^2 y^{\frac12} = \lim y^{\frac12} (n^{-1}+x)^{-1} x^{\frac12} (\textstyle\sum x_i) x^{\frac12} (n^{-1}+x)^{-1} y^{\frac12}$$
$$= \lim \textstyle\sum \big(y^{\frac12} (n^{-1}+x)^{-1} x^{\frac12} x_i^{\frac12}\big)\big(x_i^{\frac12} x^{\frac12} (n^{-1}+x)^{-1} y^{\frac12}\big).$$

Received October 6, 1967; revised May 10, 1968.

But $y_{i,n} = x_i^{\frac{1}{2}} x^{\frac{1}{2}} (n^{-1} + x)^{-1} y^{\frac{1}{2}}$ converges to an element $y_i \in B$ since

$$\begin{aligned}\|y_{i,n} - y_{i,m}\|^2 &= \|x_i^{\frac{1}{2}} x^{\frac{1}{2}} \big((n^{-1}+x)^{-1} - (m^{-1}+x)^{-1}\big) y^{\frac{1}{2}}\|^2 \\ &\leqq \|y^{\frac{1}{2}} \big((n^{-1}+x)^{-1} - (m^{-1}+x)^{-1}\big)^2 x^2 y^{\frac{1}{2}}\| = \|(u_n - u_m) y^{\frac{1}{2}}\|^2 .\end{aligned}$$

Hence $y = \sum y_i^* y_i$ with $y_i = [x_i] y_i$.

Corollary 2 (Riesz decomposition property). *If $0 \leqq y \leqq \sum x_i$, then there exist $y_i \in B$ such that $y = \sum y_i^* y_i$, $y_i y_i^* \leqq x_i$.*

Proof. With y_i as in the proof of theorem 1, we observe that

$$y_{i,n} y_{i,n}^* = x_i^{\frac{1}{2}} x^{\frac{1}{2}} (n^{-1}+x)^{-1} y (n^{-1}+x)^{-1} x^{\frac{1}{2}} x_i^{\frac{1}{2}} \leqq x_i^{\frac{1}{2}} u_n{}^2 x_i^{\frac{1}{2}} \leqq x_i .$$

Proposition 3 (Størmer). *If I and J are closed two-sided ideals in A, then $(I+J)^+ = I^+ + J^+$.*

Proof. Suppose $x \in (I+J)^+$. Then there exist self-adjoint elements $y \in I$, $z \in J$ with $x = y + z$, hence $x \leqq |y| + |z|$. By corollary 2 there exist u, v such that

$$x = u^* u + v^* v, \quad u u^* \leqq y, \quad v v^* \leqq z .$$

We have $u^* u \in I^+$ and $v^* v \in J^+$, and proposition 3 follows.

Let K denote the intersection of all dense, order-related two-sided ideals in A (see [1, section 1] for a discussion). Then K^+ is the smallest order ideal containing all elements $x \in A^+$ such that there exist elements $y \in A^+$ with $[x] \leqq y$. However the smallest order ideal J containing all elements $x \in A^+$, such that there exists $y \in K^+$ with $[x] \leqq y$, is also an invariant order ideal, and is dense in A^+ by the same argument which proves the density of K^+. Hence $J = K^+$.

With the above remarks in mind we may proceed with

Proposition 4. *If $\{x_i\}$ is a finite set of elements from K, then the order-related C^* algebra they generate is also in K.*

Proof. Each element x_i is a linear combination of elements from K^+ each of which is majorized by a finite sum of elements y_j for which $[y_j] \leqq z_j$ for some $z_j \in K^+$.

Hence the C^* algebra generated by the x_i's is contained in the order-related C^* subalgebra generated by the y_j's. But every element in this algebra is spanned by positive elements y for which, by theorem 1, we have a decomposition

$$y = \sum v_j^*[y_j]v_j \leqq \sum v_j^* z_j v_j \in K^+ .$$

Hence $y \in K^+$.

PROPOSITION 5. *If Φ is a *homomorphism from a C^* algebra A onto a C^* algebra B, and if $a \in A^+$, $b=\Phi(a)$, then*

$$\Phi\{x \in A^+ \mid x \leqq a\} = \{y \in B^+ \mid y \leqq b\} .$$

PROOF. Suppose $y \in B^+$, $y \leqq b$. Since $\Phi(A^+)=B^+$, there exists a self-adjoint $z \in A$ such that $z \leqq a$, $\Phi(z)=y$. We have $\Phi(z_+)=y$, $\Phi(z_-)=0$ and $z_+ \leqq a+z_-$, hence the sequence with elements

$$u_n = z_+^{\frac{1}{2}}(n^{-1}+z_-+a)^{-1}(z_-+a)^{\frac{1}{2}}a^{\frac{1}{2}}$$

is convergent (compare with the sequence $\{y_{in}\}$ in the proof of theorem 1). Set $x=\lim u_n^* u_n \leqq a$, and observe that

$$\Phi(x) = \lim b(n^{-1}+b)^{-1}y(n^{-1}+b)^{-1}b = [b]y[b] = y .$$

COROLLARY 6. $\Phi(K_A)=K_B$.

PROOF. Since Φ preserves spectral theory, $\Phi(K_A) \subset K_B$. On the other hand $\Phi(K_A)$ is clearly a dense, two-sided ideal in B. By proposition 5, $\Phi(K_A{}^+)$ is an order ideal, hence $\Phi(K_A)$ is order-related. Since K_B is minimal among all dense, order-related, two-sided ideals in B, we conclude $K_B \subset \Phi(K_A)$.

REFERENCES

1. Gert Kjærgård Pedersen, *Measure theory for C* algebras,* Math. Scand. 19 (1966), 131–145.
2. E. Størmer, *Two-sided ideals in C* algebras,* Bull. Amer. Math. Soc. 73 (1967), 254–257.

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