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IMMERSIONS OF COMPLEX FLAGMANIFOLDS

JØRGEN TORNEHAVE

In the following G denotes a compact connected Lie group. If H is any subgroup of G, we denote by $C(H)$ the centralizer of H in G. Moreover if X_0 is a non-zero element in the Lie algebra $\mathscr{L}(G)$ of G, and L is the one-parameter subgroup of G in the direction X_0, we put $C(X_0)=C(L)$. The purpose of this paper is to prove the following immersion theorem, where T is a maximal torus in G such that $X_0 \in \mathscr{L}(T)$:

THEOREM 1. *Let $X_0 \in \mathscr{L}(T)\setminus\{0\}$ belong to some singular hyperplane in $\mathscr{L}(T)$, and define S to be the intersection of the singular hyperplanes to which X_0 belongs. Then $G/C(X_0)$ can be immersed in R^{n-p}, where $n=\dim G$ and $p=\dim S$.*

An explanation of the terminology follows below. In case $G=U(n)$, Theorem 1 gives results on immersions of complex flagmanifolds (see Theorem 2 below).

1.

It is well known that the subgroups $C(X_0)$ are connected (they are precisely the centralizers of tori in G), and that the adjoint representation gives an embedding

$$G/C(X_0) \xrightarrow{\varphi} \mathscr{L}(G)\ ,$$

defined by

$$\varphi(gC(X_0)) \;=\; (\mathrm{Ad}\,g)X_0\ .$$

In this way $G/C(X_0)$ becomes a submanifold of $\mathscr{L}(G)$ whose tangent space at the point $(\mathrm{Ad}\,g)X_0$ is

$$\{[X,\,(\mathrm{Ad}\,g)X_0] \mid X \in \mathscr{L}(G)\}\ . \tag{1}$$

Now equip $\mathscr{L}(G)$ with an inner product $(\,.\,,\,.\,)$ such that $\mathrm{Ad}\,g$ is an isometry of $\mathscr{L}(G)$ for every $g \in G$. Then the following relation holds for all X,Y,Z in $\mathscr{L}(G)$:

$$(X\,,[Y,Z]) \;=\; ([X,Y]\,,Z)\ . \tag{2}$$

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We choose an orthonormal basis in $\mathscr{L}(G)$ such that $\operatorname{Ad}|T$ with respect to this basis is described by the matrix

$$\operatorname{Exp} X \to \begin{pmatrix} \cos 2\pi\theta_1(X) & -\sin 2\pi\theta_1(X) & & & & & & \\ \sin 2\pi\theta_1(X) & \cos 2\pi\theta_1(X) & & & & * & & \\ & & \ddots & & & & & \\ & & & \cos 2\pi\theta_m(X) & -\sin 2\pi\theta_m(X) & & & \\ & * & & \sin 2\pi\theta_m(X) & \cos 2\pi\theta_m(X) & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \tag{3}$$

for $X \in \mathscr{L}(T)$. Here θ_ν are non-zero elements of $\operatorname{Hom}_{\mathsf{R}}(\mathscr{L}(T),\mathsf{R})$ which map $\mathscr{L}(T) \cap \operatorname{Exp}^{-1}(0)$ into Z, and the number of 1's is $l = \dim T$. We put

$$S_\nu = \{X \in \mathscr{L}(T) \mid \theta_\nu(X) = 0\} .$$

$S_1, S_2, \ldots, S_m$ are the so called *singular hyperplanes* in $\mathscr{L}(T)$. These are known to be mutually different (cf. Hopf [4]) and the Weyl group W as a finite group of isometries of $\mathscr{L}(T)$ is generated by the reflections in $S_1, S_2, \ldots, S_m$. The analytic subgroup U_ν of T with Lie algebra S_ν is an $(l-1)$-dimensional torus in T (Hopf [4]).

Suppose that X_0 lies in k of the singular hyperplanes, say $S_{\nu_1}, S_{\nu_2}, \ldots, S_{\nu_k}$, and put

$$S = \bigcap_{i=1}^{k} S_{\nu_i}, \qquad U = \bigcap_{i=1}^{k} U_{\nu_1} . \tag{4}$$

If $k=0$ it is understood that $S = \mathscr{L}(T)$ and $U = T$.

Lemma. *With the above notation we have $C(X_0) = C(U)$.*

Proof. The one-parameter subgroup L of T in the direction X_0 is contained in U, and therefore

$$C(U) \subseteqq C(L) = C(X_0) .$$

Since $C(X_0)$ is connected, we only have to prove that

$$\dim C(U) \geqq \dim C(X_0) .$$

That X_0 lies in exactly k singular hyperplanes implies that

$$\dim C(X_0) = l + 2k .$$

From the definition of U and (3) one concludes that

$$\dim C(U) \geqq l+2k .$$

PROPOSITION. *The normal bundle of $G/C(X_0)$ in $\mathscr{L}(G)$ has p linearly independent cross sections, where $p=\dim S$.*

PROOF. We will define a bundle homomorphism

$$G/C(X_0)\times S \xrightarrow{\psi} \bar{\nu} ,$$

where $\bar{\nu}$ is the normal bundle of $G/C(X_0)$ in $\mathscr{L}(G)$, by

$$\psi((\mathrm{Ad}g)X_0, Y) = ((\mathrm{Ad}g)X_0, (\mathrm{Ad}g)Y) .$$

To see that ψ is well defined, we first prove that, if $h \in C(X_0)$, then $(\mathrm{Ad}h)Y=Y$, but this is true since $h \in C(U)$ and $Y \in \mathscr{L}(U)$. Secondly we prove that $(\mathrm{Ad}g)Y$ is orthogonal to the tangent space (1). This follows from (2):

$$\begin{aligned}([X, (\mathrm{Ad}g)X_0]\,{}_{\epsilon}\,(\mathrm{Ad}g)Y) &= (X\,{}_{\epsilon}\,[\mathrm{Ad}g)X_0, (\mathrm{Ad}g)Y]) \\ &= (X\,{}_{\epsilon}\,(\mathrm{Ad}g)[X_0, Y]) = 0 ,\end{aligned}$$

since X_0 and Y belong to the abelian Lie algebra $\mathscr{L}(T)$. Obviously ψ defines p linearly independent cross sections of $\bar{\nu}$.

As a corollary we have the following theorem of Borel–Hirzebruch [1].

COROLLARY. *G/T is a π-manifold.*

PROOF. If $X_0 \notin S_\nu$ for all ν, we have $C(X_0)=T$ and $S=\mathscr{L}(T)$, so ψ becomes a trivialization of $\bar{\nu}$.

Now we can prove Theorem 1. The main tool is the following theorem of Hirsch [3]:

THEOREM (Hirsch). *Let M^n be a C^∞-manifold of dimension n and $\tau(M^n)$ its tangent bundle. If η is a real k-dimensional vector bundle ($k \geqq 1$) over M^n such that $\tau(M^n)\oplus\eta$ is trivial, then M^n can be immersed in euclidean space R^{n+k} of dimension $n+k$.*

Hirsch also proved, that the immersion can be chosen to have η as normal bundle, but the above will be sufficient.

PROOF OF THEOREM 1. If X_0 belongs to at least one singular hyperplane we have a positive dimensional subbundle η of $\bar{\nu}$ complementary to the trivial p-dimensional subbundle given by the proposition. Thus

$$\tau \oplus \eta \oplus p = n$$

from which one can conclude that

$$\tau \oplus \eta = n - p .$$

Now we apply the theorem of Hirsch to get Theorem 1.

The above theorem of Borel–Hirzebruch and the theorem of Hirsch show that G/T can be immersed in R^{k+1}, where $k = \dim(G/T)$.

If X_0 belongs to only one singular hyperplane, $C(X_0)$ has dimension $l+2$ and S dimension $l-1$. In this case Theorem 1 shows that $G/C(X_0)$ is immersible in euclidean space with codimension 3.

2.

Now we shall apply Theorem 1 in the case $G = U(n)$. As the maximal torus T we take the diagonal matrices and $\mathscr{L}(T)$ is identified with R^n in the obvious way. The singular hyperplanes in R^n are then given by

$$S_{ij} = \{(x_1,\ldots,x_n) \mid x_i = x_j\}, \qquad i > j .$$

Let $X_0 \in \mathsf{R}^n$ have the first n_1 coordinates equal to y_1, the next n_2 coordinates equal to y_2 and so on. The last n_q coordinates are then equal to y_q, $\sum n_i = n$, and we assume that $y_1,\ldots,y_i,\ldots,y_q$ are mutually distinct. Then an easy matrix calculation shows that

$$C(X_0) = U(n_1) \times U(n_2) \times \ldots \times U(n_q) .$$

Moreover S is the subspace of R^n given by the condition, that the first n_1 coordinates are equal, the next n_2 coordinates are equal and so on. Since the dimension of S is q, we get

THEOREM 2. *The complex flagmanifold*

$$W(n; n_1,\ldots,n_q) = U(n)/U(n_1) \times U(n_2) \times \ldots \times U(n_q), \qquad n = \sum n_i ,$$

where $q \leqq n-1$, can be immersed in R^{n^2-q}.

If $q = n$ it is a π-manifold and can then be immersed with codimension 1.

The flagmanifold $W(n; n_1,\ldots,n_q)$ has dimension

$$n^2 - \sum_{i=1}^{q} n_i^2 ,$$

and the theorem gives an immersion in euclidean space of codimension

$$\sum_{i=1}^{q} (n_i^2 - 1) .$$

Notice that the manifolds

$$W(n+k; n_1, \ldots, n_q, 1, 1, \ldots, 1), \qquad k \text{ numbers } 1 ,$$

are immersible in euclidean space with a codimension independent of k, whereas the dimensions of the manifolds tends to infinity as $k \to \infty$.

REFERENCES

1. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces* III, Amer. J. Math. 82 (1960), 491–504.
2. R. Bott, *Morse theory and its application to homotopy theory,* (Mimeographed, Notes by A. van de Ven), Bonn, 1960.
3. M. W. Hirsch, *Immersions of manifolds,* Trans. Amer. Math. Soc. 93 (1959), 242–276.
4. H. Hopf, *Maximale Toroide und singuläre Elemente in geschlossenen Lieschen Gruppen,* Comment. Math. Helv. 15 (1942–43), 59–70.

UNIVERSITY OF AARHUS, DENMARK