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UNIFORM APPROXIMATION ON COMPACT SETS IN $\mathbf{C}^n$

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1. Introduction.

If K is a compact set in the space $\mathbf{C}^n$ of n complex variables we denote by $A(K)$ the class of functions defined on K which are uniform limits of functions holomorphic in some neighborhood of K. Thus $f \in A(K)$ means that there exists a sequence of functions f_n, each analytic in some neighborhood Ω_n of K, such that $f_n \to f$ uniformly on K. Similarly, we define $P(K)$ as the class of functions on K which are uniform limits of polynomials in the complex coordinates. Evidently

$$P(K) \subset A(K) \subset C(K)$$

where $C(K)$ as usual denotes the space of all complex valued continuous functions on K.

We shall be principally concerned with sets K lying on certain smooth submanifolds of $\mathbf{C}^n$. Let Σ be such a manifold, of real dimension k. If $x \in \Sigma$ we denote by T_x the tangent space to Σ at x, viewed as a real linear subspace of $\mathbf{C}^n$. By a complex tangent to Σ at x we mean a complex line, that is, a complex linear subspace of $\mathbf{C}^n$ of complex dimension 1, contained in T_x.

Results connecting complex tangents to Σ with the space $P(K)$ have been given for 2-dimensional Σ in Wermer [9], [10] and Freeman [1]. The general case of a smooth k-dimensional manifold in $\mathbf{C}^n$ with no complex tangents was discussed in a recent note by Nirenberg and Wells [6]. They made use of the solution of the $\bar\partial$ Neumann problem due to Kohn [4] in tubular neighborhoods of Σ. The results announced are quite complete for the case where Σ is a C^∞ manifold having no complex tangent vectors. Our methods here are similar but based on the uniform bounds for solutions of the $\bar\partial$ Neumann problem proved in Hörmander [2]. This proof was communicated to R. Wells in 1965. We also give some essentially geometrical arguments which permit the study of certain cases where complex tangents may exist. The results we prove contain the following one:

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THEOREM 1.1. *Let S be a ν-dimensional submanifold of $\mathbf{C}^n$ which is of class C^r where $r \geqq \frac{1}{2}\nu + 1$. Let K_0 be the set of those points where S has a complex tangent. Assume that K is a compact polynomially convex set $\subset S$ such that K_0 is in the interior of K relative to S. Every $u \in C(K)$ with $u|_{K_0} \in A(K_0)$ is then in $P(K)$.*

When S is a 2-dimensional disk this result is closely related to that given in the Appendix of Wermer [10]. Another approximation theorem we obtain is the following:

THEOREM 1.2. *Let X be a compact set in $\mathbf{C}^n$ and let N be a neighborhood of X. Consider a vector function $R = (R_1, \ldots, R_n)$ with values in $\mathbf{C}^n$, defined and of class C^{n+1} in N. Suppose that there is a constant $k < 1$ such that*

$$|R(z) - R(z')| \leqq k|z - z'| \quad \textit{for all } z, z' \in N .$$

Every $u \in C(X)$ is then a uniform limit of polynomials in the functions

$$z_1, \ldots, z_n, \bar{z}_1 + R_1(z), \ldots, \bar{z}_n + R_n(z) .$$

The case $n = 1$, under slightly weaker hypotheses, is Theorem 1 in Wermer [9].

2. Preliminaries.

We shall collect in this section a few definitions and facts concerning functions of real variables which will be needed later on. Let Ω be an open subset of $\mathbf{R}^N$. If r is an integer $\geqq 0$ we denote as usual by $C^r(\Omega)$ the space of functions with continuous derivatives of order $\leqq r$ in Ω. When $s < r < s+1$, where s is a non-negative integer, we use the same notation for the space of all $u \in C^s(\Omega)$ such that for $|\alpha| = s$

$$D^\alpha u(x) - D^\alpha u(y) = o(|x-y|^{r-s}) \quad \text{when } x, y \in K \Subset \Omega \text{ and } x - y \to 0 .$$

Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-order, that is, the components are non-negative integers, and $|\alpha| = \sum \alpha_i$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}$. The notation $K \Subset \Omega$ means that K is relatively compact in Ω.

A submanifold Σ of $\mathbf{R}^N$ of dimension ν is said to be of class C^r if it can be locally parametrized by a regular map $\varkappa \colon \omega \to \mathbf{R}^N$ where ω is an open subset of $\mathbf{R}^\nu$ and the mapping belongs to C^r, $r \geqq 1$. In a neighborhood of any point on Σ one can take $\varkappa$ to be the inverse of the projection of Σ on some ν-dimensional coordinate plane, identified with $\mathbf{R}^\nu$.

Denote by $d(x, \Sigma)$ the Euclidean distance from x to Σ and by $d(x, T_y)$ the distance to the tangent plane at $y \in \Sigma$. In order to be able to work with C^1 manifolds we shall need the following fact in Section 3.

Lemma 2.1. *Let Σ be a closed C^1 submanifold of an open set $\Omega \subset \mathsf{R}^N$. Then there exists a function $\varrho \in C^2(\Omega)$ such that for all $y \in \Sigma$*

$$\varrho(x) - d(x, T_y)^2 = o(|x-y|^2), \quad x \to y \in \Sigma .$$

Since $\varrho(x) - d(x, T_y)^2$ is the remainder term in Taylor's formula, this o will be uniform for y in a compact subset of Σ. Hence we have, if y is chosen to be the closest point to x in Σ, that

$$\tfrac{1}{2} d(x,\Sigma)^2 \leqq \varrho(x) \leqq 2d(x,\Sigma)^2$$

in a neighborhood of Σ.

Proof of Lemma 2.1. If Σ were in C^2 we could take $\varrho(x) = d(x,\Sigma)^2$ in a neighborhood of Σ. Under our weaker hypotheses the lemma follows instead from the extension theorem of Whitney [11] provided that we show that for x,y in compact subsets of Σ we have uniformly for $|\alpha| \leqq 2$

$$D_z^\alpha\big(d(z,T_x)^2 - d(z,T_y)^2\big)_{z=x} = o(|x-y|^{2-|\alpha|}) .$$

But this follows immediately from the fact that $T_x \to T_y$ as $x \to y$.

The following lemma is also closely related to Whitney's extension theorem and will be used in section 4 to keep down the differentiability hypotheses.

Lemma 2.2. *For $x = (x_1, \ldots, x_N) \in \mathsf{R}^N$ we set $x' = (x_1, \ldots, x_\nu)$ and $x'' = (x_{\nu+1}, \ldots, x_N)$ for a certain ν with $1 \leqq \nu < N$. For every $u \in C^r(\mathsf{R}^\nu)$ one can find $U \in C^r(\mathsf{R}^N)$ so that*

$$U(x',0) = u(x') \quad \text{and} \quad x^\alpha U \in C^{r+|\alpha|}(\mathsf{R}^N)$$

for every multi-order $\alpha = (0, \ldots, 0, \alpha_{\nu+1}, \ldots, \alpha_N)$.

Proof. Choose $\varphi \in C_0^\infty(\mathsf{R}^\nu)$ with integral 1, and set with $y^j \in \mathsf{R}^\nu$, $\nu < j \leqq N$

$$U(x) = \int u\Big(x' - \sum_{\nu+1}^{N} x_j y^j\Big) \prod_{\nu+1}^{N} \varphi(y^j)\, dy^j . \tag{2.1}$$

Since the functions

$$x \to u(x' - \textstyle\sum x_j y^j)$$

belong to a bounded set in $C^r(\mathsf{R}^N)$ when all y^j belong to a compact set, it is clear that $U \in C^r$. It is also obvious that $U(x',0) = u(x')$. To complete the proof it suffices to show that if $1 \leqq j \leqq N$ and $\nu < k \leqq N$, then $D_j x_k U$ is a sum of integrals of the same kind as used in the definition of U, for Lemma 2.2 then follows by induction with respect to $|\alpha|$. To simplify notations we may assume that $k=N$.

Let $x_N \neq 0$ and introduce

$$y' = x' - \sum_{\nu+1}^{N} x_j y^j$$

as a new variable instead of y^N. Then we obtain

$$x_N U(x) = x_N |x_N|^{-\nu} \int u(y') \prod_{\nu+1}^{N-1} \varphi(y^j)\, dy^j\, \varphi\left(\left(x' - y' - \sum_{\nu+1}^{N-1} x_j y^j\right)\Big/ x_N\right) dy' .$$

We can differentiate here under the sign of integration. Writing

$$\varphi_j(x') = \partial\varphi(x')/\partial x_j, \quad \varphi^j(x') = -x_j\varphi(x'), \qquad 1 \leqq j \leqq \nu ,$$

$$\varphi_N(x') = (1-\nu)\varphi(x') - \sum_{1}^{\nu} x_j \varphi_j(x') ,$$

we conclude that $D_j x_N U(x)$ is given by (2.1) with $\varphi(y^N)$ replaced by $\varphi_j(y^N)$ if $j \leqq \nu$ or $j=N$, and is otherwise the sum of such integrals with $\varphi(y^j)$ replaced by $\varphi^k(y^j)$ and $\varphi(y^N)$ replaced by $\varphi_k(y^N)$, the summation being made for $1 \leqq k \leqq \nu$. But integrals such as (2.1) are continuous everywhere so it follows that the conclusion is not only true when $x_N \neq 0$. This proves the lemma.

Remark. From the continuity of the map $u \to x^\alpha U$ it also follows that for $|\beta| \leqq |\alpha|$ we have $D^\beta x^\alpha U = u D^\beta x^\alpha$ when $x''=0$.

3. Construction of some domains of holomorphy.

It is well known that tubular neighborhoods with small radius of a manifold in $\mathbf{C}^n$ are pseudo-convex if there are no complex tangents to the manifold. However, we shall study certain cases where complex tangents may occur. This requires the construction of tubular neighborhoods with highly variable radius made in the following theorem.

Theorem 3.1. *Let S be a closed subset of an open set $\Omega \subset \mathbf{C}^n$, and let K_0 be a compact subset of S such that $S \setminus K_0$ is a C^1 manifold with no complex tangent. If there exists a holomorphically convex set $K_1 \subset S$ which is a neighborhood of K_0 relative to S, it follows that every compact set K with $K_0 \subset K \subset S$ is holomorphically convex. Moreover, for any neighborhood N of K_0 one can for all small $\varepsilon > 0$ find a domain of holomorphy ω_ε decreasing with ε so that*

(i) *ω_ε contains all points at distance $< \varepsilon/2$ from K,*

(ii) *all $z \in \omega_\varepsilon \cap \complement N$ have distance $< 2\varepsilon$ from S.*

Here we have called a compact set holomorphically convex if it has a fundamental system of neighborhoods which are domains of holomorphy.

PROOF. Let Ω_K be any open neighborhood of K which is contained in Ω. We shall construct domains of holomorphy containing K which are relatively compact in Ω_K. This will be done by fitting together small neighborhoods of K_1 which are domains of holomorphy with tubular neighborhoods of S away from K_0. The crucial fact is that an open set which is locally a domain of holomorphy is indeed a domain of holomorphy (see e.g. Hörmander [3, Theorems 2.6.10 and 4.2.8]).

Using Lemma 2.1 we choose a function $\varrho \in C^2(\Omega \setminus K_0)$ such that $\varrho(z) \neq 0$ when $z \notin S$ and for every $\zeta \in S \setminus K_0$

$$\varrho(z) - d(z,T_\zeta)^2 = o(|z-\zeta|^2) \quad \text{as } z \to \zeta .$$

This implies that for some open set U with $S \setminus K_0 \subset U \subset \Omega \setminus K_0$ we have

$$\tfrac{1}{2} d(z,S)^2 \leqq \varrho(z) \leqq 2\,d(z,S)^2, \quad z \in U . \tag{2.1}$$

We can write

$$d(z,T_\zeta)^2 = H_\zeta(z-\zeta) + \operatorname{Re} A_\zeta(z-\zeta)$$

where H_ζ is a hermitian symmetric form and A_ζ is an analytic polynomial. Since no complex line lies in T_ζ we obtain for every $w \neq 0$ in $\mathbf{C}^n$

$$0 < d(w,T_\zeta)^2 + d(iw,T_\zeta)^2 = 2H_\zeta(w) .$$

Thus the Levi form $H_\zeta(w)$ of ϱ at ζ is positive definite. For reasons of continuity this is still true in a neighborhood of $S \setminus K_0$. Shrinking the open set U if necessary we may therefore assume that ϱ is strictly plurisubharmonic in U.

Choose compact sets K', K'' and open sets $\Omega', \Omega'', \Omega'''$ so that

$$K_0 \Subset K' \Subset \Omega' \Subset K'' \Subset \Omega'' \Subset \Omega''' \Subset \Omega_K \cap N; \qquad \Omega''' \cap K_1 = \Omega''' \cap S . \tag{2.2}$$

Here N is the neighborhood of K_0 occurring in the statement of the theorem. Then it follows that for some constant C

$$d(z,K_1)^2 \leqq C\varrho(z), \quad z \in \Omega'' \setminus K'' . \tag{2.3}$$

In fact, in a neighborhood of the closure of $\Omega'' \setminus K''$ the functions $d(z,K_1)^2$ and $\varrho(z)$ have the same zeros and $d(z,K_1) = d(z,S)$ near the zeros, so (2.3) follows from (2.1).

The sets

$$\Omega_0 = \Omega', \quad \Omega_1 = \Omega'' \setminus K', \quad \Omega_2 = \Omega_K \setminus K''$$

are all open, and

$$\Omega_K = \Omega_0 \cup \Omega_1 \cup \Omega_2, \quad \Omega_0 \cap \Omega_2 = \emptyset . \tag{2.4}$$

We shall define our domains of holomorphy by different conditions in each of these three open sets, making sure that the various conditions agree in the intersections $\Omega_0 \cap \Omega_1$ and $\Omega_1 \cap \Omega_2$.

Choose two functions $\varphi, \psi \in C_0^\infty(U \cap \Omega_K)$ such that

$$0 \leqq \varphi \leqq 1, \quad \varphi = 1 \text{ in a neighborhood of } K \setminus \Omega', \tag{2.5}$$

$$0 \leqq \psi \leqq 1, \quad \psi = 0 \text{ in } \Omega_2, \quad \psi = 1 \text{ in } \{z; z \in \Omega' \setminus K', \varrho(z) < \eta\}, \tag{2.6}$$

for some $\eta > 0$. Since $K \setminus \Omega'$ is a compact subset of $U \cap \Omega_K$ there exist functions in $C_0^\infty(U \cap \Omega_K)$ satisfying (2.5). Similarly one can find ψ with the required properties since $\{z; z \in \Omega' \setminus K', \varrho(z) < \eta\} \Subset U \cap \Omega_K$ for sufficiently small η while Ω_2 and $\Omega' \setminus K'$ have disjoint closures.

Let c be a positive number such that

$$p_{\varepsilon,\delta}(z) = \varrho(z) - \varepsilon^2 \varphi(z) - \delta^2 \psi(z)$$

is strictly plurisubharmonic in $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$ when $0 \leqq \varepsilon \leqq c$ and $0 \leqq \delta \leqq c$. Then

$$\{z;\ z \in \Omega_K \cap U,\ p_{\varepsilon,\delta}(z) < 0\}$$

is a domain of holomorphy for such ε and δ, and is relatively compact in $\Omega_K \cap U$ since it is contained in $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$.

Let ω_1 be a domain of holomorphy containing K_1 such that

$$\varrho(z) < \min(\eta, c^2) \quad \text{when} \quad z \in \omega_1 \cap (\Omega' \setminus K'), \tag{2.7}$$

and define ω_ε as the set of points $z \in \Omega_K$ such that

$$\begin{array}{lll} p_{\varepsilon,c}(z) < 0 & & \text{if } z \in \Omega_2, \\ p_{\varepsilon,c}(z) < 0 & \text{and } z \in \omega_1 & \text{if } z \in \Omega_1, \\ z \in \omega_1 & & \text{if } z \in \Omega_0 . \end{array} \tag{2.8}$$

We shall prove in a moment that these definitions are compatible in the sets $\Omega_1 \cap \Omega_2$ and $\Omega_0 \cap \Omega_1$ when ε is sufficiently small. But first we note that since $\{z; p_{\varepsilon,c}(z) < 0\} \subset \operatorname{supp}\varphi \cup \operatorname{supp}\psi \Subset \Omega_K$, and since $\Omega_0 \cup \Omega_1 = \Omega'' \Subset \Omega_K$, it is clear that $\omega_\varepsilon \Subset \Omega_K$. In view of the remark made at the beginning of the proof we conclude that ω_ε is a domain of holomorphy, for every boundary point has a neighborhood V such that $\omega_\varepsilon \cap V$ is a domain of holomorphy.

Since $K \cap \Omega'' \subset S \cap \Omega'' = K_1 \cap \Omega'' \subset K_1 \Subset \omega_1$ and since $\varphi = 1$ in a neighborhood of $K \setminus \Omega'$ by (2.5), we have $K \subset \omega_\varepsilon$. In fact, the $\frac{1}{2}\varepsilon$ neighborhood of K belongs to ω_ε for small ε since $\varrho(z) \leqq 2d(z,K)^2$ in a neighborhood of $K \setminus \Omega'$. On the other hand, (2.9) $$\varrho(z) < \varepsilon^2\varphi(z) \leqq \varepsilon^2 \quad \text{if} \quad z \in \Omega_2 \text{ and } p_{\varepsilon,c}(z) < 0 ,$$

which in view of (2.1) proves that $\omega_\varepsilon \cap \complement N \subset \omega_\varepsilon \cap \Omega_2$ is at distance $\leqq 2\varepsilon$ from S for small ε.

It remains to verify that the definitions (2.8) are compatible for small ε. First assume that $z \in \Omega_1 \cap \Omega_2 = \Omega'' \setminus K''$ and that $p_{\varepsilon,c}(z) < 0$. By (2.9) we have $\varrho(z) < \varepsilon^2$, so we may conclude using (2.3) that $d(z,K_1)^2 < C\varepsilon^2$. For small ε this implies that $z \in \omega_1$. Hence the first two definitions (2.8) are compatible in $\Omega_1 \cap \Omega_2$.

Next let $z \in \Omega_1 \cap \Omega_0 = \Omega' \setminus K'$, and let $z \in \omega_1$. Then we know by (2.7) that $\varrho(z) < \eta$, so it follows from (2.6) that $\psi(z) = 1$. Using (2.7) again we obtain

$$\varrho(z) < c^2 \leqq \varepsilon^2\varphi(z) + c^2\psi(z) ,$$

that is, $p_{\varepsilon,c}(z) < 0$. This completes the proof of the consistency of (2.8) and thus the proof of Theorem 3.1.

Theorem 3.1 would have been false if we had only assumed that K_0 is holomorphically convex and not that there is a holomorphically convex neighborhood K_1 of K_0 in S. In fact, we have

EXAMPLE 3.2. Let $S = \{z;\, z \in \mathbf{C}^2,\ |z|^2 = 1,\ \operatorname{Im} z_2 = 0\}$ and let $K_0 = \{(0,1), (0,-1)\}$. Then $S \setminus K_0$ has no complex tangents and K_0 is holomorphically convex. However, every domain of holomorphy containing S also contains the convex hull of S by Hartog's theorem, so S is not holomorphically convex. (We owe this example to E. Bishop.)

4. An approximation theorem.

The purpose of this section is to prove the following somewhat more general version of Theorem 1.1.

THEOREM 4.1. *Let the hypotheses be as in Theorem* 3.1 *and assume in addition that $S \setminus K_0$ is of class C^r where r is so large that the dimension of $S \setminus K_0$ is at most $2r-2$. If $u \in C(K)$ and $u|_{K_0} \in A(K_0)$ it follows that $u \in A(K)$.*

It will be clear from the proof that the high differentiability assumptions are caused by the fact that we are considering approximation in maximum norms while our methods are based on L^2 estimates. If we only want to approximate in L^2 norms on K with respect to a smooth density (assuming that K lies on a manifold) it would be sufficient to assume that $S \setminus K_0$ is of class C^1. We do not know if such an improvement is possible in Theorem 4.1. However, it would not be difficult to relax the hypothesis that $\dim S \setminus K_0 \leqq 2r-2$ to assuming that $\dim S \setminus K_0 \leqq 2r$. This requires that in the proof of Theorem 3.1 one also constructs a plurisubharmonic function which is uniformly bounded in ω_ε but has a lower bound for the Levi form in $\omega_\varepsilon \cap \complement N$ which is $\geqq \varepsilon^{-2}$. We leave for the reader to make these improvements, which also require that one uses the full force of Theorem 2.2.1′ in Hörmander [2].

In order to prove Theorem 4.1 we must first show that smooth functions on K can be extended to a neighborhood so that the Cauchy–Riemann equations are satisfied to a high order on K. This is a simple algebraic fact, analogous to solving the Cauchy problem for a differential equation in terms of formal power series, but for lack of a suitable reference we shall give a proof here.

LEMMA 4.2. *Let T be a real linear subspace of $\mathbf{C}^n$ containing no complex line, and let P_T be the ring of polynomials generated by the real linear forms which vanish on T. Then the equations*

$$\partial u/\partial \bar z_j = f_j, \quad j=1,\ldots,n\,, \tag{4.1}$$

have a solution $u \in P_T$ for all $f_j \in P_T$ such that

$$\partial f_j/\partial \bar z_k = \partial f_k/\partial \bar z_j, \quad j,k=1,\ldots,n\,. \tag{4.2}$$

If f_j are homogeneous of degree μ we can choose u homogeneous of degree $\mu+1$.

PROOF. A real basis for T is linearly independent over $\mathbf{C}$ and can therefore be extended to a complex basis for $\mathbf{C}^n$. We may therefore assume that T is given by the equations $z_1 = \ldots = z_k = 0$, $\operatorname{Re} z_{k+1} = \ldots = \operatorname{Re} z_n = 0$. Every $g \in P_T$ can then be uniquely written as a finite sum

$$g = \sum g^\alpha(x) z^\alpha, \quad \alpha = (\alpha_1,\ldots,\alpha_k)\,,$$

where g_α is a polynomial in $x = \operatorname{Re} z$, and conversely every such sum belongs to P_T. Since $\partial z^\alpha/\partial \bar z_j = 0$ consideration of each monomial z^α separately reduces the proof of the lemma to the case where f_j are polynomials in x alone. Since $\partial/\partial \bar z_j$ coincides with $\frac{1}{2}\partial/\partial x_j$ when acting on functions of x alone, the lemma now follows from the Poincaré lemma for functions of the real variables x; the last statement is obvious.

Note that the lemma would be false already for constants f_j if T contains a complex line.

LEMMA 4.3. *Let S be a closed subset of an open set $\Omega \subset \mathbf{C}^n$, and let K_0 be a compact subset of S such that $S \setminus K_0$ is a C^r manifold with no complex* *tangent*, $r \geqq 1$. *Let* $u \in C^r(\Omega)$ *and* $\bar\partial u = 0$ *in a neighborhood of* K_0. *Then one can find* $v \in C^r(\Omega)$ *so that* $v = u$ *on* S *and in a neighborhood of* K_0, *and in addition*

$$\bar\partial v(z) = o\big(d(z, S)^{r-1}\big) \quad \text{as } z \to S\,, \tag{4.3}$$

uniformly on compact subsets of S.

PROOF. By induction we shall prove that for every integer $s \leqq r$ there is a function $v \in C^r(\Omega)$ such that $v = u$ on S and in a neighborhood of K_0, and such that the derivatives of $\bar\partial v$ of order $< s$ vanish on S. When s is the largest integer $\leqq r$ this statement implies Proposition 4.3. Since it is obviously true for $s = 0$ we may assume that $s \geqq 1$ and that the statement is already proved for smaller values of s. Thus we may assume that $\bar\partial u$ vanishes of order $s-1$ on S, that is, that the derivatives of order $< s-1$ vanish there.

Let $\zeta \in S \setminus K_0$ and let T_ζ be the tangent space of S at ζ regarded as a real linear subspace of $\mathbf{C}^n$. If $\bar\partial u = f_1 d\bar z_1 + \ldots + f_n d\bar z_n$ we have by Taylor's formula and the hypothesis made above

$$f_j(\zeta + z) = f_j^0(z) + o(|z|^{s-1}) \quad \text{as } z \to 0\,,$$

where f_j^0 is a (non-analytic) polynomial of degree $s-1$ which belongs to P_{T_ζ} in the notations of Lemma 4.2. We claim that

$$\partial f_j^0 / \partial \bar z_k = \partial f_k^0 / \partial \bar z_j\,.$$

This is evident if $s = 1$ since f_j^0 is then a constant. When $s > 1$ the statement follows from the fact that

$$\partial f_j(\zeta + z)/\partial \bar z_k = \partial f_k(\zeta + z)/\partial \bar z_j$$

if we equate the terms of degree $s-2$ in the Taylor expansion with respect to z of the two sides. In view of Lemma 4.2 it follows that there exists a homogeneous polynomial $u_\zeta^0 \in P_{T_\zeta}$ of degree s such that

$$f_j^o = \partial u_\zeta^0 / \partial \bar z_j, \quad j = 1, \ldots, n\,,$$

which means that

$$\bar\partial\big(u(z) - u_\zeta^0(z - \zeta)\big)$$

vanishes of order s at ζ. Clearly u_ζ^0 is unique modulo analytic polynomials in the complex linear forms vanishing on T_ζ.

After studying the problem at a single point we shall now look at the local solution in a neighborhood of a point $\zeta_0 \in S \setminus K_0$. Writing $z_k = x_k + i x_{k+n}$ we may assume that $S \setminus K_0$ in a neighborhood of ζ_0 can be described by equations of the form

$$x_j = \varphi_j(x_1,\ldots,x_\nu), \quad j=\nu+1,\ldots,2n\,,$$

where $\nu = \dim S \setminus K_0$. If $\psi_j(x) = x_j - \varphi_j(x_1,\ldots,x_\nu)$, $j=\nu+1,\ldots,2n$, the equations $\psi_j(x)=0$ define $S\setminus K_0$ in a neighborhood of ζ_0, and for any $\zeta \in S\setminus K_0$ in a neighborhood of ζ_0 the differentials of the functions ψ_j form a basis for all linear forms vanishing on T_ζ. Hence it follows from the first part of the proof that there exists a homogeneous polynomial h_ζ^0 of degree s such that

$$\bar\partial\big(u(z) - h_\zeta^0(\psi_{\nu+1}(z),\ldots,\psi_{2n}(z))\big)$$

vanishes of order s at ζ. But this amounts to a system of linear equations for the coefficients of h_ζ^0 with rank independent of ζ, equal to the number of linearly independent homogeneous polynomials in $2n-\nu$ variables of degree s minus the number of linearly independent analytic polynomials in $n-\nu$ variables which are homogeneous of degree s. To proceed we now need the following well known fact:

Let V and W be two finite dimensional vector spaces and $Q_\zeta\colon V \to W$ be a linear transformation which is a C^r function of ζ in a neighborhood of a point ζ_0, with rank independent of ζ. If $\zeta \to w(\zeta) \in W$ is a C^r function and $w(\zeta)$ is in the range of Q_ζ for every ζ, it follows that one can find a C^r function v from a neighborhood of ζ_0 to V such that $Q_\zeta v(\zeta) = w(\zeta)$.

We leave the proof of the reader. Writing $\zeta_k = \xi_k + i\xi_{k+n}$ we conclude that we can take

$$h_\zeta^0(\psi_{\nu+1},\ldots,\psi_{2n}) = \sum_{|\alpha|=s} a_\alpha(\xi_1,\ldots,\xi_\nu)\,\psi^\alpha$$

for suitable $a_\alpha \in C^{r-s}$; here $\alpha = (\alpha_{\nu+1},\ldots,\alpha_{2n})$. By Lemma 2.2 it is possible to find $A_\alpha \in C^{r-s}$ in a neighborhood of ζ_0 so that with $\xi' = (\xi_1,\ldots,\xi_\nu)$

$$A_\alpha(\xi',\varphi(\xi')) = a_\alpha(\xi') \quad\text{and}\quad A_\alpha(x)\psi(x)^\alpha \in C^r\,.$$

In view of the remark following Lemma 2.2 the function

$$v(x) = u(x) - \sum_{|\alpha|=s} A_\alpha(x)\,\psi(x)^\alpha$$

will then have the required properties in a neighborhood of ζ_0, and

$$v(z) - u(z) = O\big(d(z,S)^s\big)\,. \tag{4.4}$$

Let now $\{\chi_i\}$ be a partition of unity in Ω, $i=0,1,\ldots$, such that $\chi_0 = 1$ in a neighborhood of K_0, $\bar\partial u = 0$ in a neighborhood of $\operatorname{supp}\chi_0$, and such that for $i \neq 0$ the preceding discussion gives a function v_i with the required properties in a neighborhood of $\operatorname{supp}\chi_i$. Set $v_0 = u$ and

$$v = \sum \chi_i v_i\,.$$

Then v is equal to u on S and in the neighborhood of K_0 where $\chi_0=1$. Noting that $\sum\bar\partial\chi_i=0$ we have

$$\bar\partial v = \sum \chi_i\bar\partial v_i+\sum v_i\bar\partial\chi_i = \sum \chi_i\bar\partial v_i+\sum (v_i-u)\bar\partial\chi_i ,$$

so it follows from (4.4) that v has the desired properties.

The last preparation for the proof of Theorem 4.1 is an elementary lemma which allows us to pass from L^2 estimates to estimates in the maximum norm.

Lemma 4.4. *Let $B_\varepsilon=\{z;\, z\in \mathbf{C}^n,\ |z|<\varepsilon\}$; let $u\in L^2(B_\varepsilon)$ and $\bar\partial u=f$ in the sense of distribution theory. If f is continuous it follows that u is continuous, and we have*

$$|u(0)| \leqq C(\varepsilon^{-n}\|u\|_{L^2(B_\varepsilon)} + \varepsilon \sup_{B_\varepsilon}|f|) . \tag{4.5}$$

Proof. It suffices to prove the a priori estimate (4.5) when $u\in C^\infty$. For if we apply it to regularizations of u in balls which are relatively compact in B_ε, it follows that u is continuous and that (4.5) is valid when u merely satisfies the hypotheses in the lemma. We may also assume that $\varepsilon=1$ in proving (3.5). Choose $\chi\in C_0^\infty(B_1)$ so that $\chi=1$ in $B_{\frac12}$. With E denoting the fundamental solution of the Laplacean in $\mathbf{R}^{2n}$, we have

$$u(0) = (\chi u)(0) = \int E(x)\,\Delta(\chi u)\,dx$$

$$= \int E(x)(\Delta\chi)(x)\,u(x)\,dx+2\int E(x)(\operatorname{grad}\chi, \operatorname{grad} u)\,dx+\int E(x)\chi\,\Delta u\,dx$$

$$= -4\int \sum f_j\partial(\chi E)/\partial z_j\,dx+\int (E(x)\Delta\chi(x)-2\operatorname{div}(E(x)\operatorname{grad}\chi(x))u(x)\,dx .$$

Here we have used that $\Delta u=4\sum\partial f_j/\partial z_j$. Since the first order derivatives of E are homogeneous functions of degree $1-2n$, hence locally integrable, we obtain the estimate (4.5).

Proof of Theorem 4.1. It suffices to prove the statement assuming that u can be extended to a function which belongs to C^r in all of $\mathbf{C}^n$ and is analytic in a neighborhood N of K_0. In view of Lemma 4.3 we may assume that

$$f = \bar\partial u = o\big(d(z,S)^{r-1}\big) .$$

If ω_ε is chosen according to Theorem 3.1 and ν is the dimension of $S\setminus K_0$, we obtain

$$\|f\|_{L^2(\omega_\varepsilon)} = o(\varepsilon^{r-1}\varepsilon^{\frac12(2n-\nu)}) ,$$

for the measure of $\omega_\varepsilon \cap \complement N$ is $O(\varepsilon^{2n-\nu})$ by condition (ii) in Theorem 3.1. In view of Theorem 2.2.3 in Hörmander [2] we conclude that there is a function $w_\varepsilon \in L^2(\omega_\varepsilon)$ with $\bar\partial w_\varepsilon = f$ in ω_ε and

$$\|w_\varepsilon\|_{L^2(\omega_\varepsilon)} \leqq C\|f\|_{L^2(\omega_\varepsilon)} = o(\varepsilon^{r-1}\varepsilon^{\frac{1}{2}(2n-\nu)}) .$$

The difference $u_\varepsilon = u - w_\varepsilon$ is then analytic in ω_ε, and by Lemma 4.4 we obtain when $z \in K$, using (i) of Theorem 3.1,

$$|w_\varepsilon(z)| \leqq C(\varepsilon^{-n}\|w_\varepsilon\|_{L^2(\omega_\varepsilon)} + \varepsilon\|f\|_{L^\infty(\omega_\varepsilon)}) = o(\varepsilon^{r-1-\frac{1}{2}\nu}) \quad \text{as } \varepsilon \to 0 .$$

Since we have assumed that $\nu \leqq 2(r-1)$ it follows that u is the uniform limit of u_ε on K.

5. Polynomial approximation.

Let X be a compact space and let $f_1,\ldots,f_k \in C(X)$. Denote by $[f_1,\ldots,f_k|X]$ the class of all functions which are uniform limits on X of polynomials in $f_1,\ldots,f_k$ with complex coefficients.

It is a general problem to give conditions on $f_1,\ldots,f_k$ and X assuring that $[f_1,\ldots,f_k|X] = C(X)$. Necessary and sufficient conditions are known only when $k=1$ (Mergelyan [5]). For arbitrary compact $X \subset \mathbf{C}^n$ the Stone–Weierstrass theorem gives

$$[z_1,\ldots,z_n,\bar z_1,\ldots,\bar z_n \mid X] = C(X) .$$

We are concerned here with a certain perturbation of this result.

Let $R_1,\ldots,R_n$ be functions in $C^{n+1}(N)$ for some neighborhood N of a compact set X in $\mathbf{C}^n$. We write R for the vector valued function $(R_1, \ldots,R_n)$ from N to $\mathbf{C}^n$. If $z=(z_1,\ldots,z_n) \in \mathbf{C}^n$ we write

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} .$$

The following is Theorem 1.2 of the introduction.

THEOREM 5.1. *Assume that there is a constant $k<1$ such that*

$$|R(z)-R(z')| \leqq k|z-z'| \quad \textit{when } z,z' \in N . \tag{5.1}$$

Then it follows that

$$[z_1,\ldots,z_n,\bar z_1+R_1(z),\ldots,\bar z_n+R_n(z) \mid X] = C(X) .$$

Note that if we allow $k=1$ in (5.1) the assertion of the theorem may become false. Take for example $n=1$, $X=$ the closed unit disk and $R(z) = -\bar z$.

Let Φ denote the map of N into C^{2n} defined by

$$\Phi(z) = (z, \bar{z} + R(z)) ,$$

and let Σ be the image of N under Φ. Evidently Σ is a C^{n+1} submanifold of C^{2n}.

LEMMA 5.2. *The submanifold Σ has no complex tangents.*

PROOF. If Σ has a complex tangent, then there exist two tangent vectors to Σ differing only by a factor i. Hence one can find $\xi, \eta \in \mathsf{C}^n$ different from 0 so that at some point of N

$$d\Phi(\eta) = i d\Phi(\xi) . \tag{5.2}$$

If we write R_z for the $n \times n$ matrix whose (j,k)-th entry is $\partial R_j / \partial z_k$ and define $R_{\bar{z}}$ similarly, the equations (5.2) can be written

$$(\eta,\, \bar{\eta} + R_z \eta + R_{\bar{z}} \bar{\eta}) = i(\xi, \bar{\xi} + R_z \xi + R_{\bar{z}} \bar{\xi}) .$$

Hence $\eta = i\xi$ and

$$\bar{\xi} + R_{\bar{z}} \bar{\xi} = 0 . \tag{5.3}$$

Now we have by Taylor's formula

$$R(z + \varepsilon\theta) - R(z) = R_z \varepsilon \theta + R_{\bar{z}} \varepsilon \bar{\theta} + o(\varepsilon)$$

as $\varepsilon \to 0$ through real values for fixed $z \in N$ and $\theta \in \mathsf{C}^n$. Hence it follows from (5.1) that

$$|R_z \theta + R_{\bar{z}} \bar{\theta}| \leqq k|\theta| . \tag{5.4}$$

Replacing θ by $i\theta$ gives

$$|R_z \theta - R_{\bar{z}} \bar{\theta}| \leqq k|\theta| . \tag{5.4$'$}$$

Combining (5.4) and (5.4)$'$ we get

$$|R_{\bar{z}} \bar{\theta}| \leqq k|\theta| \quad \text{for all } \theta \in \mathsf{C}^n , \tag{5.5}$$

which contradicts (5.3) if $\xi \neq 0$. Hence Σ has no complex tangent, as claimed.

LEMMA 5.3. *The set $X^* = \Phi(X)$ is a polynomially convex compact set in C^{2n}.*

PROOF. Denote the coordinates in C^{2n} by $z_1, \ldots, z_{2n}$, and let

$$A = [z_1, \ldots, z_n, \bar{z}_1 + R_1(z), \ldots, \bar{z}_n + R_n(z) \mid X], \quad A^* = [z_1, \ldots, z_{2n} \mid X^*] .$$

The map Φ induces an isomorphism between A and A^*. To show that X^* is polynomially convex in $\mathbf{C}^{2n}$ is equivalent to showing that every homomorphism of A^* into $\mathbf{C}$ is evaluation at a point of X^*, and so to the corresponding statement about A and X.

Let h be a homomorphism of A into $\mathbf{C}$. Choose a probability measure μ on X representing h, that is, so that

$$h(f) = \int_X f\,d\mu, \quad f \in A .$$

Put $h(z_i) = \alpha_i$, $i = 1,\ldots,n$, and $\alpha = (\alpha_1,\ldots,\alpha_n)$. Choose an extension of R to a map from $\mathbf{C}^n$ to $\mathbf{C}^n$ such that (5.1) remains valid for $z, z' \in \mathbf{C}^n$. This can be done by a result of Valentine [8].

Define, for all $z \in X$,

$$f(z) = \sum_{i=1}^{n} (z_i - \alpha_i)\big((\bar{z}_i + R_i(z)) - (\bar{\alpha}_i + R_i(\alpha))\big) .$$

Since $z_i \in A$, $\bar{z}_i + R_i(z) \in A$, and the α_i and $R_i(\alpha)$ are constants, we have $f \in A$. Evidently $h(f) = 0$. Also, for $z \in X$,

$$f(z) = \sum_{i=1}^{n} |z_i - \alpha_i|^2 + \sum_{i=1}^{n} (z_i - \alpha_i)(R_i(z) - R_i(\alpha)) .$$

The modulus of the second sum is $\leqq |z-\alpha|\ |R(z) - R(\alpha)| \leqq k|z-\alpha|^2$ by (5.1). Hence $\operatorname{Re} f(z) > 0$ for $\alpha \neq z \in X$. On the other hand,

$$0 = \operatorname{Re} h(f) = \int_X \operatorname{Re} f\,d\mu .$$

It follows that $\alpha \in X$ and that μ is concentrated at α. Thus h is evaluation at α and we are done.

Proof of Theorem 5.1. We now know that X^* is a polynomially convex compact subset of Σ, and Σ is a C^{n+1} submanifold of $\mathbf{C}^{2n}$ of dimension $2n$ having no complex tangents. Hence $P(X^*) = A(X^*) = C(X^*)$ in virtue of the Oka–Weil approximation theorem and Theorem 4.1. But this means precisely that

$$[z_1,\ldots,z_n,\bar{z}_1 + R_1(z),\ldots,\bar{z}_n + R_n(z) \mid X] = C(X) .$$

One might be tempted by Theorem 5.1 to believe that if X is a compact set in some differentiable manifold, the set of all $\varphi_1,\ldots,\varphi_k \in C(X)$ such that $[\varphi_1,\ldots,\varphi_k | X] = C(X)$ is open, at least in some C^r topology.

This is not true, however, as is shown by Example 6.2 in the next section. On the other hand, we have

THEOREM 5.4. *Let X be a compact subset of a differentiable manifold M of dimension n and class $r' \geqq \max(2, 1+\frac{1}{2}n)$. Let $\varphi_1, \ldots, \varphi_k \in C^r(M)$ and assume that*

(i) $[\varphi_1, \ldots, \varphi_k | X] = C(X)$,

(ii) *the differential of the map $\varphi = (\varphi_1, \ldots, \varphi_k)$ from M to $\mathbf{C}^k$ extends to an injective map from the complexification of the tangent space at any point of M into $\mathbf{C}^k$; if $(x_1, \ldots, x_n)$ are local coordinates on M this means that $(\partial\varphi_i/\partial x_j)$, $i = 1, \ldots, k$; $j = 1, \ldots, n$, has rank n.*

Then it follows that $[\psi_1, \ldots, \psi_k | X] = C(X)$ for all $\psi_j \in C^r(M)$ sufficiently close to φ_j in the C^2 topology.

PROOF. Choose an open set M_1 with $X \subset M_1 \Subset M$ such that φ separates points in M_1. This is possible since φ separates points in X by (i) and since φ is a regular map by (ii) and so separates nearby points. Then all ψ in a C^2 neighborhood N of φ (in C^r) are also one-to-one and satisfy (ii) on M_1, which implies that $\psi(M_1)$ is a C^r manifold with no complex tangents. If $d_\psi(z)$ is the distance from $z \in \mathbf{C}^k$ to $\psi(M_1)$, it is easy to see that there is an open neighborhood Ω of $\varphi(X)$ and a C^2 neighborhood $N_1 \subset N$ of φ such that for all $\psi \in N_1$ and some positive constant c we have

a) $\{x \in M_1; \psi(x) \in \Omega\} \Subset M_1$,

b) $d_\psi^2 \in C^2(\Omega)$ and $\sum \partial^2 d_\psi^2/\partial z_j \partial \bar{z}_k\, w_j \bar{w}_k \geqq c|w|^2$ if $z \in \Omega$, $w \in \mathbf{C}^k$.

(See the beginning of the proof of Theorem 3.1.) Choose $\chi \in C_0^\infty(\Omega)$ so that $0 \leqq \chi \leqq 1$ and $\chi = 1$ in a neighborhood of $\psi(X)$ if ψ belongs to some C^2 neighborhood $N_2 \subset N_1$ of φ. From b) it follows for some $\delta > 0$ that $d_\psi^2 - \delta\chi$ is strictly plurisubharmonic in Ω when $\psi \in N_2$. The sets

$$\omega_{\psi,\varepsilon} = \{z \in \Omega; d_\psi(z)^2 < \delta\chi(z) - \varepsilon\}, \qquad 0 \leqq \varepsilon < \delta,$$

are thus domains of holomorphy having the Runge property relative to each other for fixed $\psi \in N_2$. They shrink to $\psi(M_1) \cap \chi^{-1}(1) \supset \psi(X)$ when $\varepsilon \nearrow \delta$. This is a compact set in view of a), and it belongs to $\psi(M_1)$ which we recall is a C^r manifold of dimension n with no complex tangent. Hence it follows from Theorem 4.1 that all continuous functions on $\psi(X)$ can be approximated uniformly by functions analytic in $\omega_{\psi,\varepsilon}$ for some $\varepsilon < \delta$, and so by functions analytic in $\omega_{\psi,0}$ by the Runge property.

Since $\varphi(X)$ is polynomially convex by (i), there exists a Runge domain U with $\varphi(X) \subset U \Subset \omega_{\varphi,0}$. For all ψ in a C^2 neighborhood $N_3 \subset N_2$ of φ we still have $\psi(X) \subset U \Subset \omega_{\psi,0}$. If $\psi \in N_3$ we have proved that every continuous function f on $\psi(X)$ can be uniformly approximated by a function g

which is analytic in $\omega_{\psi,0} \supset U$; and g can then be approximated by a polynomial P uniformly on the compact set $\psi(X) \subset U$. Hence polynomials are uniformly dense among the continuous functions on $\psi(X)$, which means that

$$[\psi_1, \ldots, \psi_k \mid X] = C(X), \quad \psi \in N_3 .$$

The proof is complete.

Remark. The hypothesis that ψ is close to φ in C^2 can be relaxed.

6. Examples.

Let K be a compact set in $\mathbf{C}^n$. Assume

a) K lies on a smooth submanifold of $\mathbf{C}^n$ without complex tangents,

b) K is topologically a cell.

If $\dim \Sigma = 1$ then a) and b) together imply that $P(K) = C(K)$. In other words, $P(I) = C(I)$ when I is a smooth Jordan arc in $\mathbf{C}^n$. (See Stolzenberg [7] and the references there.) Of course, when $\dim \Sigma = 1$, the absence of complex tangents is automatic.

Do a) and b) imply that $P(K) = C(K)$ in general? The following example shows that the answer is negative for $\dim \Sigma = 2$.

Example 6.1. For $z \in \mathbf{C}$ put $f(z) = -(1+i)\bar{z} + iz\bar{z}^2 + z^2\bar{z}^3$. Denote by Σ and K the images of $\mathbf{C}$ and the unit disk $\{z;\, |z| \leqq 1\}$ under the map $z \to (z, f(z)) \in \mathbf{C}^2$. Then Σ has no complex tangents, K is a disk but $P(K) \neq C(K)$.

In fact, $\partial f/\partial \bar{z}$ must have a zero if Σ has a complex tangent. But, setting $r^2 = z\bar{z}$, we have

$$\partial f/\partial \bar{z} = -(1+i) + 2ir^2 + 3r^4$$

which is never 0. Since K contains all points $(z, 0)$ with $|z| = 1$, the polynomially convex hull of K contains the disk $\{(z,0),\, |z| \leqq 1\}$. Thus K is not polynomially convex and so $P(K) \neq C(K)$ as claimed.

Example 6.2. Let X be the unit disk $\{z;\, z \in \mathbf{C},\, |z| \leqq 1\}$. Fix an integer $k > 0$. We claim that

(i) $[z, \bar{z}^{2k} \mid X] = C(X)$

(ii) there exist functions f arbitrarily close to $\bar{z}^{2k}$ in C^k norm such that

$$[z, f \mid X] \neq C(X) .$$

Set $A = [z, \bar{z}^{2k} \mid X]$. In order to prove (i) we note that since $\partial(\bar{z}^{2k})/\partial \bar{z} \neq 0$ on X except at a single point it suffices by Wermer [10] or Theorem 4.1

above to prove that every homomorphism $m: A \to \mathbf{C}$ is evaluation at some point of X. (See the proof of Theorem 5.1.) Put $m(z) = z_0$. Since m is represented by a positive measure on X we have $m(\bar{z}^{2k}) = \bar{z}_0^{2k}$. Hence for each polynomial Q

$$m(Q(z, \bar{z}^{2k})) = Q(z_0, \bar{z}_0^{2k}) ,$$

so that $m(h) = h(z_0)$ for all $h \in A$. Thus (i) holds.

To prove (ii) we note that since $\bar{z}^{2k}$ vanishes at 0 to an order $> k$ there exists a sequence $f_n \in C^k(X)$ such that $f_n = 0$ in some neighborhood U_n of 0 and $f_n \to \bar{z}^{2k}$ in C^k. Each element of $[z, f_n | X]$ is analytic on U_n, so $[z, f_n | X] \neq C(X)$ for every n. Thus (ii) holds.

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