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SUBHARMONIC AND STRONGLY SUBHARMONIC FUNCTIONS IN CASE OF VARIABLE COEFFICIENTS

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Introduction.

It is known that a subharmonic function u in the unit ball of R^n, which is majorized by a positive harmonic function, has a boundary measure. This means that there exists a measure on the boundary of the unit ball, which is the weak limit of the measures $u(r\omega)d\omega$ ($d\omega$ is the Lebesgue measure on the boundary). Further, if φ is convex and increasing on R and continuous for $t=-\infty$, then $\varphi(u)$ is also subharmonic. If in addition φ is non-negative and $\varphi(t)/t \to \infty$ as $t \to \infty$, then $\varphi(u)$ is said to be strongly subharmonic in Gårding–Hörmander's notation (see [3]. By Solomentsev [12], it is called φ-subharmonic.)

The interesting feature is that a strongly subharmonic function with a positive harmonic majorant has an absolutely continuous boundary measure. For a proof see [3] or [12]. The purpose of this paper is to extend these results to subharmonic functions in a wider sense, that is subsolutions of the Laplace–Beltrami equation on a Riemann manifold. Our treatment is in particular valid in the case of subsolutions of a selfadjoint uniformly elliptic operator of second order without constant term, defined in an open and bounded set of R^n.

It should also be noted that Privaloff and Kouznetzoff [9] extended Solomentsev's theorem to Liapounov regions in n-dimensional euclidean space.

A brief outline of the paper goes as follows. After preliminary definitions and lemmas we consider local properties of harmonic and subharmonic functions and discuss equivalent ways of defining subharmonic functions. We prove that if u is subharmonic, so is $\varphi(u)$, with such a φ as described above. This was proved by Moser [7] in the special case, when u has square integrable first derivatives. In section 5 we prove by a simple method, the existence of a boundary measure of a subharmonic function which has a positive harmonic majorant. Riesz'

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representation formula is derived in section 6, and section 7 contains our main theorem which asserts that $\varphi(u)$ has an absolutely continuous boundary measure under the same assumptions of φ as in the classical case when $\varphi(u)$ has a harmonic majorant. The proof mainly follows the classical one. For part of the proof we need a Fatou theorem about pointwise convergence on the boundary of the Poisson integral of a measure. Such a theorem was recently proved by Widman [14]. However, it is possible to derive, without having to regularize the coefficients, a somewhat weaker result (lemma 7.2), which is sufficient for our purposes.

Finally, in section 8 we analyse the connection between quasibounded resp. singular harmonic functions (defined by Parreau [8], see also Heins [4]) and boundary measures.

The subject of this paper was suggested to me by Lars Gårding. I wish to thank him for his valuable advice and generous interest.

1. Preliminaries.

A function f defined in an open set ω of R^n is said to be of class C^k if it has k continuous derivatives there and of class H^k if its derivatives of order k are Hölder continuous. If in addition f has compact support, f is of class $C_0{}^k$ or $H_0{}^k$. Let V be an n-dimensional manifold of class H^2. When $\Omega \subset V$ is open, we say that f belongs to $C^k(\Omega)$ if f, defined in Ω, is in C^k in a fixed (and then in each) coordinate-system. The classes $C^k(\bar{\Omega})$, $C_0{}^k(\Omega)$, $H^k(\Omega)$, $H^k(\bar{\Omega})$ and $H_0{}^k(\Omega)$, $k=0,1,2$, are defined correspondingly.

Let the topology on V be given by a positive definite metric $\sum_{j,k=1}^n g_{jk}(x)\,dx^j dx^k$ of class H^1. If $s(x,y)$ denotes the geodetic distance and $\omega \subset V$, where $\bar{\omega}$ is compact, then $s(x,y)$ belongs to $H^2(\omega \times \omega)$.

The inverse of the matrix $(g_{jk}(x))$ is denoted $(g^{jk}(x))$, and $g(x) = \det g_{jk}(x)$. The volume element is then

$$dV(x) = g(x)^{\frac{1}{2}}\, dx^1 \wedge \ldots \wedge dx^n > 0 ,$$

and the Laplace–Beltrami operator on the manifold is

$$\Delta = g(x)^{-\frac{1}{2}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(g(x)^{\frac{1}{2}}\, g^{jk}(x)\, \frac{\partial}{\partial x^k}\right).$$

The scalar product $(\operatorname{grad} f, \operatorname{grad} s)$ is denoted f_s, and so we have

$$f_s(x) = \sum_{j,k=1}^{n} g^{jk}(x)\, \frac{\partial f(x)}{\partial x^j}\, \frac{\partial s(x)}{\partial x^k}.$$

Let S be a hypersurface of class H^1, defined by $s(x)=0$, where $\operatorname{grad} s(x) \neq 0$. Its volume element $dS=\delta(s(x))dV(x)$ is defined by $dS \wedge ds = dV$ on S. A substitution $s(x) \to \lambda(x)s(x)$ changes dS into $\lambda^{-1}(x)dS$. Since $\operatorname{grad} s \neq 0$ on S, each compact part of S may be covered by a finite number of spheres which have the property that, singling out one coordinate, say x_1, one can express the part of the boundary contained in each of these spheres in the form

$$x_1 = h(x_2,\ldots,x_n) ,$$

where h has Hölder continuous first derivatives.

A region is said to be of class H^1 if its boundary $\partial\Omega$ is a H^1-surface.

Let Ω be an H^1-region such that $\bar{\Omega}$ is compact. The smoothness property of the boundary and the Hölder continuity of the coefficients ensure that the following *Dirichlet problem*:

$$\begin{aligned} \Delta f &= h \quad \text{in } \Omega , \\ f &= g \quad \text{on } \partial\Omega , \end{aligned}$$

where $h \in H^0(\Omega) \cap C^0(\bar{\Omega})$ and $g \in H^1(\partial\Omega)$, has a unique solution $f \in H^2(\Omega) \cap H^1(\bar{\Omega})$. (For the local theory, see [6]. The global existence follows from standard arguments.)

Maximum principle. (See e.g. Courant–Hilbert [1, p. 326 ff.]) Let $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$. If $\Delta f \geqq 0$ in Ω and f has a maximum at an interior point x_0, then f is constant in the connected component of Ω, which contains x_0. It follows that if $\Delta f \geqq 0$ in Ω and $f=0$ on $\partial\Omega$, then $f \leqq 0$ in $\bar{\Omega}$; and if further f is not identically zero, then f is strictly negative in Ω and $\operatorname{grad} f \neq 0$ on $\partial\Omega$.

LEMMA 1.1. *If Ω is of class H^1, there exists a function $s \in H^2(\Omega) \cap H^1(\bar{\Omega})$ such that $s(x) > 0$ in Ω, $s(x)=0$ and* $\operatorname{grad} s(x) \neq 0$ *on $\partial\Omega$, and $\Delta s(x)/|\operatorname{grad} s(x)|$ arbitrarily large close to $\partial\Omega$.*

PROOF. Take $h \in H_0{}^0(\Omega)$, where $h \leqq 0$ and not identically zero in any connected component of Ω. The solution f of the Dirichlet problem $\Delta f = h$ in $\Omega, f=0$ on $\partial\Omega$ is then by the maximum principle strictly positive in Ω and $\operatorname{grad} f \neq 0$ on $\partial\Omega$. Put $\chi(t)=\exp(kt)-1$, where k is a constant. Then $s(x)=\chi(f(x))$ has the required properties. In particular

$$\Delta s(x)/|\operatorname{grad} s(x)| = k|\operatorname{grad} f(x)| + \Delta f(x)\,|\operatorname{grad} f(x)|^{-1} ,$$

which is arbitrarily large by proper choice of k.

In the following Ω always denotes an H^1-region such that $\bar{\Omega}$ is compact with boundary $s(x)=0$, where s is chosen as in lemma 1.1.

Definition. Let $A(\Omega)$ be the class of functions $f \in H^2(\Omega) \cap H^1(\bar{\Omega})$ such that $\Delta f(x) = O(s(x)^{-\gamma})$ in a neighbourhood of $\partial\Omega$, for some γ with $0 < \gamma < 1$, and $A_0(\Omega)$ the subclass of $A(\Omega)$, whose elements satisfy $f = 0$ on $\partial\Omega$.

Green's function. Under our assumptions on $\partial\Omega$ and the coefficients, Green's function $G(x,y)$ for Ω belonging to Δ exists. See for example [6]. We list some properties of G.

1) $G(x,y)$ satisfies the equation $\Delta_x G(x,y) = 0$ in $\Omega - \{y\}$, and $G(x,y) = 0$ when $x \in \partial\Omega$, $y \in \Omega$.

2) If y is fixed in Ω, G and its first and second derivatives with respect to x are continuous in $\Omega - \{y\}$.

3) $\int G(x,y)\Delta v(x)\, dV(x) = -v(y)$ if $v \in C_0^2(\Omega)$.

4) $G(x,y) = G(y,x)$.

5) If s denotes the geodetic distance between x and y,

$$G(x,y) = O(s^{2-n}), \quad \frac{\partial G}{\partial x^i}(x,y) = O(s^{1-n}), \quad \frac{\partial^2 G}{\partial x^i \partial x^k}(x,y) = O(s^{-n})$$

uniformly on each compact subdomain of Ω.

A consequence of 1) and 3) and the maximum principle for subharmonic functions to be discussed in section 3 is that $G \geqq 0$.

When $f \in A(\Omega)$, we have Green's formula

$$f(x) = \int G_s(x,y) f(y)\, \delta(s(y))\, dV(y) - \int G(x,y)\, \Delta f(y)\, dV(y)\,.$$

In particular

$$u(x) = \int G_s(x,y) f(y)\, \delta(s(y))\, dV(y) \tag{1.1}$$

solves $\Delta u = 0$ in Ω, $u = f \in H^1(\partial\Omega)$ on $\partial\Omega$. For fixed $x \in \Omega$ the harmonic (or Poisson) measure

$$dP(x,y) = G_s(x,y)\, \delta(s(y))\, dV(y)$$

has a continuous density on $\partial\Omega$.

In section 5 we need the following mean value theorem for right derivatives.

Lemma 1.2. *If h is continuous in the closed interval $[a,b]$ and the right derivative h'_+ exists in the open interval (a,b), then there exist ξ and η in (a,b) and $\lambda = \lambda(\xi,\eta)$ and $\mu = \mu(\xi,\eta)$ with $\lambda > 0$, $\mu > 0$ and $\lambda + \mu = 1$ such that*

$$h(b) - h(a) = (b-a)(\lambda h'_+(\xi) + \mu h'_+(\eta))\,.$$

PROOF. It is sufficient to study the case $h(b)=h(a)$. We make use of the following elementary fact: If h is continuous in $[a,b]$, and h'_+ is $\geqq 0$ in (a,b), then h is increasing in $[a,b]$, and analogously if $h'_+ \leqq 0$. Now, if there was no ξ in (a,b) with $h'_+ < 0$, h would be increasing, and, since $h(b)=h(a)$, constant. Then there is nothing to prove. In the same way there is an $\eta \in (a,b)$ with $h'_+(\eta) > 0$. Put

$$\lambda(\xi,\eta) = \frac{h'_+(\eta)}{h'_+(\eta)-h'_+(\xi)} > 0 , \qquad \mu(\xi,\eta) = \frac{-h'_+(\xi)}{h'_+(\eta)-h'_+(\xi)} > 0 ,$$

and we are through with the lemma.

2. Local properties of harmonic functions.

DEFINITION. A function u is said to be harmonic in Ω if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. A measure dU is said to be harmonic in Ω if

$$f \in C_0^2(\Omega) \Rightarrow \int \Delta f(x)\, dU(x) = 0 .$$

EXAMPLE. A measure $dU(x)=u(x)dV(x)$ with u harmonic is a harmonic measure.

By the analogue of Weyl's lemma (see e.g. Friedrichs [2]) we obtain

LEMMA 2.1. *Every harmonic measure has a unique harmonic density.*

If we put $f \equiv 1$ in (1.1) and take $v \in C_0^2(\Omega)$, we get

$$\begin{aligned} 0 &= \int \Delta v(x) \left(\int G_s(x,y)\, \delta(s(y))\, dV(y)\right) dV(x) \\ &= \int \left(\int G_s(x,y)\, \Delta v(x)\, dV(x)\right) \delta(s(y))\, dV(y) \end{aligned}$$

which implies

$$\int G_s(x,y)\, \Delta v(x)\, dV(x) = 0 \quad \text{for each } y \in \partial\Omega .$$

By lemma 2.1, $G_s(x,y)$ for fixed y on $\partial\Omega$ a.e. equals a harmonic function in Ω. But since it is continuous, it is harmonic. Renewed application of lemma 2.1 shows that if $f \in C^0(\partial\Omega)$,

$$u(x) = \int f(y)\, dP(x,y)$$

is harmonic. Approximating f by functions in $H^1(\partial\Omega)$ we see that u has continuous boundary values f. We have proved

COROLLARY. *If $f \in C^0(\partial\Omega)$, then $u(x) = \int f(y)dP(x,y)$ solves the Dirichlet problem $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$.*

Mean values. Let $B_r(x)$ be the ball $s(x,y) \leqq r$ (s denotes the geodetic distance) with center x and when $f \in C^0(\partial\Omega)$, let $M_r f(x)$ be the value at x of the solution of Dirichlet's problem $\Delta u = 0$ in $B_r(x)$ with boundary values f. It is clear that the mapping $f \to M_r f(x)$ is linear, reproduces constants and that $a \leqq M_r f(x) \leqq b$, where a and b are $\min f$ and $\max f$ on B_r respectively. If $a < b$, strict inequalities hold.

LEMMA 2.2. *Let f be continuous from above, and (f_n) a sequence of continuous functions decreasing to f. Then*

$$M_r f(x) = \lim_{n\to\infty} M_r f_n(x)$$

defines $M_r f$ uniquely and the map $f \to M_r f(x)$ is positive and linear. If $f(y) \leqq b$ on ∂B_r and $M_r f(x) = b$, then f is constant $= b$ on ∂B_r.

The proof is standard.

Lemmas 2.1 and 2.2 also give

COROLLARY. *A monotone, locally integrable limit of harmonic functions is harmonic and the convergence is locally uniform.*

PROOF. Let (u_n) be a sequence of harmonic functions in Ω, decreasing (or increasing) to u. It is clear that u a.e. in Ω equals a harmonic function u_0. We shall prove that $u = u_0$ everywhere in Ω. We have for small r

$$u_0(x) = M_r u_0(x) = M_r u(x) = \lim_{n\to\infty} M_r u_n(x) = \lim_{n\to\infty} u_n(x) = u(x) .$$

The uniform convergence follows by Dini's theorem.

3. Local properties of subharmonic functions.

DEFINITION. A function u is said to be strictly subharmonic in Ω if it is continuous from above, not identically $-\infty$ and satisfies the maximum principle with respect to harmonic functions in all subdomains ω of Ω: if v is harmonic in ω and $\overline{\lim}(u-v) \leqq 0$ on $\partial\omega$, then $u \leqq v$ in ω.

Note that a strictly subharmonic function is locally integrable.

That subharmonicity is a local property follows from the following lemma.

LEMMA 3.1. *An upper semicontinuous function u is strictly subharmonic in Ω if and only if $u(x) \leqq M_r u(x)$ for every $x \in \Omega$ and all sufficiently small r.*

The proof can be carried out almost as the classical one. (See e.g. [5, p. 16].)

It is clear that instead of $M_r u(x)$, we may take the mean value of u over any domain, containing x, with sufficiently regular boundary for the Dirichlet problem to be solvable.

It is also easy to see that the condition in lemma 3.1 is equivalent to the following:

If K is a compact set, $K \subset \Omega$, if v is harmonic and $u-v$ takes its maximum at an interior point of K, then $u-v$ is constant in a neighbourhood of the point.

Example. If $u \in C^2(\Omega)$, then u is strictly subharmonic if and only if $\Delta u \geqq 0$ in Ω. This follows from the maximum principle formulated in section 1.

We now generalize this example.

Definition. A measure $dU(x)$ in Ω is said to be subharmonic if

$$0 \leqq f \in C_0^2(\Omega) \ \Rightarrow \ \int \Delta f(x)\, dU(x) \geqq 0 \, .$$

It is clear that this is a local property.

Example. If $dU(x) = u(x) dV(x)$, where $u \in C^2(\Omega)$, then dU is subharmonic if and only if u is strictly subharmonic.

Theorem 3.2. *Every subharmonic measure has a unique strictly subharmonic density, locally the limit of a decreasing sequence of strictly subharmonic functions in* $C^2(\Omega)$.

Proof.

$$\int \Delta f(x)\, dU(x) \; = \int f(x)\, d\sigma(x) \quad \forall f \in C_0^2(\Omega)$$

defines a non-negative local measure $d\sigma$ in Ω. The map $dU \to d\sigma$ is positive linear. Let $\omega \subset \bar{\omega} \subset \Omega$ and let G be Green's function of Ω. Green's formula yields

$$f(x) \; = \; -\int G(x,y)\, \Delta f(y)\, dV(y)$$

and we get

$$\int \Delta f(x)\, dU(x) = -\int \left(\int_\omega G(x,y)\, \Delta f(y)\, dV(y) \right) d\sigma(x)$$

$$= -\int \Delta f(y) \left(\int_\omega G(x,y)\, d\sigma(x) \right) dV(y) ,$$

for every $f \in C_0^2(\omega)$. By lemma 2.1 there exists a harmonic function h_ω in ω such that dU has the density

$$(3.1) \qquad u(x) = h_\omega(x) - \int_\omega G(x,y)\, d\sigma(y)$$

in ω. Now let $\chi \in C_0^\infty(1,2)$, $\chi \geqq 0$, $\int_{-\infty}^{\infty} \chi(t)\,dt = 1$ and let

$$\chi^{(T)}(t) = \chi(t/T)\,T^{-1} \quad \text{for } T > 0 .$$

Put

$$\chi_1^{(T)}(t) = \int_0^t \chi^{(T)}(\tau)\, d\tau \quad \text{and} \quad \chi_2^{(T)}(t) = \int_0^t \chi_1^{(T)}(\tau)\, d\tau .$$

If $\psi_T(t) = t - \chi_2^{(T)}(t)$, then ψ_T belongs to C^2. Moreover

$$\psi_T(t) = t \qquad \text{when } t \leqq T ,$$
$$\psi_T(t) = \text{const.} = C_T \quad \text{when } t \geqq 2T ,$$

$T \leqq C_T \leqq 2T$. For a fixed $x \in \omega$, $y \to \psi_T(G(x,y))$ belongs to C^2 and (differentiations with respect to y)

$$\Delta \psi_T(G(x,y)) = \psi_T'' \,|\text{grad}\, G|^2 + \psi_T' \Delta G$$
$$= -\chi^{(T)}(G(x,y))\, |\text{grad}\, G(x,y)|^2 \leqq 0 \quad \text{if } y \neq x$$

and

$$\Delta \psi_T(G(x,y)) = 0 \text{ in a neighbourhood of } x .$$

Put

$$u_T(x) = h_\omega(x) - \int_\omega \psi_T(G(x,y))\, d\sigma(y) .$$

Then $u_T \in C^2$, $\Delta u_T(x) \geqq 0$ and when $T \to \infty$, $u_T(x)$ decreases to $u(x)$. Also, $u_T(x) \leqq M_r u_T(x)$ gives $u(x) \leqq M_r u(x)$ so that u is a strictly subharmonic density. Since u is continuous from above,

$$\overline{\lim}_{r\to 0} M_r u(x) \leqq u(x) \quad \text{so that} \quad u(x) = \lim_{r \to 0} M_r u(x) .$$

If $u(x) = u'(x)$ a.e. with u' strictly subharmonic, then $u(y) = u'(y)$ a.e. on almost all $\partial B_r(x)$ (when x is fixed). Hence $M_r u(x) = M_r u'(x)$ for almost all r, which implies $u(x) = u'(x)$.

Our description of subharmonic functions is complete with

THEOREM 3.3. *If u is strictly subharmonic, then $u(x)dV(x)$ is a subharmonic measure.*

The theorem is proved in the next section. In the sequel, by a subharmonic function, we understand a strictly subharmonic function. By theorem 3.3 it may be considered as the density of a subharmonic measure.

We also obtain the following corollary.

COROLLARY. *Let $\varphi(t)$ be an increasing convex function on R, continuous for $t=-\infty$. Then, if u is subharmonic, so is $\varphi(u)$.*

EXAMPLES. $\varphi(t)=\max(0,t)$; $\varphi(t)=\exp(pt)$, where $p>0$.

PROOF OF COROLLARY. That $\varphi(u)$ is upper semicontinuous is obvious. Consider a sequence of functions u_n, $n=1,2,\ldots$, continuous on $\partial B_r(x)$ and decreasing to u there. Let u_n also denote their harmonic extensions to $B_r(x)$. We have

$$u(x) \leqq u_n(x) = \int_{\partial B_r(x)} u_n(y)\, dP(x,y)$$

which gives

$$\varphi(u(x)) \leqq \varphi(u_n(x)) \leqq \int_{\partial B_r(x)} \varphi(u_n(y))\, dP(x,y)\,,$$

the last step by Jensen's inequality. When n tends to ∞ we get

$$\varphi(u(x)) \leqq M_r\varphi(u)(x)\,.$$

Below we will need

LEMMA 3.4. *Let $s \in C^1(\Omega)$, $\operatorname{grad} s \neq 0$ and let u be subharmonic. Then, for $f \in C_0{}^0(\Omega)$,*

$$t \to F(t) = \int u(x) f(x)\, \delta(s(x)-t)\, dV(x)$$

is continuous and, if (χ_n) is a sequence of functions in $C_0^\infty(R)$, tending to the Dirac measure δ,

$$F(t) = \lim_{n\to\infty} \int u(x) f(x)\, \chi_n(s(x)-t)\, dV(x)\,.$$

PROOF. Using (3.1) it is sufficient to prove the statement when

$$u(x) = \int_\omega G(x,y)\, d\sigma(y)$$

and $f \in C_0{}^0(\omega)$. But then, by the properties of G,

$$F(t,y) = \int_\omega G(x,y)\, f(x)\, \delta(s(x)-t)\, dV(x)$$

is continuous, and

$$F(t,y) = \lim_{n\to\infty} \int_\omega G(x,y)\, f(x)\, \chi_n(s(x)-t)\, dV(x)$$

where the convergence is locally uniform in y. Since

$$F(t) = \int_\omega F(t,y)\, d\sigma(y) ,$$

the rest is clear.

4. Proof of Theorem 3.3.

Let ω be a subset of the Riemann manifold V. Throughout this section ω is supposed to be small enough. Consider the class $\mathscr{A}$ of all functions $f(x,y) \in H^2(\omega \times \omega)$, where $f(x,y) = 1 + O(s^2(x,y))$.

Example. $f(x,y) = 1 + s^2(x,y) \in \mathscr{A}$.

Lemma 4.1. *If $f \in \mathscr{A}$ and $g(x,y) = 1 + s^2(x,y)$, then*

$$\nabla_x f(x,y) \cdot \nabla_x g(x,y) - \nabla_y f(x,y) \cdot \nabla_y g(x,y) = o(s^2(x,y)) .$$

(For shortness we write ∇ instead of grad and $\nabla f \cdot \nabla g$ instead of $(\mathrm{grad} f, \mathrm{grad} g)$.)

Proof. Take a fixed coordinate system and put $f(x,y) = 1 + s^2(x,y) f_1(x,y)$, where f_1 is bounded. We have

$$\begin{aligned} &\nabla_x f(x,y) \cdot \nabla_x g(x,y) \\ &= \sum_{j,k=1}^{n} g^{jk}(x) \left(2s(x,y) \frac{\partial s}{\partial x^j}(x,y)\, f_1(x,y) + s^2(x,y) \frac{\partial f_1}{\partial x^j}(x,y) \right) 2s(x,y) \frac{\partial s}{\partial x^k}(x,y) \\ &= 4s^2(x,y)\, f_1(x,y)\, |\nabla_x s(x,y)|^2 + 2s^3(x,y)\, \nabla_x f_1(x,y) \cdot \nabla_x s(x,y) . \end{aligned}$$

Now $|\nabla_x s|^2 = 1$ and a computation shows that $s(\nabla_x f_1 \cdot \nabla_x s)$ tends to zero with s. Consequently

$$\nabla_x f(x,y) \cdot \nabla_x g(x,y) = 4s^2(x,y)\, f_1(x,y) + o(s^2(x,y)) .$$

In the same way

$$\nabla_y f(x,y) \cdot \nabla_y g(x,y) = 4s^2(x,y)\, f_1(x,y) + o(s^2(x,y))$$

and the lemma follows.

LEMMA 4.2. *If f and g belong to $\mathscr{A}$, then*

$$\nabla_x sf\cdot\nabla_x sg - \nabla_y sf\cdot\nabla_y sg = o(s^2) .$$

PROOF. The left member has four terms:

$$\begin{aligned} s^2(\nabla_x f\cdot\nabla_x g-\nabla_y f\cdot\nabla_y g) &= O(s^4),\\ sg(\nabla_x f\cdot\nabla_x s-\nabla_y f\cdot\nabla_y s) &= \tfrac{1}{2}g(\nabla_x f\cdot\nabla_x(1+s^2)-\nabla_y f\cdot\nabla_y(1+s^2)) = o(s^2),\\ sf(\nabla_x g\cdot\nabla_x s-\nabla_y g\cdot\nabla_y s) &= o(s^2) \text{ analogously},\\ fg(\nabla_x s\cdot\nabla_x s-\nabla_y s\cdot\nabla_y s) &= 0 .\end{aligned}$$

It was mentioned in section 1 that a symmetric fundamental solution exists. We write it in the form

$$G(x,y) = c_n s(x,y)^{2-n}F(x,y) .$$

Here $F\in H^2(\omega\times\omega)$, $F(x,x)=1$ and $c_n^{-1}=(n-2)e_n$, where e_n denotes the surface of the unit sphere in R^n. ($n>2$ is assumed. If $n=2$, the singularity is logarithmic.)

Our first aim is to show that $F\in\mathscr{A}$. Introduce normal coordinates with center x, put

$$\alpha(x,y) = \tfrac{1}{2}\log g(y) \quad\text{and}\quad \nabla_y\alpha(x,y)\cdot\nabla_y s(x,y) = \alpha_s^{(y)}(x,y) .$$

By use of the equalities

$$\begin{aligned} s^2(x,y) &= \textstyle\sum_{i,k}g_{ik}(x)\, y_i y_k ,\\ \textstyle\sum_i g_{ik}(x)\, y_i &= \textstyle\sum_i g_{ik}(y)\, y_i , \qquad k=1,\ldots,n ,\end{aligned}$$

we obtain for $x\neq y$

$$\Delta_y s(x,y)^{2-n} = (2-n)g(y)^{-\frac{1}{2}}\, s(x,y)^{-n}\textstyle\sum_i y_i \frac{\partial}{\partial y^i}g(y)^{\frac{1}{2}} .$$

Since

$$\alpha_s^{(y)}(x,y) = g(y)^{-\frac{1}{2}}\, s(x,y)^{-1}\textstyle\sum_i y_i \frac{\partial}{\partial y^i}\, g(y)^{\frac{1}{2}}$$

we get

$$\Delta_y s(x,y)^{2-n} = (2-n)s(x,y)^{1-n}\alpha_s^{(y)}(x,y)$$

and so

$$\begin{aligned} 0 = \Delta_y G(x,y) = c_n(2-n)s(x,y)^{1-n}\big(&F(x,y)\alpha_s^{(y)}(x,y) + \\ &+s/(2-n)\Delta_y F(x,y) + 2F_s^{(y)}(x,y)\big) .\end{aligned}$$

In the centre x of the normal coordinate system $\partial g_{ik}/\partial y^j=0$, so that $F_s^{(y)}(x,y)$ tends to zero with s. From this we conclude that all the first derivatives of F are zero on the diagonal $x=y$ and consequently $F\in\mathscr{A}$. If ω is small enough, we may also assume $F(x,y)\geqq\text{constant}>0$.

Next, put

$$G(x,y) = c_n \sigma(x,y)^{2-n} .$$

It is clear that $\sigma(x,y) = s(x,y)H(x,y)$ with $H \in \mathscr{A}$, so that

$$\nabla_x \sigma \cdot \nabla_x \sigma - \nabla_y \sigma \cdot \nabla_y \sigma = o(s^2) = o(\sigma^2) . \tag{4.1}$$

Consider Green's spheres

$$C_r(x) : \sigma(x,y) \leqq r ,$$

contained in ω. Green's function for $C_r(x)$ is

$$G_r(x,y) = c_n\big(\sigma(x,y)^{2-n} - r^{2-n}\big) ,$$

and the Poisson measure at the centre x is by section 1

$$\begin{aligned} dP_r(x,y) &= |\nabla_y G_r(x,y) \cdot \nabla_y \sigma(x,y) \delta(\sigma(x,y) - r)\, dV(y)| \\ &= e_n^{-1} \sigma(x,y)^{1-n} (\nabla_y \sigma \cdot \nabla_y \sigma)\, \delta\big(\sigma(x,y) - r\big)\, dV(y) . \end{aligned} \tag{4.2}$$

Let

$$M_r f(x) = \int_{\partial C_r} f(y)\, dP_r(x,y)$$

be the value at x of the harmonic function which has continuous boundary values f on ∂C_r. Now suppose f in H^2. According to Green's formula

$$f(x) = M_r f(x) - c_n \int_{C_r(x)} \big(\sigma(x,y)^{2-n} - r^{2-n}\big) \Delta f(y)\, dV(y) .$$

We want to prove that

$$\lim_{r \to 0} r^{-2}\big(M_r f(x) - f(x)\big) = \Delta f(x)/(2n) \tag{4.3}$$

uniformly when $x \in \omega$. Introduce normal coordinates with centre x such that $g^{ik}(x) = \delta^{ik}$, which implies $dV(y) = \big(1 + o(1)\big) dy$ as $r \to 0$. Since $\Delta f(y) = \Delta f(x) + o(1)$, uniformly in x, it is sufficient to prove

$$\lim_{r \to 0} r^{-2} c_n \int_{C_r(x)} \big(\sigma(x,y)^{2-n} - r^{2-n}\big)\, dy = 1/(2n) ,$$

but since $\sigma(x,y)/s(x,y) = 1 + O(r^2)$ (also uniformly), we have only to verify that

$$\lim_{r \to 0} r^{-2} c_n \int_{B_r(x)} \big(s(x,y)^{2-n} - r^{2-n}\big)\, dy = 1/(2n) ,$$

where $B_r(x)$ is the geodetic sphere $s(x,y) \leqq r$. But

$$\int_{B_r(x)} (s(x,y)^{2-n} - r^{2-n})\, dy = \int_0^r (s^{2-n} - r^{2-n})\, e_n s^{n-1}\, ds\,,$$
$$= e_n r^2 (n-2)/(2n)\,,$$

so that (4.3) follows.

Let u be strictly subharmonic so that $u(x) \leqq M_r u(x)$ when $C_r(x) \subset \omega$. If $0 \leqq f \in H_0^2(\omega)$, we have by (4.2)

$$0 \leqq \int (M_r u(x) - u(x)) f(x)\, dV(x)$$
$$= \int u(y) \left\{ \int e_n^{-1} \sigma(x,y)^{1-n} |\nabla_y \sigma(x,y)|^2 f(x)\, \delta(\sigma(x,y) - r)\, dV(x) - f(y) \right\} dV(y)\,,$$

when r is small. The difference between the last expression and

$$\int u(y)\,(M_r f(y) - f(y))\, dV(y)$$

is, since σ is symmetric,

$$e_n^{-1} \int u(y) \left\{ \int \sigma(x,y)^{1-n} (|\nabla_y \sigma|^2 - |\nabla_x \sigma|^2) f(x)\, \delta(\sigma(x,y) - r)\, dV(x) \right\} dV(y)\,.$$

According to (4.1) and the fact that u is locally integrable, this is $o(r^2)$. Thus

$$r^{-2} \int u(y)\,(M_r f(y) - f(y))\, dV(y) \geqq o(1)\,.$$

By (4.3) we conclude

$$\int u(y) \Delta f(y)\, dV(y) \geqq 0$$

and the proof is finished.

5. Existence of boundary measures.

Let W denote the class of all subharmonic functions in Ω having non-negative harmonic majorants there. We choose $s(x)$ according to lemma 1.1 such that $\Delta s(x) \geqq 0$ for $s(x) \leqq$ some s_1. Put

$$M_s(fu)(t) = \int |\mathrm{grad}\, s(x)|^2 f(x)\, u(x)\, \delta(s(x) - t)\, dV(x)\,.$$

If u is subharmonic, there exists a local measure $d\sigma = d\sigma_u$ in Ω, $d\sigma \geqq 0$, such that

(5.1) $$\int u(x) \Delta f(x)\, dV(x) = \int f(x)\, d\sigma(x), \quad \forall f \in C_0^2(\Omega)\,.$$

If $f \in A(\Omega)$ (defined in section 1) and χ is a C^2-function on the positive real axis such that $\chi(s)=0$ when s is small, $\chi(s(x))f(x) \in C_0^2(\Omega)$ and we have, with $f_s(x)=(\operatorname{grad} f(x), \operatorname{grad} s(x))$,

$$\begin{aligned}(5.2)\qquad &\int \chi''(s(x))\,|\operatorname{grad} s(x)|^2 f(x)\,u(x)\,dV(x)+\\ &+\int \chi'(s(x))\big(\Delta s(x)f(x)+2f_s(x)\big)\,u(x)\,dV(x)+\\ &+\int \chi(s(x))\,\Delta f(x)\,u(x)\,dV(x)\\ &\qquad=\int \chi(s(x))f(x)\,d\sigma(x)\,.\end{aligned}$$

For a fixed $t>0$, we choose $\chi=\chi_\varepsilon$ by taking

$$\chi_\varepsilon'' \geqq 0, \quad \chi_\varepsilon'' \in C_0^\infty(t-\varepsilon, t+\varepsilon), \quad \int_{-\infty}^{\infty} \chi_\varepsilon''(s)\,ds = 1$$

and $\chi'=\chi=0$ for small s. Denote by θ the function which is 0 for $s \leqq 0$ and 1 for $s>0$. When ε tends to 0, we get from (5.2)

$$\begin{aligned}(5.3)\qquad &\int \delta(s(x)-t)\,|\operatorname{grad} s(x)|^2 f(x)\,u(x)\,dV(x)+\\ &+\int \theta(s(x)-t)\,\big(\Delta s(x)f(x)+2f_s(x)\big)\,u(x)\,dV(x)+\\ &+\int (s(x)-t)\,\theta(s(x)-t)\,\Delta f(x)\,u(x)\,dV(x)\\ &\qquad=\int (s(x)-t)\,\theta(s(x)-t)\,f(x)\,d\sigma(x)\,.\end{aligned}$$

From lemma 3.4 we know that the mean value $M_s(fu)(t)$ is continuous for $t>0$. This could also be seen from (5.3). Our purpose is to show that $M_s(fu)(t)$ has a limit when t tends to zero, if u has a non-negative harmonic majorant.

First let u be harmonic and non-negative and take f identically 1 in (5.3). Since $d\sigma=0$, we have for small $t>0$

(5.4)

$$M_s(u)(t) + \int_{s\leqq s_1} \theta(s(x)-t)\,\Delta s(x)\,u(x)\,dV(x) + \int_{s>s_1} \Delta s(x)\,u(x)\,dV(x) = 0$$

and we see that $M_s(u)(t)$ is an increasing function of t. Since it is positive, it has a limit when t tends to zero. Now, let $f \in A(\Omega)$ and consider the second integral of (5.3):

$$\int \theta(s(x)-t)(f(x)\Delta s(x)+2f_s(x))u(x)\,dV(x)$$

$$= \int \theta(s(x)-t)\,|\mathrm{grad}\, s(x)|^2 \left[\frac{f(x)\Delta s(x)+2f_s(x)}{|\mathrm{grad}\, s(x)|^2}\right] u(x)\,dV(x)\,.$$

We may restrict ourselves to the case when the function within the bracket is non-negative. The integral then increases when t tends to zero and is dominated by $C\cdot\int M_s(u)(s)ds<\infty$. (In the following C denotes various constants.) In the same way, we find that the third integral in (5.3) has a limit and consequently $\lim_{t\to+0} M_s(fu)(t)$ exists. Clearly the limit is linear in f and since we have

$$|M_s(fu)(t)| \leqq M_s(u)(t)\max_{s(x)=t}|f|\,,$$

it follows that

$$|\lim_{t\to+0} M_s(fu)(t)| \leqq C\max_{s(x)=0}|f| \quad \forall f\in A(\Omega)\,.$$

By approximation we get the same for $f\in C(\bar{\Omega})$ and thus we have proved

THEOREM 5.1. *If u is harmonic and non-negative in Ω, then*

$$\lim_{t\to+0} M_s(fu)(t)$$

exists and is a positive measure on $\partial\Omega$.

Next, we extend theorem 5.1 to the class W.

THEOREM 5.2. *If $u\in W$, then*

$$\int s(x)d\sigma(x) < \infty \tag{5.5}$$

and $\lim_{t\to+0} M_s(fu)(t)$ exists and is a measure on $\partial\Omega$.

PROOF. Let $u\leqq v$, $v\geqq 0$ and harmonic. Choosing $f\equiv 1$ in (5.3), we have, when t is small,

$$\int (s(x)-t)\,\theta(s(x)-t)\,d\sigma(x)$$

$$= M_s(u)(t) + \int_{\Delta s\geqq 0} \theta(s(x)-t)\Delta s(x)u(x)\,dV(x) + \int_{\Delta s<0} \Delta s(x)u(x)\,dV(x)$$

$$\leqq M_s(v) + \int \theta(s(x)-t)\Delta s(x)v(x)\,dV(x) + C\,,$$

which is bounded, and (5.5) follows. Further

$$M_s(u)(t) \geqq -\int \theta(s(x)-t)\Delta s(x)v(x)\,dV(x) + C\,,$$

so that $M_s(u)(t)$ is bounded from below. Let $u^+ = \max(u,0)$, $u = u^+ - u^-$. Then

$$M_s(|u|)(t) = 2M_s(u^+)(t) - M_s(u)(t)$$

is bounded and we may proceed as in the proof of theorem 5.1 to conclude that the second and third terms in (5.3) have limits when t tends to zero. The convergence of the right side follows from (5.5).

Since

$$|M_s(fu)(t)| \leqq M_s(|u|)(t) \max_{s(x)=t} |f|,$$

we see as before that the limit is a measure on $\partial\Omega$. As $\operatorname{grad} s(x) \neq 0$ near the boundary,

$$\lim_{t\to+0} \int \delta(s(x)-t) f(x) u(x)\, dV(x)$$

exists too. That means that the measures $\delta(s(x)-t)u(x)\, dV(x)$, $t>0$ and small, have a weak limit as t tends to zero. The limit is called *the boundary measure of* u.

From (5.3) we obtain by taking the right derivative with respect to t

$$\text{(5.6)} \quad -M'_{s+}(fu)(t) + \int \delta(s(x)-t)u(x)(f(x)\Delta s(x) + 2f_s(x))\, dV(x) + \\ + \int \theta(s(x)-t)\, u(x) \Delta f(x)\, dV(x) = \int \theta(s(x)-t) f(x)\, d\sigma(x)\,.$$

Let $f \in A_0(\Omega)$ (defined in section 1). Then $f(x) = O(s(x))$ and so $\int_\Omega f(y)\, d\sigma(y)$ exists. Since $\Delta f(x) = O(s(x)^{-\gamma})$, $0<\gamma<1$, the third term in (5.6) is dominated by

$$C \int_0^{\max s} s^{-\gamma} ds\,.$$

Hence, for such an f, each term in (5.6) has a limit when $t \to +0$, except possibly the first, and consequently also the first. If we denote by $d\mu_s$ the boundary measure of u and put $\lim_{t\to+0} M'_{s+}(fu)(t) = A(f)$, we get

$$-A(f) + 2\int_{\partial\Omega} f_s(y)\, d\mu_s(y) + \int_\Omega u(y) \Delta f(y)\, dV(y) = \int_\Omega f(y)\, d\sigma(y)\,.$$

According to lemma 1.2, $M'_{s+}(fu)(0)$ exists and is equal to $A(f)$, and so we have $A(f) = \lim_{t\to+0} M_s(fu)(t)/t$. On the other hand, let $x \to x_t$ be the mapping from $\partial\Omega$ orthogonally onto the level surfaces $s(x)=t$. Then

$$x_t = x + ty + o(t) \quad \text{as } t \to +0\,,$$

where $y = (y_1, \ldots, y_n)$,

$$y_j = \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial s}{\partial x^k}(x) \,|\mathrm{grad}\, s(x)|^{-2}$$

and $o(t)$ is uniform in x. Further

$$f(x_t) = t f_s(x) |\mathrm{grad}\, s(x)|^{-2} + o(t), \quad o(t) \text{ uniform in } x ,$$

so that

$$M_s(fu)(t)/t \to \int_{\partial\Omega} f_s(y)\, d\mu_s(y) \quad \text{as } t \to +0 .$$

Thus $A(f) = \int f_s(y) d\mu_s(y)$ and we have obtained

$$\int_{\Omega} u(y) \Delta f(y)\, dV(y) + \int_{\partial\Omega} f_s(y)\, d\mu_s(y) = \int_{\Omega} f(y)\, d\sigma(y) . \tag{5.7}$$

To sum up, we have proved

THEOREM 5.3. *If $u \in W$, then*

$$\lim_{t\to+0} \int u(x) f(x) \delta(s(x)-t)\, dV(x) = \int_{\partial\Omega} f(y)\, d\mu_s(y)$$

exists when $f \in C(\bar{\Omega})$, and

$$\int u(y) \Delta f(y)\, dV(y) + \int f_s(y)\, d\mu_s(y) = \int f(y)\, d\sigma(y)$$

when $f \in A_0(\Omega)$.

REMARK. It is clear that $d\mu_s$ depends on the choice of s (lemma 1.1), but it follows from (5.7) that $\int f_s d\mu_s$ is not depending on s.

We now show that $d\mu_s$ and $d\sigma$ determine u uniquely.

THEOREM 5.4. *If $d\sigma \geqq 0$ is a measure in Ω with $\int_\Omega s(x) d\sigma(x) < \infty$ and $d\mu_s$ a measure on $\partial\Omega$, then*

$$-\int \Delta f(y)\, dU(y) = \int_{\partial\Omega} f_s(y)\, d\mu_s(y) - \int_{\Omega} f(y)\, d\sigma(y), \quad f \in A_0(\Omega),$$

defines a unique subharmonic measure dU with a density u in the class W, such that $d\sigma = d\sigma_u$ and $d\mu_s$ is the boundary measure of u.

PROOF. We start with the special case $d\sigma = 0$ and $d\mu_s \geqq 0$. Given $h \in H_0{}^0(\Omega)$, there is a unique $f \in A_0(\Omega)$ with $\Delta f = h$ and the map dU:

$$\Delta f \to -\int_{\partial\Omega} f_s(y)\, d\mu_s(y)$$

is linear. Further, $h \geqq 0$ implies, because of the maximum principle, $f \leqq 0$ in Ω, and so $f_s \leqq 0$ on $\partial\Omega$. Consequently dU is a positive measure. If $f \in C_0^2(\Omega)$, then f_s is zero on $\partial\Omega$ and so $\int \Delta f(x) dU(x) = 0$. Hence $dU(x) = u(x) dV(x)$ with u harmonic and non-negative. If its boundary measure is $d\nu_s$, we have by theorem 5.3

$$\int_\Omega u(y) \Delta f(y)\, dV(y) + \int_{\partial\Omega} f_s(y)\, d\nu_s(y) = 0 \quad \forall f \in A_0(\Omega)\,.$$

We conclude

$$\int_{\partial\Omega} f_s(y)(d\mu_s(y) - d\nu_s(y)) = 0$$

for every $f \in A_0(\Omega)$. To show that $d\mu_s = d\nu_s$, let $\Omega_1 \supset \bar{\Omega}$ and $F \in H^2(\Omega_1)$ and put $f(x) = s(x)F(x)$ with s as before. Then $s \in H^2(\Omega) \cap H^1(\bar{\Omega})$, $\Delta s \in H^0(\bar{\Omega})$, and $\operatorname{grad} s \neq 0$ on $\partial\Omega$. We see that

$$\Delta f(x) = \Delta s(x) F(x) + 2F_s(x) + s(x) \Delta F(x)$$

belongs to $H^0(\bar{\Omega})$ so that $f \in A_0(\Omega)$. Further $f_s(x) = |\operatorname{grad} s(x)|^2 F(x)$ on $\partial\Omega$. If $\varphi \in C^0(\partial\Omega)$ and $\varepsilon > 0$ are given, we take F so that

$$\left| F(x) - \varphi(x) |\operatorname{grad} s(x)|^{-2} \right| \leqq \varepsilon |\operatorname{grad} s(x)|^{-2}$$

on $\partial\Omega$. Thus

$$\int_{\partial\Omega} \varphi(x)(d\mu_s - d\nu_s) = 0$$

for each continuous φ, so that $d\mu_s = d\nu_s$.

The extension to the case $d\sigma = 0$, $d\mu_s$ arbitrary, is trivial. In the case $d\sigma \geqq 0$, $d\mu_s = 0$, we proceed as above. Given $h \in H_0^0(\Omega)$, there is a unique $f \in A_0(\Omega)$ with $\Delta f = h$ and the map

$$\Delta f \to \int f d\sigma$$

is negative and linear and thus defines a negative measure, which is seen to be subharmonic. We conclude that there exists a subharmonic function v, $v \leqq 0$, with boundary measure $d\nu_s$ such that

$$\int \Delta f(y) v(y)\, dV(y) = \int f(y)\, d\sigma(y) \quad \forall f \in A_0(\Omega)\,.$$

By theorem 5.3 it follows that $d\nu_s = 0$. A combination of the two special cases gives the existence part of the theorem. The uniqueness is obvious.

6. Riesz' representation and least harmonic majorants.

Let $u \in W$ and denote its boundary measure $d\mu_s$. With G and ψ_T as in section 3 we have by theorem 5.3

$$\begin{aligned} v_T(x) &= -\int_\Omega u(y)\Delta\psi_T(G(x,y))\,dV(y) \\ &= \int_{\partial\Omega} G_s(x,y)\,d\mu_s(y) - \int_\Omega \psi_T(G(x,y))\,d\sigma(y)\,. \end{aligned}$$

Since $\Delta_y\psi_T(G(x,y)) \leqq 0$, v_T is subharmonic. When $T \to \infty$, $v_T(x)$ decreases to $v(x)$, say, where

$$v(x) = \int_{\partial\Omega} G_s(x,y)\,d\mu_s(y) - \int_\Omega G(x,y)\,d\sigma(y)\,.$$

Here $v=u$, for if $f \in C_0^2(\Omega)$ and

$$F(y) = \int_\Omega G(x,y)f(x)\,dV(x)\,,$$

then $F \in A_0(\Omega)$ and $\Delta F = -f$. Further

$$F_s(y) = \int_\Omega G_s(x,y)f(x)\,dV(x)$$

and we get

$$\begin{aligned} \int_\Omega v(x)f(x)\,dV(x) &= \int_{\partial\Omega} F_s(y)\,d\mu_s(y) - \int_\Omega F(y)\,d\sigma(y) \\ &= -\int_\Omega u(y)\Delta F(y)\,dV(y) = \int_\Omega u(y)f(y)\,dV(y)\,. \end{aligned}$$

Since u and v are subharmonic, we conclude $u=v$. We have proved

THEOREM 6.1 (Riesz' representation theorem). *If $u \in W$, then*

$$u(x) = \int_{\partial\Omega} G_s(x,y)\,d\mu_s(y) - \int_\Omega G(x,y)\,d\sigma(y)\,,$$

where $d\mu_s$ is the boundary measure of u and $d\sigma = \Delta u$.

Since the greatest harmonic minorant of $\int_\Omega G(x,y)\,d\sigma(y)$ is zero, we get

COROLLARY 1. *If $u \in W$, then $\int G_s(x,y)\,d\mu_s(y)$, where $d\mu_s$ is the boundary measure of u, is the least harmonic majorant of u in Ω.*

Theorem 5.4 and the representation theorem immediately give

COROLLARY 2. *If $d\mu_s$ is a measure on $\partial\Omega$, then $\int G_s(x,y)\,d\mu_s(y)$ is the harmonic function in Ω which has the boundary measure $d\mu_s$.*

7. Main theorem.

THEOREM 7.1. *Let u be subharmonic in Ω, let φ be a non-negative convex increasing function on R such that*

$$\varphi(t)/t \to \infty \quad \text{as } t \to +\infty\,, \tag{7.1}$$

$\varphi(-\infty)=\lim_{t\to-\infty}\varphi(t)$, and assume that $\varphi(u)$ has a harmonic majorant. Then u has a boundary measure

$$d\mu_s(x) = \lambda(x)\delta(s(x))\,dV(x) + d\mu_s'(x)\,,$$

where the singular part $d\mu_s'$ is $\leqq 0$, and the boundary measure of $\varphi(u)$ is absolutely continuous and equals $\varphi(\lambda(y))\delta(s(y))dV(y)$.

PROOF. Formula (7.1) implies that u has a non-negative harmonic majorant, since $\varphi(u)$ has one. Hence u has a boundary measure $d\mu_s$. Let $E\subset\partial\Omega$ be open with Lebesgue measure $m(E)$ and let f be continuous in $\bar{\Omega}$, $0\leqq f(y)\leqq 1$ when $y\in E$ and $f=0$ on $\partial\Omega\cap\complement E$. Then we have, following [3],

$$\int \delta(s(x)-t)u(x)f(x)\,dV(x) \leqq \sup_{\tau\geqq T}\tau/\varphi(\tau)\int\delta(s(x)-t)\varphi(u(x))f(x)\,dV(x) + {} + T\int\delta(s(x)-t)f(x)\,dV(x)\,.$$

When t tends to zero we get

$$\int f(x)\,d\mu_s(x) \leqq C\left(\sup_{\tau\geqq T}\tau/\varphi(\tau) + Tm(E)\right). \tag{7.2}$$

Putting $T=(m(E))^{-\frac{1}{2}}$ the right side tends to zero with $m(E)$. Hence μ^+ is absolutely continuous and so $d\mu_s'\leqq 0$. Hence

$$u(x) \leqq \int G_s(x,y)\lambda(y)\delta(s(y))\,dV(y)$$

and Jensen's inequality implies

$$\varphi(u(x)) \leqq \int G_s(x,y)\varphi(\lambda(y))\delta(s(y))\,dV(y)\,.$$

The right member has by corollary 2 of section 6 the boundary measure $\varphi(\lambda(y))\delta(s(y))dV(y)$. If $d\nu_s$ is the boundary measure of $\varphi(u)$, then $d\nu_s$ is absolutely continuous and

$$dv_s(y) \leqq \varphi(\lambda(y))\delta(s(y))\,dV(y)\,. \tag{7.3}$$

For the opposite inequality we require a result concerning pointwise convergence on the boundary. Let v be harmonic and belong to W. Then v has a boundary measure

$$d\mu_s(y) = \lambda(y)\delta(s(y))\,dV(y) + d\mu_s'^{+}(y) - d\mu_s'^{-}(y)\,,$$

where $\lambda \in L^1(\partial\Omega)$, and $v(x) = \int G_s(x,y)\,d\mu_s(y)$.

LEMMA 7.2. *To each sequence* (t_j) *with* $\lim_{j\to\infty} t_j = 0$ *there is a subsequence* $(t_{j'})$ *such that*

$$v(x_{t_{j'}}) \to \lambda(x) \quad \text{a.e. } on\ \partial\Omega\,. \tag{7.4}$$

PROOF. Let us first assume that $d\mu_s'^{+} = d\mu_s'^{-} = 0$. Put

$$\int \delta(s(x)-t_j)v(x)\,dV(x) = \int v(x_{t_j})\gamma(x_{t_j})\delta(s(x))\,dV(x)\,,$$

where $\gamma(x_{t_j})$ tends to 1 as $j\to\infty$. It is sufficient to prove

$$\int |v(x_{t_j})\gamma(x_{t_j}) - \lambda(x)|\,\delta(s(x))\,dV(x) \to 0 \quad \text{as } j\to\infty\,. \tag{7.5}$$

For (7.5) implies that there exists a subsequence $(t_{j'})$ such that

$$\sum_{j'=1}^{\infty} \int |v(x_{t_{j'}})\gamma(x_{t_{j'}}) - \lambda(x)|\,\delta(s(x))\,dV(x) < \infty\,.$$

By Beppo Levi's theorem we conclude that

$$\sum_{j'=1}^{\infty} |v(x_{t_{j'}})\gamma(x_{t_{j'}}) - \lambda(x)| \in L^1(\partial\Omega)\,,$$

and so $v(x_{t_{j'}})\gamma(x_{t_{j'}}) - \lambda(x)$ tends to zero for almost every $x \in \partial\Omega$.

To prove (7.5), put $\lambda = \lambda_1 + \lambda_2$, where λ_1 is continuous and

$$\int |\lambda_2(y)|\,\delta(s(y))\,dV(y) < \varepsilon\,.$$

We have

$$\begin{aligned}
&\int |v(x_{t_j})\gamma(x_{t_j}) - \lambda(x)|\,\delta(s(x))\,dV(x) \\
&\quad\leqq \int \left|\gamma(x_{t_j})\int G_s(x_{t_j},y)\lambda_1(y)\delta(s(y))\,dV(y) - \lambda_1(x)\right|\delta(s(x))\,dV(x) + \\
&\qquad + \int |\lambda_2(y)|\left(\int \delta(s(x)-t_j)G_s(x,y)\,dV(x)\right)\delta(s(y))\,dV(y) + \\
&\qquad + \int |\lambda_2(y)|\,\delta(s(y))\,dV(y)\,.
\end{aligned}$$

The first integral tends to zero. If we may establish

$$\int \delta(s(x)-t)\, G_s(x,y)\, dV(x) \leqq C \tag{7.6}$$

for every small t and the constant not depending on y, the rest is less than a constant times ε.

For the proof of (7.6) we remind ourselves that the mean value $M_s(u)(t)$ of a non-negative harmonic function over the level surface $s(x)=t$ is increasing when t is small. Taking a fixed t_0 such that $\Delta s \geqq 0$ and $|\mathrm{grad}\, s| \neq 0$ for $s<t_0$, we have by (5.4)

$$\int \delta(s(x)-t)\, G_s(x,y)\, |\mathrm{grad}\, s(x)|^2\, dV(x) \leqq \int \delta(s(x)-t_0)\, G_s(x,y)\, |\mathrm{grad}\, s(x)|^2\, dV(x)$$
$$= -\int \theta(s(x)-t_0)\, G_s(x,y)\, \Delta s(x)\, dV(x)\,.$$

Now the Harnack inequality (Serrin [11]) gives

$$G_s(x,y) \leqq MG(x_0,y)$$

for each x in the compact K: $s(x) \geqq t_0$ ($x_0 \in K$ fixed), and M does not depend on y. Hence

$$\int \delta(s(x)-t)\, G_s(x,y)\, |\mathrm{grad}\, s(x)|^2\, dV(x)$$
$$\leqq M\, G_s(x_0,y) \int_{s \geqq t_0} |\Delta s(x)|\, dV(x) \leqq C \max_{y \in \partial\Omega} G_s(x_0,y)\,,$$

and (7.6) follows.

To complete the proof of the lemma it is sufficient to suppose $d\mu_s = d\mu_s' \leqq 0$, which implies $v \leqq 0$. Then $e^{v(x)}$ is subharmonic and bounded. By the first part of this section it has an absolutely continuous boundary measure $\mathscr{H}(x)\delta(s(x))\, dV(x)$. Consequently, for each sequence (t_j) there exists a subsequence $(t_{j'})$, such that

$$e^{v(x_{t_{j'}})} \to \mathscr{H}(x) \quad \text{a.e. on } \partial\Omega\,,$$

that is

$$v(x_{t_{j'}}) \to \log \mathscr{H}(x) = \varrho(x)\,.$$

We shall show that $\varrho(x)$ equals zero a.e. By Fatou's lemma

$$\int f(x)\, d\mu_s(x) = \lim_{j' \to \infty} \int \delta(s(x)-t_{j'})\, v(x) f(x)\, dV(x)$$
$$\leqq \int \overline{\lim}_{j' \to \infty}\, v(x_{t_{j'}})\, \gamma(x_{t_{j'}}) f(x)\, \delta(s(x))\, dV(x)$$
$$= \int \varrho(x) f(x)\, \delta(s(x))\, dV(x)$$

for each $f \in C(\bar{\Omega})$, $f \geqq 0$. Since $d\mu_s$ is singular, it follows that $\varrho(x) = 0$ a.e., and we are through with the proof of the lemma.

If $u \in W$ and has the boundary measure $d\mu_s$,

$$h(x) = \int G_s(x,y)\, d\mu_s(y)$$

is by corollary 1, section 6, the least harmonic majorant of u. Further

$$\int \delta(s(x)-t)(u(x)-h(x))\, dV(x) \to 0 \quad \text{as } t \to +0 .$$

By the same argument which led to (7.4) there exists a sequence (t_j) such that for a.e. $x \in \partial\Omega$

$$u(x_{t_j}) - h(x_{t_j}) \to 0 \quad \text{as } j \to \infty .$$

Applying lemma 7.2 to h we conclude that there exists a subsequence $(t_{j'})$ with the property that

$$u(x_{t_{j'}}) \to \lambda(x) \quad \text{as } j' \to \infty ,$$

a.e. on $\partial\Omega$.

Proof of Theorem 7.1 continued. By Fatou's lemma

$$\int f(x)\, d\nu_s(x) \geqq \int \underline{\lim}_{j' \to \infty} \varphi(u(x_{t_{j'}})) f(x_{t_{j'}}) \delta(s(x)-t_{j'})\, dV(x)$$
$$= \int \varphi(\lambda(x)) f(x) \delta(s(x))\, dV(x)$$

for each continuous $f \geqq 0$ and so

$$d\nu_s(x) \geqq \varphi(\lambda(x)) \delta(s(x))\, dV(x) .$$

Remark. As observed by Gårding and Hörmander [3], Solomentsev's theorem may be used to give a short proof of the following theorem by F. and M. Riesz [10]: Let f be analytic in the unit disc. If

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \quad \text{is bounded} ,$$

then f has an absolutely continuous boundary measure $f(e^{i\theta})\, d\theta$. Further $f(re^{i\theta}) \to f(e^{i\theta})$ a.e. θ and

$$\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|\, d\theta \to 0 \quad \text{as } r \to 1 .$$

A corresponding theorem concerning harmonic gradients in a half-space has been proved by Stein and Weiss [13]. If $(u_1,\ldots,u_n)$ is a harmonic gradient, i.e. if there exists a function U, harmonic in the classical sense, such that $\partial U/\partial x_j = u_j$, then $|u| = (\sum_1^n u_j^2)^{\frac{1}{2}}$ is strongly subharmonic. This follows from the fact that $|u|^p$ is subharmonic for $p \geqq (n-2)/(n-1)$, which is proved in [13]. From [13] we also get the following result: If $u=(u_1,\ldots,u_n)$ is a harmonic gradient in the half-space $\{(x_1,\ldots,x_n) \in R^n;\ x_n > 0\}$ and

$$\int_{R^{n-1}} |u(z,y)|\, dz \leqq C \quad \text{for every } y > 0\,, \tag{7.7}$$

then u has an absolutely continuous boundary measure $u(z)dz$ on $x_n = 0$ and

$$\int_{R^{n-1}} |u(z,y)| - u(z)|\, dz \to 0 \quad \text{as } y \to 0\,.$$

Since (7.7) implies the existence of a harmonic majorant of $|u|$ in the half-space, we may apply our main theorem to convenient smooth domains having a part of the boundary contained in $x_n = 0$ to get an alternative proof of the result of Stein and Weiss.

8. Quasibounded and singular harmonic functions.

Parreau [8] has introduced the notions of quasibounded and singular harmonic functions. Our form of the definition is found in [4]:

Definition. A non-negative harmonic function p is said to be *quasi-bounded* if there exists an increasing sequence (b_n) of non-negative bounded harmonic functions such that

$$p = \lim_{n\to\infty} b_n;$$

p is called *singular* provided that the only non-negative bounded harmonic function dominated by p is the constant 0.

As M. Heins [4] has pointed out in the case of harmonic functions in the open unit disc (or n-dimensional ball), quasibounded (resp. singular) harmonic functions are precisely those given by Poisson integrals of non-negative absolutely continuous (resp. singular) boundary measures.

Let W' denote the class of all non-negative harmonic functions in Ω and let φ be as in section 7. Denote by LHMu the least harmonic majorant of a subharmonic function u. We recall some result of [4].

(i) If $p \in W'$, it admits a unique representation $p = q + s$, where q is quasibounded and s singular.

(ii) If $p \in W'$ and $\varphi(p)$ has a harmonic majorant, then p and $\mathrm{LHM}\varphi(p)$ are quasibounded.

(iii) Let u be subharmonic and suppose that $\varphi(u)$ has a harmonic majorant. Then $\varphi(u^+)$ and u^+ have one. Further, if $\mathrm{LHM}u = v$ and $\mathrm{LHM}u^+ = w$, then $\varphi(w)$ has a harmonic majorant and

$$\mathrm{LHM}\varphi(u) = \mathrm{LHM}\varphi(v)$$

and

$$\mathrm{LHM}\varphi(u^+) = \mathrm{LHM}\varphi(w) .$$

(iv) If h is harmonic and belongs to W, it has a representation $h = p_1 - p_2$, where $p_i \in W'$. The component terms are least, if we choose $p_1 = \mathrm{LHM}u^+$ and $p_2 = p_1 - h$.

Now, if u is subharmonic and $\varphi(u) \in W$, by (iii) the function u has a harmonic majorant. Application of (iv) to $\mathrm{LHM}u = v$ gives $v = Q - s$, where Q is a difference of quasibounded non-negative harmonic functions and s is non-negative and singular.

So far, all verifications only use the definition of quasibounded and singular harmonic functions, and so they are valid in our case.

From (i)–(iv) it follows that

$$\mathrm{LHM}\varphi(u) = \mathrm{LHM}\varphi(Q) ,$$

and if Q has the boundary measure $Q^*(y)\delta(s(y))dV(y)$, it is possible to conclude, by help of Riesz' representation formula that the boundary measure of $\mathrm{LHM}\varphi(u)$ is $\varphi(Q^*(y))\delta(s(y))dV(y)$. This gives an alternative proof for this part of the Solomentsev theorem, and consequently for the corresponding part of ours. It remains to prove the following lemma.

Lemma 8.1. *If $u \in W'$, then*

(a) *u is quasibounded if and only if its boundary measure is absolutely continuous,*

(b) *u is singular if and only if its boundary measure is singular.*

Proof. To prove (b), suppose $u \in W'$ with singular boundary measure $d\mu_s$. If h is bounded and non-negative, its boundary measure $d\nu_s$ is absolutely continuous. If h is dominated by u, we get

$$0 \leqq d\nu_s \leqq d\mu_s ,$$

and so $d\nu_s$ is singular. Consequently h is zero.

Conversely, suppose that $u \in W'$ is singular. We have

$$u(x) = \int G_s(x,y)\, d\mu_s(y)\,,$$

where $d\mu_s(y) = \lambda(y)\,\delta(s(y))\,dV(y) + d\mu_s'(y)$ with $\lambda \in L^1(\partial\Omega)$ and $d\mu_s' \geqq 0$ and singular. Since

$$0 \leqq \min(\lambda(y),m)\,\delta(s(y))\,dV(y) \leqq d\mu_s(y)\,,$$

where m is a positive integer, we get

$$0 \leqq b_m(x) = \int G_s(x,y)\,\min(\lambda(y),m)\,\delta(s(y))\,dV(y) \leqq u(x)\,.$$

It follows that b_m is 0, and so (by lemma 7.2) that $\min(\lambda(y),m) = 0$ a.e. We conclude $\lambda = 0$ a.e., and $d\mu_s$ is singular.

For the proof of (a), assume that q is quasibounded and let (b_n) be a sequence of non-negative bounded harmonic functions increasing to q. Then each b_n has an absolutely continuous boundary measure and

$$b_n(x) = \int G_s(x,y)\,\lambda_n(y)\,\delta(s(y))\,dV(y)\,,$$

where (λ_n) is an increasing sequence of L^1-functions on $\partial\Omega$. Since

$$\lim_{n\to\infty} \int G_s(x,y)\,\lambda_n(y)\,\delta(s(y))\,dV(y) = q(x) < \infty\,.$$

Beppo Levi's theorem asserts that $\lim_{n\to\infty}\lambda_n = \lambda \in L^1(\partial\Omega)$ and

$$q(x) = \int G_s(x,y)\,\lambda(y)\,\delta(s(y))\,dV(y)\,.$$

Conversely, if q has an absolutely continuous boundary measure and $q = q' + s$ is the representation of q with q' quasibounded and s singular, from what we just have proved, q' has an absolutely continuous boundary measure. Then, so has s and by (b), s is zero.

REFERENCES

1. R. Courant and D. Hilbert, *Methods of mathematical physics* II, Interscience Publishers, New York, 1962.
2. K. O. Friedrichs, *On the differentiability of the solutions of linear elliptic differential equations*, Comm. Pure Appl. Math. 6 (1953), 299–325.
3. L. Gårding and L. Hörmander, *Strongly subharmonic functions*, Math. Scand. 15 (1964), 93–96.
4. M. Heins, *On the theorem of Szegö–Solomentsev*, Math. Scand. 20 (1967), 281–289.
5. L. Hörmander, *An introduction to complex analysis in several variables*, D. Van Nostrand Comp., Princeton, N. J., 1966.
6. C. Miranda, *Equazioni alle derivate parziali di tipo ellittico* (Ergebn. Math. Grenzgeb. N. F. 2), Springer-Verlag, Berlin · Göttingen · Heidelberg, 1955.
7. J. Moser, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. 13 (1960), 457–468.

8. M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann*, Ann. Inst. Fourier (Grenoble) 3 (1951), 103–197.
9. I. I. Privaloff, and P. Kouznetzoff, *Sur les problèmes limites et les classes différentes de fonctions harmoniques et subharmoniques définies dans un domaine arbitraire*, Mat. Sbornik 48 (1939), 345–76. (Russian.)
10. F. Riesz und M. Riesz, *Über die Randwerte einer analytischen Funktion*, C. R. Congr. Math. Scand., Stockholm 1916, 27–44.
11. J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. 4 (1954/55), 292–308.
12. D. Solomentsev, *On some classes of subharmonic functions*, Izv. Akad. Nauk SSSR, Ser. Mat. 5–6 (1938), 571–582. (Russian.)
13. E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables* I, Acta Math. 103 (1960), 25–62.
14. K.-O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations*, Thesis, Uppsala, 1966.

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