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FUNCTIONALS ON THE SPACE OF SOLUTIONS TO A DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS. THE FOURIER AND BOREL TRANSFORMATIONS

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0. Introduction.

A classical theorem of Pólya [9] relates estimates for an entire function F of exponential type in one variable with analytic continuations of a function, the Borel transform of F, which is defined in a neighborhood of infinity. It is natural to regard this as a result connecting on one hand the behavior of the Fourier transform

$$\hat{\mu}(\zeta_1,\zeta_2) = \mu\big((x_1,x_2) \mapsto e^{-i(x_1\zeta_1+x_2\zeta_2)}\big)$$

of a measure or distribution μ for vectors (ζ_1,ζ_2) satisfying

$$\zeta_1+i\zeta_2 = 0 ,$$

and on the other hand the location of the singularities of the potential U_μ of μ,

$$U_\mu(z) = \mu\left(w \mapsto \frac{1}{\pi(z-w)}\right), \qquad z=x_1+ix_2 \in \mathsf{C},\ |z| \text{ large} .$$

In both cases we need to know only how μ operates on the analytic functions.

Similarly, the results of V. K. Ivanov [6] concerning the singularities of the potential

$$U_\mu(x) = \mu\left(y \mapsto \frac{1}{4\pi|x-y|}\right), \qquad x \in \mathsf{R}^3,\ |x| \text{ large} ,$$

of a measure μ in R^3 can be interpreted as giving a relation between the Fourier transform $\hat{\mu}$ and the "Borel transform" U_μ of a functional on the space of harmonic functions in R^3, that is, the space of solutions to

$$\Delta u = \sum_1^3 \frac{\partial^2 u}{\partial x_j^{\,2}} = 0 .$$

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Ivanov's results have been generalized by Kozmanova and Mesis [7] to solutions in R^3 of the equation

$$\Delta u + cu = 0 ,$$

where c is an arbitrary complex number.

The purpose of the present paper is to prove that the Fourier transform $\hat{T}$ and the Borel transform, or outer potential, U_T of T are connected in the expected manner if T is a continuous linear functional on any of the spaces

$$\mathscr{D}'_P = \{u \in \mathscr{D}'(\mathsf{R}^n) \; ; \; P(-D)u = 0\}, \quad \mathscr{E}_P = \{u \in \mathscr{E}(\mathsf{R}^n) \; ; \; P(-D)u = 0\} ,$$

of distribution or C^∞ solutions to a differential equation with constant coefficients. Roughly speaking, if all derivatives $D^\alpha \hat{T}(\zeta)$ of $\hat{T}$, wherever they are defined, are bounded for all $\varepsilon > 0$ by an expression

$$C_\varepsilon (1 + |\zeta|)^{N_\varepsilon} \exp(\sup_{x \in K} \langle x, \operatorname{Im} \zeta \rangle + \varepsilon |\operatorname{Im} \zeta|) ,$$

where K is a convex compact set, then U_T can be continued as a solution of $P(D)V = 0$ to the complement of K. (The converse holds trivially.) We express the latter condition by introducing the notion of carrier. We thus obtain an (incomplete) analogue of the Paley–Wiener theorem for functionals on $\mathscr{D}'_P$ or $\mathscr{E}_P$ which by an immediate generalization extends to functionals on the space of solutions in an arbitrary convex open subset of R^n. For precise statements see Theorem 4.1. In somewhat vague terms this result can be described as follows: If $\hat{T}$ is the restriction of a function in $\hat{\mathscr{E}}'$ to the set of zeros of P considered as an algebraic variety with multiplicities, then there exists an extension of $\hat{T}$ to all of C^n preserving a given bound for $\hat{T}$. Our result is therefore a particular instance of Ehrenpreis' fundamental principle. It is hoped that the paper may nevertheless be of some interest in view of the fact that the method of proof is completely elementary, the main tool being the estimates for residue integrals given in Section 3.

I am indebted to Lennart Carleson for pointing out that the original proof of Theorem 4.1 could be significantly simplified.

Notation. We shall use the standard notation of distribution theory. Thus, for example, $C^\infty(\Omega) = \mathscr{E}(\Omega)$ will denote the space of arbitrarily many times continuously differentiable functions in an open set Ω in R^n; $C_0^\infty(\Omega) = \mathscr{D}(\Omega)$ is the subspace of functions vanishing outside some compact subset of Ω. With their usual topologies, these spaces have duals $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$, respectively. We shall often write $\mathscr{D}'$ for $\mathscr{D}'(\mathsf{R}^n)$, etc. For the definition of the topologies in all of these spaces see Schwartz

[10]. We write $\operatorname{supp} u$ for the support of u, the smallest closed set outside which the function or distribution u vanishes. It is convenient to denote by $\mathscr{D}(M)$ the set of all functions in $\mathscr{D}$ whose supports are contained in an arbitrary set $M \subset \mathsf{R}^n$. We shall also use the symbol $\check{u}$, defined by $\check{u}(x) = u(-x)$ for functions and by $\check{u}(\varphi) = u(\check{\varphi})$, $\varphi \in \mathscr{D}$, for distributions. If A and B are sets in R^n, we denote by $A+B$ their vector sum $\{x+y; x \in A, y \in B\}$. In contradistinction to this, $A \setminus B$ will stand for the set $A \cap \complement B$ where $\complement B$ is the complement of B. We shall often regard C^n as the complex dual of R^n and write $\langle x, \zeta \rangle = \sum_1^n x_j \zeta_j$ for the bilinear pairing of these spaces.

1. Preliminaries.

Let us equip the space

$$\mathscr{D}'_P = \{u \in \mathscr{D}'(\mathsf{R}^n)\ ;\ P(-D)u = 0\} \tag{1.1}$$

of all distribution solutions to a constant coefficient differential equation with the topology induced by the strong topology in $\mathscr{D}'(\mathsf{R}^n) = \mathscr{D}'$. Here $P(-D) = P(-D_1, \ldots, -D_n)$ where P is a polynomial and $D_j = -i\,\partial/\partial x_j$. Then a continuous linear form T on $\mathscr{D}'_P$ can always be extended to all of $\mathscr{D}'$ in view of the Hahn–Banach theorem; hence there exists a function $\varphi \in \mathscr{D}$ such that

$$T(u) = u(\varphi) \quad \text{for every } u \in \mathscr{D}'_P \tag{1.2}$$

(reflexivity of $\mathscr{D}$, see Schwartz [10, p. 75]). We shall say that φ represents T. Two functions φ_1 and φ_2 represent the same functional if and only if

$$u(\varphi_1 - \varphi_2) = 0 \quad \text{for every } u \in \mathscr{D}'_P\,. \tag{1.3}$$

This is obviously true if $\varphi_1 - \varphi_2 = P(D)\psi$ for some function $\psi \in \mathscr{D}$. Conversely, if (1.3) holds, a well-known theorem of Malgrange [8] shows that there exists a function $\psi \in \mathscr{D}$ such that $\varphi_1 - \varphi_2 = P(D)\psi$.

In analogy with this we also consider functionals on the space of C^∞ solutions to $P(-D)u = 0$,

$$\mathscr{E}_P = \{u \in \mathscr{E}(\mathsf{R}^n)\ ;\ P(-D)u = 0\}\,. \tag{1.4}$$

Here $\mathscr{E}_P$ has the topology induced by $\mathscr{E}(\mathsf{R}^n) = \mathscr{E}$ which implies that every continuous linear functional T on $\mathscr{E}_P$ may be extended to a distribution $\mu \in \mathscr{E}'$:

$$T(u) = \mu(u) \quad \text{if } u \in \mathscr{E}_P\,.$$

Again, we shall say that μ represents T if this holds. It follows that μ_1 and μ_2 represent the same functional if and only if $\mu_1 - \mu_2 = P(D)\nu$ for some $\nu \in \mathscr{E}'$. In short, the Malgrange theorem shows that $(\mathscr{E}_P)' = \mathscr{E}'/P(D)\mathscr{E}'$ and that $(\mathscr{D}'_P)' = \mathscr{D}/P(D)\mathscr{D}$, a result which remains valid in a $P(-D)$-convex open subset of R^n (see Definition 3.5.1 in Hörmander [5]).

Definition 1.1. *We shall say that a continuous linear form T on $\mathscr{D}'_P$ or $\mathscr{E}_P$ (in short, a functional T) is carried by a compact set $K \subset \mathsf{R}^n$ if for every neighborhood L of K there exists a representative of T with support contained in L.*

Every functional on $\mathscr{D}'_P$ defines a functional on $\mathscr{E}_P$ by restriction, for the topology in $\mathscr{E}_P$ is stronger than that induced by $\mathscr{D}'_P$. Conversely, let S be a functional on $\mathscr{E}_P$. Then we may define $S*\varphi$ for $\varphi \in \mathscr{D}$ by

$$(S*\varphi)(u) = S(\check{\varphi}*u), \quad u \in \mathscr{D}'_P .$$

This means that $\mu*\varphi$ represents $S*\varphi$ if μ represents S. It follows that $S*\varphi$ is continuous in $\mathscr{D}'_P$ for convolution with a fixed function in $\mathscr{D}$ maps $\mathscr{D}'_P$ continuously into $\mathscr{E}_P$. It is also clear that $S*\varphi$ is carried by $K+\operatorname{supp}\varphi$ if S is carried by K.

When P is hypoelliptic, that is, when $\mathscr{D}'_P \subset \mathscr{E}$, we can express the notion of carrier in several equivalent ways:

Lemma 1.2. *Suppose that P is hypoelliptic. Then $\mathscr{D}'_P = \mathscr{E}_P$ with identity also of the topologies. The following three conditions on a compact set $K \subset \mathsf{R}^n$ and a functional T on $\mathscr{D}'_P = \mathscr{E}_P$ are equivalent.*

(i) *K carries T as a functional on $\mathscr{D}'_P$, that is, T has representatives in $\mathscr{D}$ with support in an arbitrarily prescribed neighborhood of K.*
(ii) *For every neighborhood L of K there is a constant C such that $|T(u)| \leqq C\sup_L |u|$ for every $u \in \mathscr{D}'_P = \mathscr{E}_P$.*
(iii) *K carries T as a functional on $\mathscr{E}_P$, that is, for every compact neighborhood L of K there exists a representative $\mu \in \mathscr{E}'$ of T with $\operatorname{supp}\mu \subset L$.*

Proof. Let E be a fundamental solution for $P(D)$, that is, a distribution E in R^n such that $P(D)E = \delta$, the Dirac measure at the origin. Then the restriction of E to $\mathsf{R}^n \setminus \{0\}$ is a C^∞ function since $P(D)$ is hypoelliptic. Define

$$\varphi = P(D)(1-f)E, \tag{1.5}$$

where $f \in \mathscr{D}$ and $f = 1$ near 0. Then $(1-f)E$ is in $\mathscr{E}$, hence $\varphi \in \mathscr{D}$. The map

$$\mathscr{D}'_P \ni u \mapsto u*\check{\varphi} \in \mathscr{E}_P$$

which is always continuous is equal to the identity in view of our choice of φ. In fact,

$$u-u*\check{\varphi} = u*P(-D)\check{E}-u*\check{\varphi} = u*P(-D)\check{f}\check{E} = P(-D)u*\check{f}\check{E}\ ,$$

where the last expression is zero if $P(-D)u=0$. Hence the topology in $\mathscr{D}'_P$ is stronger than that in $\mathscr{E}_P$ which proves that they are in fact equally strong. In particular, the continuous functionals on $\mathscr{D}'_P$ and $\mathscr{E}_P$ are the same.

A slight modification of this argument shows that (iii) implies (i). In fact, suppose that (iii) holds and that L is an arbitrary neighborhood of K. Choose $\varepsilon>0$ such that $K+B_{2\varepsilon}\subset L$ where $B_\varepsilon=\{x\in \mathsf{R}^n;\ |x|\leqq\varepsilon\}$. Then $K+B_\varepsilon$ is a neighborhood of K and we can by assumption find $\mu\in\mathscr{E}'$ such that μ represents T and $\operatorname{supp}\mu\subset K+B_\varepsilon$. Define

$$\psi = \mu*\varphi\in\mathscr{D},$$

where φ is given by (1.5). Then

$$\int\psi(x)\,u(x)\,dx = \mu*\varphi*\check{u}(0) = \mu(\check{\varphi}*u) = \mu(u),\qquad u\in\mathscr{E}_P\ ,$$

since $\check{\varphi}*u=u$ if $P(-D)u=0$. Hence ψ represents T and

$$\operatorname{supp}\psi \subset \operatorname{supp}\mu+\operatorname{supp}\varphi \subset K+B_\varepsilon+\operatorname{supp}\varphi\ .$$

However, it is clear that $\operatorname{supp}\varphi\subset\operatorname{supp}f$ and that we can take f with the prescribed properties such that $\operatorname{supp}f\subset B_\varepsilon$. This gives

$$\operatorname{supp}\psi \subset K+2B_\varepsilon \subset L$$

and proves (i).

Now suppose that (i) holds and take $\varphi\in\mathscr{D}$ representing T with $\operatorname{supp}\varphi\subset L$. Then

$$|T(u)| = \left|\int\varphi(x)\,u(x)\,dx\right| \leqq \int|\varphi|\,dx\ \sup_L|u|,\qquad u\in\mathscr{E}_P\ .$$

This shows that (i) implies (ii).

Finally, if (ii) holds, the Hahn–Banach theorem shows that T can be extended to a measure μ with support in a given neighborhood L of K, hence (iii) is valid. This concludes the proof of Lemma 1.2.

2. The Borel transformation.

Let $E\in\mathscr{D}'$ be a fundamental solution for a differential operator $P(D)$ with constant coefficients. We define the potential U_φ of a function $\varphi\in\mathscr{D}$ by

$$U_\varphi = U_{\varphi,E} = \varphi*E\ .$$

Now suppose that φ_1 and φ_2 represent the same functional T on the space $\mathscr{D}'_P$ of solutions to $P(-D)u=0$. By a theorem of Malgrange which we have already invoked in Section 1 this implies that $\varphi_1 - \varphi_2 = P(D)\psi$ for some function $\psi \in \mathscr{D}$. Hence

$$U_{\varphi_1} - U_{\varphi_2} = (\varphi_1 - \varphi_2) * E = P(D)\psi * E = \psi,$$

so that $U_{\varphi_1} = U_{\varphi_2}$ outside some compact set. (By the convolution theorem we even know that the support of ψ is contained in the convex hull of $\operatorname{supp}\varphi_1 \cup \operatorname{supp}\varphi_2$.) The germ $\gamma_\infty U_\varphi$ of U_φ at infinity is therefore independent of the particular representative φ we have chosen for T. Note that U_φ is a C^∞ function and satisfies $P(D)U_\varphi = 0$ outside the support of φ. We shall call $U_T = \gamma_\infty U_\varphi$ the *potential of* T or the *Borel transform of* T. This is justified by the terminology in the following two special cases.

EXAMPLE 1. Let $n = 2$, $P(D) = \partial/\partial\bar{z} = \frac{1}{2}(iD_1 - D_2)$. Then $\mathscr{D}'_P$ is the space of entire functions in $\mathsf{R}^2 = \mathsf{C}$ and T is called an analytic functional. As our choice of fundamental solution we take

$$E = \frac{1}{\pi(x_1 + ix_2)}.$$

If $\varphi \in \mathscr{D}(\mathsf{R}^2)$ and $|x| > \sup(|y|\,;\, y \in \operatorname{supp}\varphi)$ we get

$$U_\varphi(x) = \frac{1}{\pi}\int \frac{\varphi(y)\,dy}{x_1 + ix_2 - (y_1 + iy_2)} = \frac{1}{\pi}\int \varphi(y) \sum_0^\infty \frac{(y_1 + iy_2)^k}{(x_1 + ix_2)^{k+1}}\,dy$$

$$= \frac{1}{\pi}\sum_0^\infty \frac{A_k}{(x_1 + ix_2)^{k+1}},$$

where

$$A_k = \int \varphi(y)\,(y_1 + iy_2)^k\,dy = T(y \mapsto (y_1 + iy_2)^k),$$

T being the analytic functional represented by φ. Apart from the factor $1/\pi$ this is the usual Borel transform of T; equivalently, πU_φ is the Borel transform of the entire function $\zeta \mapsto T(x \mapsto e^{(x_1 + ix_2)\zeta})$.

EXAMPLE 2. Take $P(D) = \Delta = -\sum_1^n D_j^2$ and

$$E = -\frac{1}{(n-2)\,\omega_n\,|x|^{n-2}} \qquad \text{when } n \neq 2,$$

$$E = \frac{1}{2\pi}\log|x| \qquad \text{when } n = 2,$$

where ω_n is the area of the $(n-1)$-dimensional unit sphere. Then $\varphi * E$ is the usual Newtonian potential of the mass distribution φ. (Newton himself proved that $\gamma_\infty(E*\varphi)$ is determined by the mass $\int\varphi dx$ of a rotationally symmetric planet φ, hence by the harmonic functional it represents.)

The Borel transform can also be defined for functionals on $\mathscr{E}_P$. Then we put $U_T=\gamma_\infty(\mu*E)$ where μ is a distribution with compact support which represents T. We therefore have to consider germs of distributions rather than functions. With only formal changes, Theorems 2.1 and 2.2 below as well as their proofs remain valid.

THEOREM 2.1. *Let $P(D)$ be an arbitrary differential operator with constant coefficients and T a functional on $\mathscr{D}'_P$ (see (1.1)). Suppose that*

(iv) *there exists a function $V \in \mathscr{E}(\complement K)$ such that $P(D)V=0$ in $\complement K$ and $\gamma_\infty V = U_T$.*

Then T is carried by K. Conversely, if K carries T and either P is elliptic or K is convex, then (iv) *follows.*

PROOF. Assume that (iv) holds and let L be an arbitrary compact neighborhood of K. We shall then find a representative φ of T with support in L. Take $f \in \mathscr{D}(L)$ such that $f=1$ near K and define

$$\varphi = P(D)(1-f)V \in \mathscr{D}(L)\,. \tag{2.1}$$

We shall prove that φ represents T, that is,

$$u(\varphi) = u(\psi) \quad \text{when } u \in \mathscr{D}'_P\,,$$

if $\psi \in \mathscr{D}$ is an arbitrary representative of T.

Now if ψ represents T we have $\gamma_\infty U_\psi = U_T = \gamma_\infty V$, hence $(1-f)V=V=U_\psi$ in a neighborhood of infinity. We thus obtain

$$\varphi-\psi = P(D)\big((1-f)V-U_\psi\big) \in P(D)\mathscr{D}\,,$$

so that φ and ψ represent the same functional T. This proves that T is carried by K.

Conversely, assume that K carries T and let K_j, $j \in \mathsf{N}$, be a sequence of compact neighborhoods of K such that $K_j \supset K_{j+1}$ and $\bigcap K_j = K$. In case K is convex we take all K_j convex, too. Choose a representative $\varphi_j \in \mathscr{D}(K_j)$ for each j and denote by

$$V_j = U_{\varphi_j} = \varphi_j * E$$

the potential of φ_j. It is clear that $P(D)V_j=0$ in $\complement K_j$. Now V_j-V_{j+1} is zero in a neighborhood of infinity and satisfies $P(D)(V_j-V_{j+1})=0$ in $\complement K_j$. Hence, if P is elliptic, $V_j-V_{j+1}=0$ in the unbounded component

Ω_j of $\complement K_j$ and we may therefore define $V(x) = V_j(x)$ if $x \in \Omega_j$. Since $\bigcup \Omega_j$ is equal to the unbounded component of $\complement K$, we may take $V = 0$ in the other components of $\complement K$ to satisfy (iv) in this case.

Assume now, on the other hand, that K and K_j are convex. By the convolution theorem we know that the supports of $P(D)(V_j - V_{j+1})$ and $V_j - V_{j+1}$ have the same convex hull, for $V_j - V_{j+1} \in \mathscr{D}$. But the support of $P(D)(V_j - V_{j+1})$ is contained in the convex set K_j, hence V_j and V_{j+1} agree in $\complement K_j$. Taking $V(x) = V_j(x)$ if $x \notin K_j$ we have completed the proof of Theorem 2.1.

It is clear from the proof that the theorem holds in a more general setting. Let for example Ω be a $P(-D)$-convex open set in R^n (see Definition 3.5.1 in Hörmander [5]). Then if γ denotes the operation of taking the germ at $\partial\Omega \cup \{\infty\}$ we may define (with an obvious extension of our notation): $U_T = \gamma(E * \varphi)$ if $\varphi \in \mathscr{D}(\Omega)$ represents a functional T on $\mathscr{D}'_P(\Omega)$. The following condition is then necessary and sufficient for a compact set $K \subset \Omega$ to carry T.

(iv)′ *For every compact neighborhood L of K there is a function $V \in \mathscr{E}(\Omega \setminus L)$ such that $P(D)V = 0$ in $\Omega \setminus L$ and $\gamma V = U_T$.*

Similarly for functionals on $\mathscr{E}_P(\Omega)$.

There need not exist a smallest convex compact set K such that condition (iv) of Theorem 2.1 holds. However, if P is elliptic and $n > 1$, it is easy to see that this is true, in other words, there exists a smallest convex carrier of T. By contrast, if P is hyperbolic, no functional $\neq 0$ on $\mathscr{D}'_P$ or $\mathscr{E}_P$ has a smallest convex carrier.

Let us note that the germ of the function V in (2.1) is not uniquely determined by the functional T it defines and that the Borel transform of T depends on the choice of fundamental solution E. This makes the Borel transformation somewhat unsatisfactory from a logical point of view. We shall therefore introduce the space $B_P^{\cdot}$ as follows. Let B_P be the space of all germs at infinity of C^∞ solutions to the equation $P(D)V = 0$ outside some compact set, and define $B_P^{\cdot}$ as B_P modulo the class of germs of C^∞ solutions to $P(D)V = 0$ in the whole space R^n. We write $U^{\cdot}$ for the class in $B_P^{\cdot}$ of a germ $U \in B_P$. Then we can state

THEOREM 2.2. (Grothendieck). *The map α*

$$(\mathscr{D}'_P)' \ni T \mapsto \alpha(T) = U_T^{\cdot} \in B_P^{\cdot}$$

of the space of all continuous linear forms on $\mathscr{D}'_P$ into the space $B_P^{\cdot}$ of equivalence classes of germs of solutions to $P(D)V = 0$ is a bijection which is independent of the fundamental solution E.

The theorem is contained in the paper [4] from 1953. At that time, of course, one had to assume the existence of a fundamental solution. See also Bengel [1].

PROOF OF THEOREM 2.2. Define a map $\beta: B_P^{\cdot} \to (\mathscr{D}'_P)'$ as follows. For every $U^{\cdot} \in B_P^{\cdot}$ choose a compact set K and a function $V \in \mathscr{E}(\complement K)$ such that $P(D)V = 0$ in $\complement K$ and $(\gamma_\infty V)^{\cdot} = U^{\cdot}$. Then choose $f \in \mathscr{D}$ such that $f = 1$ near K and define φ by (2.1). The functional

$$\mathscr{D}'_P \ni u \mapsto u(\varphi)$$

will be denoted by $\beta(U^{\cdot})$. Note that formally $\beta(U^{\cdot})$ depends on our choice of V and f. However, we shall prove that $\alpha \circ \beta$ is the identity in $B_P^{\cdot}$ and that $\beta \circ \alpha$ is the identity in $(\mathscr{D}'_P)'$. Thus it will follow that α and β are inverses of each other, in particular α does not depend on E since β does not, and β does not depend on V or f any more than α.

We first establish that $\beta \circ \alpha = I$. Let $T \in (\mathscr{D}'_P)'$. Then $\beta(\alpha(T))$ is represented by

$$\varphi = P(D)(1-f)V, \quad \text{where} \quad (\gamma_\infty V)^{\cdot} = (U_T)^{\cdot},$$

that is, $\gamma_\infty(V + W) = U_T$ for some function $W \in \mathscr{E}(\mathsf{R}^n)$ with $P(D)W = 0$. It follows from the first part of the proof of Theorem 2.1 that $P(D)(1-f)(V + W)$ represents T. But it is clear that $P(D)(1-f)W = P(D)(-fW)$ is orthogonal to $\mathscr{D}'_P$, hence $\varphi = P(D)(1-f)V$ represents T, that is, $\beta(\alpha(T)) = T$.

Next, if $U^{\cdot} \in B_P^{\cdot}$ is given, we choose V such that $\gamma_\infty V = U$ and $P(D)V = 0$ outside some compact set. We then have to show that the potential U_φ of $\varphi = P(D)(1-f)V$ satisfies

$$(\gamma_\infty U_\varphi)^{\cdot} = \alpha(\beta(U^{\cdot})) = U^{\cdot},$$

that is, $U_\varphi - V = W$ in a neighborhood of infinity for some function W such that $P(D)W = 0$ in the whole space. It is clear that

$$W = U_\varphi - (1-f)V \in \mathscr{E}$$

has all desired properties for

$$P(D)W = P(D)(E * \varphi - (1-f)V) = \delta * \varphi - \varphi = 0.$$

Therefore $\alpha \circ \beta = I$. The theorem is proved.

3. Estimates for residue integrals.

The proof of Theorem 4.1 will require estimates for residue integrals of the form

$$\int_\Gamma \frac{F(\zeta_1,\ldots,\zeta_n)}{P(\zeta_1,\ldots,\zeta_n)}\,d\zeta_n\,,$$

where $P(\zeta_1,\ldots,\zeta_n)$ is a polynomial in ζ_n with $\zeta_1,\ldots,\zeta_{n-1}$ regarded as parameters. This is altogether a problem in one complex variable but we have to be careful to obtain the right kind of uniformity in $\zeta_1,\ldots,\zeta_{n-1}$.

To describe the kind of result we need we first formulate a special case of the main estimate (3.6).

LEMMA 3.1. *Let $P(\tau)=c\prod_1^m(\tau-\tau_j)$ be a polynomial in one complex variable with discriminant*

$$D = \prod_{j<k}(\tau_j-\tau_k)^2\,.$$

Suppose that all the roots τ_j lie in some circle with radius R and that Γ is a curve surrounding k of them. Then, if f is analytic near $\tau_1,\ldots,\tau_m$ and satisfies

$$|f(\tau)| \leqq M \quad \text{when } P(\tau)=0\,, \tag{3.1}$$

we get

$$\left|\int_\Gamma \frac{f(\tau)\,d\tau}{P(\tau)}\right| \leqq \frac{2\pi k M(2R)^{\frac12(m-1)(m-2)}}{|c|\,|D|^{\frac12}}\,. \tag{3.2}$$

PROOF. If Γ surrounds τ_1 only, Cauchy's integral formula gives

$$\left|\int_\Gamma \frac{f(\tau)\,d\tau}{P(\tau)}\right| = \frac{2\pi|f(\tau_1)|}{|c|\prod_{j\neq1}|\tau_1-\tau_j|}\,.$$

Multiplying both numerator and denominator by the $\frac12(m-1)(m-2)$ factors missing to make the denominator equal to $|c|\,|D|^{\frac12}$ and using $|\tau_i-\tau_j|\leqq 2R$ we obtain

$$\left|\int_\Gamma \frac{f(\tau)\,d\tau}{P(\tau)}\right| \leqq \frac{2\pi|f(\tau_1)|\,(2R)^{\frac12(m-1)(m-2)}}{|c|\,|D|^{\frac12}}\,.$$

The integral to be estimated is a sum of k similar terms. This proves (3.2).

The next two lemmas play an auxiliary role in the proof of Lemma 3.4.

LEMMA 3.2. *Suppose that a polynomial P satisfies*

$$|P(\tau)| \leqq C(|\tau|+R)^m, \quad \tau\in\mathbf{C}\,.$$

Then we obtain the following estimates for its derivatives:

$$|P^{(k)}(\tau)| \leqq Ce^k \frac{m!}{(m-k)!}(|\tau|+R)^{m-k}, \quad \tau \in \mathbf{C},\ 0 \leqq k \leqq m\,.$$

PROOF. By Cauchy's integral formula we have

$$|P'(\tau)| \leqq r^{-1}\sup_{|\sigma|=r}|P(\tau+\sigma)| \leqq Cr^{-1}(|\tau|+r+R)^m\,.$$

Choosing $r=(|\tau|+R)/(m-1)$ to minimize the right hand side we obtain

$$|P'(\tau)| \leqq C\left(1+\frac{1}{m-1}\right)^{m-1} m(|\tau|+R)^{m-1} \leqq Cem(|\tau|+R)^{m-1}\,.$$

(If $m=1$ we get $|P'| \leqq C$.) The general result now follows by iteration.

LEMMA 3.3. *Let P be a polynomial satisfying*

$$|P(\tau)| \leqq C(|\tau|+R)^m, \quad \tau \in \mathbf{C}\,.$$

Then

$$\left|\left(\frac{\partial}{\partial\tau}\right)^k\left(\frac{1}{P}\right)(\tau)\right| \leqq A_{mk}C^k\frac{(|\tau|+R)^{k(m-1)}}{|P(\tau)|^{k+1}} \tag{3.3}$$

when $P(\tau) \neq 0$, $k \geqq 0$. Here A_{mk} is a constant which depends only on m and k; in fact, we may take

$$A_{mk} = e^k m^k (2k-1)!!\,.$$

PROOF. We claim that

$$(\partial/\partial\tau)^k P^{-1} = Q_k P^{-k-1}$$

where Q_k is a polynomial. Indeed, $Q_0=1$ and

$$Q_{k+1} = Q_k'P-(k+1)Q_kP'\,,$$

so the assertion follows by induction. Now suppose that

$$|Q_k(\tau)| \leqq C^k e^k m^k (2k-1)!!\,(|\tau|+R)^{k(m-1)} \tag{3.4}$$

holds for a certain k (it certainly does for $k=0$). Then Lemma 3.2 implies

$$|Q_k'(\tau)| \leqq C^k e^{k+1} m^k k(m-1)(2k-1)!!\,(|\tau|+R)^{k(m-1)-1}\,.$$

We also have

$$|P'(\tau)| \leqq Cem(|\tau|+R)^{m-1}$$

so the recursion formula for Q_{k+1} gives

$$|Q_{k+1}(\tau)| \leqq C^{k+1}e^{k+1}m^k(2k-1)!!\,(2km+m-k)(|\tau|+R)^{(k+1)(m-1)}$$
$$\leqq C^{k+1}e^{k+1}m^{k+1}(2k+1)!!\,(|\tau|+R)^{(k+1)(m-1)}\,.$$

Thus (3.4) is proved for all $k \geqq 0$ which implies the desired estimate for $Q_k P^{-k-1} = (\partial/\partial\tau)^k P^{-1}$.

We can now prove a more general estimate for residue integrals.

LEMMA 3.4. *Let $P = P_1^{m_1} \dots P_s^{m_s}$ be a polynomial in one complex variable with all its zeros contained in a disk of radius $R \geqq \frac{1}{2}$. Define $Q = P_1 \dots P_s$ and let D be the discriminant of Q. If Γ is a cycle which surrounds (once) some of the zeros of P and f is analytic near those zeros and satisfies*

$$|f^{(j)}(\tau)| \leqq M \quad \textit{when } P_k(\tau) = 0 \quad \textit{and} \quad 0 \leqq j < m_k,\ 1 \leqq k \leqq s\,, \tag{3.5}$$

for those τ which lie inside Γ, then

$$\left|\int_\Gamma \frac{f(\tau)\,d\tau}{P(\tau)}\right| \leqq \frac{A_m M (2R)^{\frac{1}{2}m^4}}{|c|\;|D|^{\frac{1}{2}\mu^2}}\,. \tag{3.6}$$

Here c is the leading coefficient of P, m the degree of P and $\mu = \sup(m_k; P_k$ is non-constant$) \leqq m$. The constant A_m depends on m only.

When $m_k = 1$ for all k we get an estimate which is somewhat less sharp than that of Lemma 3.1. This lack of precision, however, is inessential for our purposes.

PROOF OF LEMMA 3.4. Let $\tau_1, \dots, \tau_n$ be the zeros of Q. It suffices to prove (3.6) when these are all different, for otherwise $D = 0$. The cycle Γ is contained in the domain of f and has winding number one or zero with respect to $\tau_1, \dots, \tau_n$. We may therefore assume that Γ encloses just one τ_j, say τ_1, a zero of P_1. Then we get

$$I = \int_\Gamma \frac{f(\tau)\,d\tau}{P(\tau)} = \int_\Gamma \frac{f(\tau)\,d\tau}{(\tau-\tau_1)^{m_1}S(\tau)} = \frac{2\pi i}{(m_1-1)!}\left(\frac{\partial}{\partial\tau}\right)^{m_1-1}(fS^{-1})(\tau_1)\,,$$

where $S(\tau) = P(\tau)(\tau-\tau_1)^{-m_1}$. Hence

$$|I| \leqq \frac{2\pi}{(m_1-1)!}\sum_{j=0}^{m_1-1}\binom{m_1-1}{j} M\,|(\partial/\partial\tau)^j S^{-1}(\tau_1)|$$

for the derivatives of f which occur in the expansion of $(\partial/\partial\tau)^{m_1-1}(fS^{-1})$ are taken care of by (3.5), with $k=1$. By Lemma 3.3 we can estimate the derivatives of S^{-1} by

$$|(\partial/\partial\tau)^j S^{-1}(\tau)| \leqq A_{m-m_1,j}|c|^j(|\tau|+R)^{j(m-m_1-1)}|S(\tau)|^{-j-1}.$$

To estimate $S(\tau)$, we note that

$$|D|^{\frac{1}{2}} = \prod_{j\neq 1}|\tau_1-\tau_j| \prod_{1<j<k}|\tau_k-\tau_j|,$$

whereas

$$|S(\tau_1)| = |c| \prod_{j\neq 1}|\tau_1-\tau_j|^{a_j},$$

where $a_j=m_k$ if τ_j is a zero of P_k (k is determined by j since the τ_j's are all different). Hence

$$\frac{1}{|S(\tau_1)|} = \frac{1}{|D|^{\frac{1}{2}\mu}}\frac{|D|^{\frac{1}{2}\mu}}{|S(\tau_1)|} = \frac{1}{|c|\;|D|^{\frac{1}{2}\mu}}\prod_{j\neq 1}|\tau_1-\tau_j|^{\mu-a_j}\prod_{1<j<k}|\tau_k-\tau_j|^{\mu}$$
$$\leqq \frac{(2R)^{\frac{1}{2}\mu n(n-1)-m+m_1}}{|c|\;|D|^{\frac{1}{2}\mu}}.$$

Inserted in the estimate for the derivatives of S^{-1} this gives

$$|(\partial/\partial\tau)^j S^{-1}(\tau_1)| \leqq A_{m-m_1,j}|c|^{-1}(2R)^{\frac{1}{2}\mu n(n-1)(j+1)-m+m_1-j}|D|^{-\frac{1}{2}\mu(j+1)}.$$

Using also the trivial estimate

$$|D| \leqq (2R)^{n(n-1)},$$

the last inequality can be simplified to

$$|(\partial/\partial\tau)^j S^{-1}(\tau_1)| \leqq A_{m-m_1,j}|c|^{-1}(2R)^{\frac{1}{2}\mu^2 n(n-1)-m+m_1-j}|D|^{-\frac{1}{2}\mu^2}.$$

We shall now finally use the assumption that $2R\geqq 1$. Since $-m+m_1-j \leqq 0$ we get

$$|(\partial/\partial\tau)^j S^{-1}(\tau_1)| \leqq B_m|c|^{-1}(2R)^{\frac{1}{2}\mu^2 n(n-1)}|D|^{-\frac{1}{2}\mu^2}, \quad 0\leqq j<m_1.$$

(One also has $-m+m_1-j\geqq -m+1$ so it is easy to obtain a bound for R small.)

To finish the proof we only have to note that $\frac{1}{2}\mu^2 n(n-1)$ is not greater than $\frac{1}{2}m^4$. The lemma is proved.

We shall also need the following two well-known estimates.

LEMMA 3.5. *Let*

$$P(\zeta) = \sum_0^q c_j(\zeta')\,\zeta_n^{q-j}$$

be a polynomial in n variables $(\zeta',\zeta_n)=(\zeta_1,\ldots,\zeta_{n-1},\zeta_n)$ of degree m and suppose that c_0 is independent of ζ' and different from zero. We can then find a constant c which depends only on $c_0,\ldots,c_q$ and m such that every *root σ of $P(\theta',\sigma)=0$ has distance not greater than ε from the set of roots τ of $P(\zeta',\tau)=0$ if*

$$0<\varepsilon\leqq 1 \quad \textit{and} \quad |\theta'-\zeta'| \leqq c(1+|\zeta'|)^{-r}\varepsilon^q ,$$

where $r=mq-q^2+q-1\leqq m^2$.

Proof. Let M_ε be the set

$$\{\sigma\in \mathbf{C}\,;\ |\sigma-\tau|\leqq\varepsilon \text{ for some } \tau\in\mathbf{C} \text{ such that } P(\zeta',\tau)=0\}\,.$$

What we have to prove is that the zeros of $\sigma\mapsto P(\theta',\sigma)$ are contained in M_ε if θ' is close to ζ'. Note that on the boundary of M_ε we have $|P(\zeta',\tau)|\geqq|c_0|\varepsilon^q$. It is therefore sufficient to find a δ such that

$$|\theta'-\zeta'| \leqq \delta \quad \text{implies} \quad |P(\theta',\tau)-P(\zeta',\tau)|<|c_0|\varepsilon^q \text{ when } \tau\in M_\varepsilon\,.$$

Now if $|\theta'-\zeta'|\leqq 1$ we have

$$\begin{aligned} (3.7) \qquad |P(\theta',\tau)-P(\zeta',\tau)| &\leqq \sum_1^q |c_j(\theta')-c_j(\zeta')|\ |\tau|^{q-j} \\ &\leqq \sum_1^q C_j(2+|\zeta'|^{\gamma_j-1})\ |\theta'-\zeta'|\ |\tau|^{q-j}\,, \end{aligned}$$

where γ_j is the degree of c_j and we have chosen C_j to make

$$|\operatorname{grad} c_j(\zeta')| \leqq C_j(1+|\zeta'|)^{\gamma_j-1}\,.$$

But it is well-known that the zeros of $P(\zeta',\tau)$ satisfy $|\tau|\leqq A(1+|\zeta'|)^\gamma$ for some constant A where $\gamma=\sup\gamma_j/j\leqq m-q+1$; hence $|\tau|\leqq (A+1)(1+|\zeta'|)^\gamma$ if $\tau\in M_\varepsilon$ and $\varepsilon\leqq 1$. Inserting this estimate in (3.7) we get since $\gamma_j\leqq j\gamma$,

$$|P(\theta',\tau)-P(\zeta',\tau)| \leqq |\theta'-\zeta'|\,(2+|\zeta'|)^{q\gamma-1}\sum_1^q C_j(A+1)^{q-j}\,.$$

The desired estimate therefore follows if c is not greater than 1 and satisfies

$$c\,2^{q\gamma-1}\sum_1^q C_j(A+1)^{q-j} < |c_0|\,.$$

Lemma 3.6. *Let F be an analytic function defined in a neighborhood of $\{\zeta+\tau N\,;\ \tau\in\mathbf{C}, |\tau|\leqq\delta\}$ for some $\zeta,N\in\mathbf{C}^n$ and some $\delta>0$. If P is a polynomial in n variables with principal part p, it follows that*

$$|p(\delta N)F(\zeta)| \leqq \sup_{|\tau|=\delta}|P(\zeta+\tau N)\,F(\zeta+\tau N)|\,.$$

For a proof we refer to Malgrange [8, Lemme 1, p. 286] or Hörmander [5, Lemma 3.1.2].

Combining Lemmas 3.4, 3.5 and 3.6 we can now prove

LEMMA 3.7. *Let P be a polynomial of degree m in n complex variables decomposed as a product $P = P_1^{m_1} \ldots P_s^{m_s}$, where $P_1, \ldots, P_s$ are irreducible and mutually prime and $m_k \geqq 1$. Assume that*

$$P(\zeta) = \sum_0^q c_j(\zeta')\, \zeta_n^{q-j}\,,$$

where c_0 is independent of ζ' and different from zero. Let θ' be an arbitrary vector in $\mathbf{C}^{n-1}$, and let $\Gamma(\theta')$ be a cycle in the complex ζ_n-plane with distance at least $\varepsilon_1(\theta')$ to all zeros of $\tau \mapsto P(\theta', \tau)$, $0 < \varepsilon_1(\theta') \leqq 1$. The winding numbers of $\Gamma(\theta')$ with respect to the zeros are supposed to be ± 1 or 0. Finally let G be an analytic function such that

$$(3.8) \qquad |(\partial/\partial\zeta_n)^j G(\zeta)| \leqq M(\zeta'), \qquad 0 \leqq j < m_k\,,$$

when

$$|\zeta' - \theta'| \leqq \varepsilon_2(\theta')$$

and ζ is a zero of some P_k, $1 \leqq k \leqq s$, such that ζ_n lies inside $\Gamma(\theta')$. Then

$$(3.9) \qquad \left| \int_{\Gamma(\theta')} \frac{G(\theta', \tau)\, d\tau}{P(\theta', \tau)} \right| \leqq C \frac{(1 + |\theta'|)^a}{\varepsilon_0(\theta')^b} \sup\nolimits_{|\zeta'-\theta'| \leqq \varepsilon_2(\theta')} M(\zeta')\,,$$

where $\varepsilon_0 = \inf(\varepsilon_1, \varepsilon_2)$, C is a constant depending only on P, and a, b are numbers depending only on the degree m of P. In particular C, a, b are independent of θ' as well as of the functions ε_1, ε_2.

PROOF. By Lemma 3.5 no zero of $\tau \mapsto P(\zeta', \tau)$ lies on $\Gamma(\theta')$ when

$$|\zeta' - \theta'| \leqq C_1 (1 + |\theta'|)^{-m^2} \varepsilon_1(\theta')^q = \delta_1(\theta') \leqq 1$$

for some constant C_1 depending only on P. Hence

$$F(\theta', \zeta') = \int_{\Gamma(\theta')} \frac{G(\zeta', \tau)\, d\tau}{P(\zeta', \tau)}$$

is an analytic function of ζ' in a neighborhood of the set

$$\{\zeta' \in \mathbf{C}^{n-1}\,;\ |\zeta' - \theta'| \leqq \delta_1(\theta')\}\,.$$

From Lemma 3.4 we obtain if D denotes the discriminant of $P_1 \ldots P_s$ regarded as a polynomial in ζ_n,

$$|F(\theta',\zeta')| \leqq \frac{A_q M(\zeta')(2R(\zeta'))^{\frac{1}{2}q^4}}{|c_0|\,|D(\zeta')|^{\frac{1}{2}\mu^2}}, \qquad |\zeta'-\theta'| \leqq \delta_2(\theta') = \inf(\delta_1(\theta'),\varepsilon_2(\theta'))\,.$$

Here $\mu = \sup m_k \leqq q \leqq m$, and $R(\zeta') \geqq \frac{1}{2}$ is chosen so large that all zeros of $P(\zeta',\tau)$ satisfy $|\tau| \leqq R(\zeta')$; it suffices to take $R(\zeta') = C_2(1+|\zeta'|)^{m-q+1}$. It follows that

$$|D(\zeta')|^{\frac{1}{2}\mu^2}|F(\theta',\zeta')| \leqq CM(\zeta')(1+|\zeta'|)^{a'}, \quad |\zeta'-\theta'| \leqq \delta_2(\theta')\,,$$

for some constant C. Here a' depends on m and q only (and therefore has an upper bound depending only on m). Since

$$|D(\zeta')| \leqq \left(2C_2(1+|\zeta'|)^{m-q+1}\right)^{q(q-1)}\,,$$

we moreover get if ν is the smallest integer $\geqq \frac{1}{2}\mu^2$,

$$\text{(3.10)} \qquad |D(\zeta')^\nu F(\theta',\zeta')| \leqq C' M(\zeta')(1+|\zeta'|)^{a''}, \quad |\zeta'-\theta'| \leqq \delta_2(\theta')\,,$$

where a'' depends only on m. Now D is not identically zero since $P_1 \ldots P_s$ is free from multiple factors (see e.g. van der Waerden [12, §§ 34, 35]). Choose a vector $N' \in \mathbf{C}^{n-1}$ such that $|N'| = 1$ and $d(N') \neq 0$, d denoting the principal part of D. Then Lemma 3.6 gives

$$\text{(3.11)} \qquad |d(\delta_2(\theta')N')^\nu F(\theta',\theta')| \leqq \sup_{|\zeta'-\theta'| \leqq \delta_2(\theta')} |D(\zeta')^\nu F(\theta',\zeta')|\,.$$

Combining (3.10) and (3.11) we obtain

$$\delta_2(\theta')^{\nu c}|d(N')|^\nu |F(\theta',\theta')| \leqq C' \sup_{|\zeta'-\theta'| \leqq \delta_2(\theta')} M(\zeta')(1+|\zeta'|)^{a''}\,,$$

where c is the degree of the discriminant D. Now (3.8) follows because c and $\nu \leqq \frac{1}{2}(\mu^2+1)$ have bounds depending only on m.

In the applications of this lemma we shall always let ε_1 and ε_2 be temperate functions of the form $B(1+|\theta'|)^{-A}$. For vectors θ' in an arbitrary subset S of $\mathbf{C}^{n-1}$ the estimate (3.9) will therefore display only a "temperate loss" in comparison with (3.8), provided (3.8) holds in a "temperate neighborhood" of S.

4. The Fourier transformation.

If φ is a function in $\mathscr{D}$ we define its Fourier transform $\hat{\varphi}$ by

$$\hat{\varphi}(\zeta) = \int \varphi(x)\, e^{-i\langle x,\zeta\rangle}\, dx, \qquad \zeta \in \mathbf{C}^n\,.$$

Suppose now that φ_1 and φ_2 both represent a functional T on the space $\mathscr{D}'_P$ (see (1.1)). As noted in Section 1 this implies that $\varphi_1 - \varphi_2 = P(D)\psi$ for some function $\psi \in \mathscr{D}$. This means that $\hat{\varphi}_1(\zeta) - \hat{\varphi}_2(\zeta) = P(\zeta)\hat{\psi}(\zeta)$. Conversely, if $(\hat{\varphi}_1 - \hat{\varphi}_2)/P$ is analytic it follows that $(\hat{\varphi}_1 - \hat{\varphi}_2)/P = \hat{\psi}$ for some function $\psi \in \mathscr{D}$ and thus φ_1 and φ_2 represent the same functional. Similarly we find that two distributions μ_1 and $\mu_2 \in \mathscr{E}'$ represent the same functional on $\mathscr{E}_P$ (for definition see (1.4)) if and only if $\hat{\mu}_1 - \hat{\mu}_2 = P\hat{\nu}$ for some $\nu \in \mathscr{E}'$, and this, in turn, is equivalent to $(\hat{\mu}_1 - \hat{\mu}_2)/P$ being analytic. Here $\hat{\mu}$ is defined by

$$\hat{\mu}(\zeta) = \mu(x \mapsto e^{-i\langle x,\zeta\rangle}), \qquad \zeta \in \mathbf{C}^n .$$

We thus have, if $A(\mathbf{C}^n)$ denotes the space of entire functions in $\mathbf{C}^n$,

$$\hat{\mathscr{D}}/P\cdot\hat{\mathscr{D}} = \hat{\mathscr{D}}/P\cdot A(\mathbf{C}^n) \cap \hat{\mathscr{D}}$$

and

$$\hat{\mathscr{E}}'/P\cdot\hat{\mathscr{E}}' = \hat{\mathscr{E}}'/P\cdot A(\mathbf{C}^n) \cap \hat{\mathscr{E}}' ,$$

both of which are subspaces of $A(\mathbf{C}^n)/P\cdot A(\mathbf{C}^n)$. We define the Fourier transform $\hat{T}$ of a functional T on $\mathscr{D}'_P$ or $\mathscr{E}_P$ as the equivalence class of the Fourier transform of an arbitrary representative of T. In both cases $\hat{T}$ is an element of $A(\mathbf{C}^n)/P\cdot A(\mathbf{C}^n)$ which determines and is determined by T.

In particular $\hat{T}$ has well-defined values in the set of zeros of P. However, $\hat{T}$ is not determined by these values if P has multiple factors. It is therefore important to note that some derivatives $D^\alpha \hat{T}(\zeta)$ may be well-defined if $P(\zeta)=0$ and $|\alpha|$ is not too large. In fact, if P is divisible by $P_1^{m_1}$ and $P_1(\zeta)=0$, then $D^\alpha P\hat{\psi}(\zeta)$ vanishes when $|\alpha| < m_1$. Hence multiples of P do not affect the value of $D^\alpha \hat{\varphi}(\zeta)$ which means that $D^\alpha \hat{T}(\zeta)$ is defined.

Theorem 4.1 below will give a correspondence of the Paley–Wiener type where the estimate of $\hat{T}$ is valid exactly for those derivatives of $\hat{T}$ which can be formed in general, viz. $D^\alpha \hat{T}(\zeta)$ if ζ is a zero of P_k and $|\alpha| < m_k$ where $P = P_1^{m_1} \dots P_s^{m_s}$. If P does not have multiple factors knowledge of the values of $\hat{T}$ at the zeros of P is sufficient. This is for instance the case in Ivanov's theorem [6] where $P(D)$ is the Laplace operator in three-dimensional space. Let us briefly indicate Ivanov's method of proof to compare it with that of Theorem 4.1. Ivanov does not consider functionals on the harmonic functions explicitly, but expresses them as outer potentials of measures, which is equivalent as we have seen in Section 2.

Let $\varphi \in \mathscr{D}$ represent $T \in \mathscr{D}'_P$ where $P(D) = \Delta$, and let $n \geqq 2$. Then

$$\hat{T}(\zeta) = \int_\Omega \varphi(x)\, e^{-i\langle x,\zeta\rangle}\, dx$$

if $\sum \zeta_j^2 = 0$ and Ω contains the support of φ. We suppose that Ω has a piecewise smooth boundary $\partial\Omega$ and shall define a differential form ω with C^∞ coefficients such that $d\omega = \varphi(x)e^{-i\langle x,\zeta\rangle}dx_1\wedge\ldots\wedge dx_n$. Then

$$\hat{T}(\zeta) = \int_\Omega d\omega = \int_{\partial\Omega}\omega$$

provided Ω and $\partial\Omega$ are given suitable orientations. Ivanov's idea is now to blow up Ω to a half-space containing $\operatorname{supp}\varphi$. If ω is suitably chosen, the integral over $\partial\Omega$ (which consists of a hemisphere and a disk) will tend to the integral of ω over the boundary of the half-space; that is, the contribution from the hemisphere tends to zero. If the half-space in question is $\{x; x_n > 0\}$, Ivanov's choice of ω becomes

$$\omega(x,\zeta) = \frac{1}{\zeta_n}e^{-i\langle x,\zeta\rangle}\left[\zeta_n\sum\frac{\partial U}{\partial x_j}*dx_j + \frac{\partial U}{\partial x_n}\sum\zeta_j*dx_j - \sum\zeta_j\frac{\partial U}{\partial x_j}*dx_n\right],$$

where $U = \varphi * E$, E being the standard fundamental solution for Δ which was given in Example 2 of Section 2, and where

$$*dx_j = (-1)^{j-1}dx_1\wedge\ldots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\ldots\wedge dx_n .$$

Note that ω is only defined for $\zeta_n \neq 0$ so we have to use different choices of ω to cover the whole set of zeros of $\sum\zeta_j^2 = 0$. It is easy to see that

$$d\omega = \Delta U\, e^{-i\langle x,\zeta\rangle}dx_1\wedge\ldots\wedge dx_n = \varphi\, e^{-i\langle x,\zeta\rangle}dx_1\wedge\ldots\wedge dx_n .$$

The important property of ω is that, after letting the radius of the hemisphere tend to infinity, we obtain, provided $x_n > 0$ when $x \in \operatorname{supp}\varphi$,

$$\hat{T}(\zeta) = -2\int e^{-i\langle x',\zeta'\rangle}\frac{\partial U}{\partial x_n}(x',0)\,dx', \quad \zeta = (\xi', -i|\xi'|),\ \xi' \in \mathsf{R}^{n-1}, \tag{4.1}$$

where the integral is abolutely convergent. (Actually, also the opposite hemisphere is needed to get (4.1).) This means that $\xi' \mapsto \hat{T}(\xi', -i|\xi'|)$ is the Fourier transform of $x' \mapsto -2(\partial/\partial x_n)U(x',0)$ so the Fourier inversion formula gives

$$-2\frac{\partial U}{\partial x_n}(x',0) = (2\pi)^{-n+1}\int_{\mathsf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\hat{T}(\xi', -i|\xi'|)\,d\xi' ,$$

or, after a translation,

(4.2)

$$-2\frac{\partial U}{\partial x_n}(x) = (2\pi)^{-n+1}\int_{\zeta=(\xi',-i|\xi'|)} e^{i\langle x,\zeta\rangle}\hat{T}(\zeta)\,d\xi', \quad x_n < \inf(y_n; y \in \operatorname{supp}\varphi) .$$

It is this formula which defines the continuation of the outer potential of T once a bound for $\hat{T}$ is known.

For general differential operators one cannot hope for a formula like (4.1). Our method will therefore be to generalize (4.2) directly. Note that the integral in (4.2) involves the zero of $\tau \mapsto P(\xi', \tau)$ which has negative imaginary part. In general there will be more than one zero taken into account and the value of $\hat{T}$ at the single zero will be replaced by a residue integral operating over several zeros of $\tau \mapsto P(\xi', \tau)$. This, in short, is the motivation for Lemma 3.7.

THEOREM 4.1. *Let P be a non-constant polynomial and write $P = P_1^{m_1} \ldots P_s^{m_s}$ where $P_1, \ldots, P_s$ are irreducible and mutually prime, $m_k \geqq 1$. Then a functional T on the space $\mathscr{D}'_P$ (or $\mathscr{E}_P$) of distribution (or C^∞) solutions to $P(-D)u = 0$ in R^n is carried by a convex compact set $K \subset \mathsf{R}^n$ if and only if the following condition holds:*

(v) *For every $\varepsilon > 0$ there are constants C_ε and N_ε such that*

$$|D^\alpha \hat{T}(\zeta)| \leqq C_\varepsilon (1+|\zeta|)^{N_\varepsilon} \exp\big(H_K(\operatorname{Im}\zeta) + \varepsilon |\operatorname{Im}\zeta|\big), \tag{4.3}$$

when

$$P_k(\zeta) = 0, \quad |\alpha| < m_k, \quad k = 1, \ldots, s. \tag{4.4}$$

Here $D^\alpha \hat{T}(\zeta)$ is defined as $D^\alpha \hat{\varphi}(\zeta)$ for an arbitrary representative φ of T. We have also written

$$H_K(\eta) = \sup_{x \in K} \langle x, \eta \rangle, \quad \eta \in \mathsf{R}^n,$$

for the supporting function of K.

PROOF OF THEOREM 4.1. Suppose that T is a functional on $\mathscr{E}_P$ and carried by K. This means that for every $\varepsilon > 0$ we can find a representative $\mu \in \mathscr{E}'$ of T such that $\operatorname{supp} \mu \subset K + \varepsilon B$ (B is the unit ball). Hence, by the Paley–Wiener theorem (see e.g. Hörmander [5, Theorem 1.7.7]), we obtain for some constants C and N

$$|\hat{\mu}(\zeta)| \leqq C(1+|\zeta|)^N \exp\big(H_K(\operatorname{Im}\zeta) + \varepsilon |\operatorname{Im}\zeta|\big),$$

so that (4.3) holds for $\alpha = 0$, ζ arbitrary. The necessity of (v) therefore follows from the estimates

$$|D^\alpha \hat{\mu}(\zeta)| \leqq C_\alpha \sup_{|\theta| \leqq 1} |\hat{\mu}(\zeta + \theta)|.$$

If T is a functional on $\mathscr{D}'_P$ we have representatives in $\mathscr{D} \subset \mathscr{E}'$ so the same argument applies. (We even get the better estimate (4.5) below.)

We shall first prove the sufficiency of the condition (v) when T is a functional on $\mathscr{D}'_P$. Let us note that in this case it follows from (v) that there are constants $C_{\varepsilon q}$ depending on $\varepsilon > 0$ and $q \geqq 0$ such that

$$|D^\alpha \hat{T}(\zeta)| \leqq C_{\varepsilon q}(1+|\zeta|)^{-q} \exp\big(H_K(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|\big) \tag{4.5}$$

when (4.4) holds. In fact, if φ represents T we get

$$|D^\alpha \hat{\varphi}(\zeta)| \leqq C_q(1+|\zeta|)^{-q}\, e^{a|\operatorname{Im}\zeta|} \tag{4.6}$$

provided a is so large that $|x| \leqq a$ when $\varphi(x) \neq 0$ (see e.g. Hörmander [5, Theorem 1.7.7]). We suppose in addition that $|x| \leqq a$ when $x \in K$. Then if $\varepsilon|\operatorname{Im}\zeta| \geqq (N_\varepsilon+q)\log(1+|\zeta|)$ we get from (4.3)

$$|D^\alpha \hat{T}(\zeta)| \leqq C_\varepsilon(1+|\zeta|)^{-q} \exp\big(H_K(\operatorname{Im}\zeta)+2\varepsilon|\operatorname{Im}\zeta|\big) . \tag{4.7}$$

On the other hand, if $\varepsilon|\operatorname{Im}\zeta| \leqq (N_\varepsilon+q)\log(1+|\zeta|)$ we must have

$$a|\operatorname{Im}\zeta| \leqq 2a(N_\varepsilon+q)\varepsilon^{-1}\log(1+|\zeta|) + H_K(\operatorname{Im}\zeta) ,$$

thus (4.6) with q replaced by r implies

$$|D^\alpha \hat{T}(\zeta)| \leqq C_r(1+|\zeta|)^{-r+2a(N_\varepsilon+q)/\varepsilon} \exp\big(H_K(\operatorname{Im}\zeta)\big) ,$$

so that, with $r = q + 2a(N_\varepsilon+q)/\varepsilon$, we obtain (4.7) with another constant. Changing notation in (4.7) we have proved (4.5).

To establish the sufficiency of (v), we shall prove that condition (iv) of Theorem 2.1 follows from (v) with (4.3) replaced by the apparently stronger estimate (4.5). To this end we shall construct, for a dense set of unit vectors η on the unit sphere in R^n:

A. *A fundamental solution $E_\eta \in \mathscr{D}'$ for the operator $P(D)$.*
B. *A C^∞ function V_η which is defined in $\{x;\ \langle x,\eta\rangle > H_K(\eta)\}$ and satisfies $P(D)V_\eta = 0$ there and which moreover coincides with $E_\eta * \varphi$ when $\langle x,\eta\rangle > H_K(\eta)$ and $|x| > r$. Here φ is an arbitrary representative of T and r a fixed large number.*

As a motivation for our interest in this construction, let us first prove that (iv) follows if E_η and V_η with the described properties can be found.

Let E be an arbitrary fundamental solution for $P(D)$ and define

$$W_\eta = V_\eta + (E - E_\eta) * \varphi .$$

This is a C^∞ function in $\{x;\ \langle x,\eta\rangle > H_K(\eta)\}$ and $P(D)W_\eta = 0$ there. When $\langle x,\eta\rangle > H_K(\eta)$ and $|x| > r$ we have $W_\eta(x) = E * \varphi(x)$. Hence W_η defines a continuation of the Borel transform U_T of T. If it can be proved that $W_{\eta(1)}$ and $W_{\eta(2)}$ coincide where they are both defined we will thus have a continuation of U_T to the whole complement of K, that is, (iv) holds. The missing argument is provided by the following lemma.

LEMMA 4.2. *Let there be given, for a dense set M of unit vectors η, solutions W_η of the equation $P(D)W_\eta = 0$ defined when $\langle x,\eta\rangle > H_K(\eta)$, K a given convex compact set in R^n. Assume moreover that $W_\eta(x) = F(x)$ when $\langle x,\eta\rangle > H_K(\eta)$ and $|x| > r$, where F is a function independent of η defined for $|x| > r$. Then $W_{\eta^{(1)}} = W_{\eta^{(2)}}$ in their common domain of definition for any two vectors $\eta^{(1)}, \eta^{(2)} \in M$.*

We have formulated this lemma for functions. With obvious changes in language only, the lemma and its proof are valid for distributions.

When P is elliptic the lemma follows at once in view of the uniqueness of the continuation of a real analytic function. In general it may happen that the conclusion is false if we have just two solutions $W_{\eta^{(1)}}$ and $W_{\eta^{(2)}}$ defined in half-spaces.

PROOF OF LEMMA 4.2. We may assume that r is so large that $H_K(\eta) \leqq r|\eta|$. We shall first prove that $W_{\eta^{(1)}}(a) = W_{\eta^{(2)}}(a)$ if $\langle a,\eta^{(1)}\rangle > H_K(\eta^{(1)})$ and $2r|\eta^{(1)} - \eta^{(2)}| < \langle a,\eta^{(1)}\rangle - H_K(\eta^{(1)})$.

Let Ω_j be the half-space (possibly the whole space R^n)

$$\{x\,;\, \langle x,\eta^{(j)}\rangle > H_K(\eta^{(j)})\}, \quad j = 1,2\,.$$

Consider the half-space

$$\Omega = \{x\,;\; \langle x,\eta^{(1)}\rangle > H_K(\eta^{(1)}) + 2r|\eta^{(1)} - \eta^{(2)}|\}$$

which contains a. It is then obvious that $x \in \Omega_1 \cap \Omega_2$ if $x \in \Omega$ and $|x| \leqq r$. Define $W = W_{\eta^{(1)}} - W_{\eta^{(2)}}$ in $\{x \in \Omega;\, |x| \leqq r\}$ and $W = 0$ elsewhere in Ω. This defines a solution of $P(D)W = 0$ in all of Ω, for when $|x| > r$ we have $W_{\eta^{(1)}}(x) = W_{\eta^{(2)}}(x)$ whenever both members happen to be defined. Since W is zero outside a bounded set in the halfspace Ω we must have $W = 0$ in all of Ω in view of a well-known theorem on unique continuation of solutions to a constant coefficient differential equation (cf. the proof of Lemma 3.4.3 in Hörmander [5]). In particular $W_{\eta^{(1)}}(a) = W_{\eta^{(2)}}(a)$.

What we have proved shows of course that $\eta \mapsto W_\eta(a)$ is a locally constant function in $\{\eta \in M;\, \langle a,\eta\rangle > H_K(\eta)\}$ where M denotes the dense set of unit vectors for which W_η is given. But we have also proved that this function is uniformly continuous when $\langle a,\eta\rangle - H_K(\eta)$ stays away from zero. Hence $\eta \mapsto W_\eta(a)$ can be extended to a locally constant function in $\{\eta \in \mathsf{R}^n;\, |\eta| = 1$ and $\langle a,\eta\rangle > H_K(\eta)\}$, a set which is connected in all non-trivial cases. Therefore $W_\eta(a)$ is independent of η. Lemma 4.2 is proved.

PROOF OF THEOREM 4.1, CONTINUED. We now turn to the construction of fundamental solutions E_η for $P(D)$. Suppose that $p(\eta) = c \neq 0$, where

p is the principal part of P. This is true for a dense set of vectors on the unit sphere in R^n. After a change of coordinates we may assume that $\eta = (0, \ldots, 0, 1)$. We then obtain

$$P(\zeta) = c\zeta_n{}^m + \text{terms of order less than } m \text{ in } \zeta_n .$$

We shall define a distribution $E = E_\eta$ by the Hörmander–Trèves formula (see Trèves [11] or the exposition in Friedman [2] or Gel'fand and Šilov [3])

$$\check{E}(\varphi) = (2\pi)^{-n} \int_{\mathsf{R}^{n-1}} d\xi' \int_{\eta_n = H(\xi')} \frac{\hat{\varphi}(\xi', \xi_n + i\eta_n)\, d\xi_n}{P(\xi', \xi_n + i\eta_n)}, \qquad \varphi \in \mathscr{D} . \tag{4.8}$$

Here H is a real-valued function in R^{n-1} which satisfies (A and a are constants):

1. $0 \leqq H(\xi') \leqq A$ for $\xi' \in \mathsf{R}^{n-1}$;
2. H is constant in every interval in some subdivision of R^{n-1} into a union of such;
3. $|P(\xi', \xi_n + i\eta_n)| \geqq a > 0$ when $\eta_n = H(\xi')$, $\xi_n \in \mathsf{R}$, $\xi' \in \mathsf{R}^{n-1}$.

The surface $\{\xi + i\eta \in \mathsf{C}^n ;\ \eta' = 0,\ \eta_n = H(\xi')\}$ is called Hörmander's steps.

For definiteness in later arguments it is advantageous to choose H as follows. For every $\xi' \in \mathsf{R}^{n-1}$, at least one of the $m+1$ strips

$$\{\tau \in \mathsf{C} ;\ |\mathrm{Im}\, \tau - k| < \tfrac{1}{2}\}, \qquad k = 0, \ldots, m ,$$

must be free from zeros of $\tau \mapsto P(\xi', \tau)$. We define $H(\xi') = k$ for some k such that the strip

$$\{\tau \in \mathsf{C} ;\ |\mathrm{Im}\, \tau - k| < \tfrac{1}{4}\}$$

does not contain any root of $P(\xi', \tau) = 0$. It is possible to do this in a way consistent with requirement 2 above. We may then take $a = |c|/4^m$. Also, in the application of Lemma 3.7 to follow, we may take the quantity $\varepsilon_1(\theta')$ of that lemma equal to $\frac{1}{4}$.

It is easy to prove that E is in fact a distribution and that $P(D)E = \delta$.

By a translation of φ we obtain

$$\begin{aligned} U(x) = E * \varphi(x) &= \check{E}(y \mapsto \varphi(x+y)) \\ &= (2\pi)^{-n} \int_{\mathsf{R}^{n-1}} e^{i\langle x', \xi' \rangle} f(\xi', x_n)\, d\xi' , \end{aligned} \tag{4.9}$$

where

$$f(\xi', x_n) = \int_{\mathrm{Im}\, \tau = H(\xi')} \frac{\hat{\varphi}(\xi', \tau)\, e^{ix_n \tau}\, d\tau}{P(\xi', \tau)} . \tag{4.10}$$

Now let $\varphi \in \mathcal{D}$ be a representative of T. We shall define a continuation of $\gamma_\infty U$ by means of the formula

$$V(x) = (2\pi)^{-n} \int_{\mathbf{R}^{n-1}} e^{i\langle x', \xi' \rangle} g(\xi', x_n)\, d\xi' , \tag{4.11}$$

where

$$g(\xi', x_n) = \int_{\Gamma(\xi')} \frac{\hat{\varphi}(\xi', \tau)\, e^{ix_n\tau}\, d\tau}{P(\xi', \tau)} . \tag{4.12}$$

Here $\Gamma(\xi')$ denotes a closed simple curve surrounding precisely those zeros of $P(\xi', \tau)$ which satisfy $\operatorname{Im} \tau > H(\xi') \geqq 0$. We claim that V is in C^∞ when $x_n > H_K(0, \ldots, 1)$ and that $U(x) = V(x)$ when $x_n > H_K(0, \ldots, 1)$ and $|x| > r$ for some r which does not depend on the direction $\eta = (0, \ldots, 1)$ we have chosen. (It follows from Holmgren's uniqueness theorem that this is true if η is non-characteristic and r is allowed to depend on η, but this result is insufficient for our purposes.)

We shall first prove that V is indeed a C^∞ function when $x_n > H_K(0, \ldots, 1)$. It follows from (4.5) that

$$\begin{aligned} &|(\partial/\partial\tau)^j (\hat{\varphi}(\zeta', \tau) e^{ix_n\tau})| \\ &\leqq C_{\varepsilon q}(1 + |x_n|)^{m-1} (1 + |\zeta'| + |\tau|)^{-q} \exp\big(H_{K+\varepsilon B}(\operatorname{Im} \zeta', \operatorname{Im} \tau) - x_n \operatorname{Im} \tau\big), \end{aligned}$$

when $P_k(\zeta', \tau) = 0$, $0 \leqq j < m_k$, $1 \leqq k \leqq s$. When $\operatorname{Im} \tau \geqq 0$ we get

$$\begin{aligned} &|(\partial/\partial\tau)^j (\hat{\varphi}(\zeta', \tau) e^{ix_n\tau})| \\ &\quad \leqq C_{\varepsilon q}(1 + |x_n|)^{m-1}(1 + |\zeta'|)^{-q} \exp\big(H_{K+\varepsilon B}(\operatorname{Im} \zeta', 0) + \\ &\qquad + \operatorname{Im} \tau (H_K(0, \ldots, 1) + \varepsilon - x_n)\big) \end{aligned} \tag{4.13}$$

at the zeros in question. Hence, if $x_n \geqq H_K(0, \ldots, 1) + \varepsilon$, the left hand side of (4.13) is majorized by

$$M_{\varepsilon q}(\zeta', x_n) = C_{\varepsilon q}(1 + |x_n|)^{m-1} (1 + |\zeta'|)^{-q} \exp\big(H_{K+\varepsilon B}(\operatorname{Im} \zeta', 0)\big) .$$

Applying Lemma 3.7 we obtain

$$\begin{aligned} &\left| \int_{\Gamma(\zeta')} \frac{\hat{\varphi}(\zeta', \tau)\, e^{ix_n\tau}}{P(\zeta', \tau)} d\tau \right| \\ &\leqq C'_{\varepsilon q}(1 + |x_n|)^{m-1} (1 + |\zeta'|)^{-q+a} \exp\big(H_{K+\varepsilon B}(\operatorname{Im} \zeta', 0)\big) , \end{aligned} \tag{4.14}$$

for any closed curve $\Gamma(\zeta')$ in the upper half-plane with distance to the zeros of $P(\zeta', \tau)$ bounded away from zero. (Note that $\log M_{\varepsilon q}(\zeta', x_n)$ is Lipschitz continuous in ζ' so that the supremum operation in (3.9), with $\varepsilon_2 = 1$, is absorbed by a multiplicative constant.) In particular we get by taking $\zeta' = \xi'$ real in (4.14),

$$(4.15) \qquad |g(\xi', x_n)| \leqq C'_{\varepsilon q}(1+|x_n|)^{m-1}(1+|\xi'|)^{-q+a}, \quad \xi' \in \mathsf{R}^{n-1}.$$

Now it is easy to see that the function V defined by (4.11) is infinitely differentiable in $R^{n-1} \times I$, where I is a bounded interval, if

$$(4.16) \qquad \xi' \mapsto \sup_{x_n \in I} |\xi'|^j |(\partial/\partial x_n)^k g(\xi', x_n)|$$

is integrable in R^{n-1} for all $j, k \geqq 0$. We have just proved that this is true for $k = 0$ if $x_n \geqq H_K(0, \ldots, 1) + \varepsilon$ when $x_n \in I$. However, we note that

$$(\partial/\partial x_n)^k g(\xi', x_n) = \int_{\Gamma(\xi')} \frac{(i\tau)^k \hat{\varphi}(\xi', \tau)\, e^{ix_n\tau}\, d\tau}{P(\xi', \tau)}$$

and that $\zeta_n{}^k \hat{\varphi}(\zeta)$ satisfies (4.5) if $\hat{\varphi}(\zeta)$ does. The estimate (4.15) is therefore valid also with g replaced by $(\partial/\partial x_n)^k g$ which proves that (4.16) is integrable for arbitrary $j, k \geqq 0$. It follows that V is C^∞ when $x_n > H_K(0, \ldots, 1) + \varepsilon$, hence when $x_n > H_K(0, \ldots, 1)$. It is also clear that $P(-D)V = 0$ there.

Next we shall prove that $U(x) = V(x)$ if $x_n > r$ where r is so large that $K \cup \operatorname{supp} \varphi \subset rB$. This will follow without any use of Lemma 3.7. In fact, if $\operatorname{Im} \tau \geqq 0$ we have

$$|\hat{\varphi}(\xi', \tau) e^{ix_n\tau}| \leqq C(1+|\tau|)^{-2}\, e^{(r-x_n)\operatorname{Im}\tau}.$$

This inequality shows immediately that the contour in (4.10) can be changed to that in (4.12) without affecting the value of the integral provided $x_n > r$. Therefore $f(\xi', x_n) = g(\xi', x_n)$ and consequently $U(x) = V(x)$.

If P is elliptic it now follows that $U(x) = V(x)$ when $x_n > H_K(0, \ldots, 1)$ and $|x'| > r$. In the general case we have to prove this separately; we are done if we prove that $\xi' \mapsto f(\xi', x_n) - g(\xi', x_n)$ can be extended to an entire function of exponential type at most r. The extension is given by

$$h(\zeta', x_n) = \int_{\operatorname{Im}\tau = A(1+|\zeta'|)} \frac{\hat{\varphi}(\zeta', \tau)\, e^{ix_n\tau}}{P(\zeta', \tau)}\, d\tau,$$

where A is so large that $P(\zeta', \tau) \neq 0$ when $\operatorname{Im} \tau \geqq A(1+|\zeta'|)$. Obviously $\zeta' \mapsto h(\zeta', x_n)$ is entire and to estimate its type we define the integration contours $\operatorname{Im} \tau = H(\xi')$ and $\Gamma(\xi')$ for complex ξ' while preserving their properties, in particular $\Gamma(\zeta')$ shall contain precisely those zeros of $P(\zeta', \tau)$ for which $\operatorname{Im} \tau > H(\zeta')$. We then have

$$h(\zeta',x_n) = \left[\int_{\operatorname{Im}\tau=H(\zeta')} - \int_{\Gamma(\zeta')}\right] \frac{\hat{\varphi}(\zeta',\tau)\, e^{ix_n\tau}}{P(\zeta',\tau)}\, d\tau\,. \tag{4.17}$$

The first integral in (4.17) is easy to estimate; we have

$$|\hat{\varphi}(\zeta',\tau)e^{ix_n\tau}| \leqq C_q(1+|\zeta'|+|\tau|)^{-q}\, e^{r|\operatorname{Im}\zeta'|+(r-x_n)\operatorname{Im}\tau}$$

when $\operatorname{Im}\tau \geqq 0$. In view of the fact that $|P|$ and $x_n \operatorname{Im}\tau$ are bounded from below on the contour of integration we obtain

$$\left|\int_{\operatorname{Im}\tau=H(\zeta')} \frac{\hat{\varphi}(\zeta',\tau)\, e^{ix_n\tau}}{P(\zeta',\tau)}\, d\tau\right| \leqq C_q'\, e^{r|\operatorname{Im}\zeta'|} \int_{\mathbf{R}} (1+|\zeta'|+|t|)^{-q}\, dt\,. \tag{4.18}$$

The second integral in (4.17) is estimated by means of Lemma 3.7. The result is (4.14). Combining (4.14) and (4.18) we see that $\zeta' \mapsto h(\zeta',x_n)$ is indeed the Fourier transform of a function with support in $\{x'; |x'| \leqq r\}$. But this function is precisely $x' \mapsto U(x) - V(x)$. This proves our claim that $U(x)$ and $V(x)$ agree for $x_n > H_K(0,\ldots,1)$, $|x'| > r$, and thus completes the proof of Theorem 4.1 for functionals on $\mathscr{D}'_P$.

If T is a functional on $\mathscr{E}_P$ we can use the reasoning above as follows. Let $\mu \in \mathscr{E}'$ represent T and let $U_\mu = \mu * E$ be the potential of μ with respect to the fundamental solution E defined by (4.8). Let $U_{\mu,j} = U_\mu * \varphi_j$ be regularizations of U_μ where $\varphi_j(x) = j^n\varphi(jx)$, $\varphi \in \mathscr{D}(B)$ being a fixed function with integral one. Then we obtain if $\operatorname{supp}\mu \subset rB$, $x_n > r + 1/j$ and $\Gamma(\xi')$ is defined as in (4.12),

$$\begin{aligned} U_{\mu,j}(x) &= E * \mu * \varphi_j(x) \\ &= (2\pi)^{-n} \int_{\mathbf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\, d\xi' \int_{\Gamma(\xi')} \frac{\hat{\mu}(\xi',\tau)\, \hat{\varphi}_j(\xi',\tau)\, e^{ix_n\tau}\, d\tau}{P(\xi',\tau)}. \end{aligned} \tag{4.19}$$

Since $\mu * \varphi_j \in \mathscr{D}$ and has Fourier transform $\hat{\mu}\hat{\varphi}_j$ satisfying (v) with K replaced by $K + B/j$, the arguments above show that the right hand side of (4.19) defines a smooth function V_j when $x_n > H_K(0,\ldots,1) + 1/j$ which moreover agrees with $U_{\mu,j}$ when $x_n > H_K(0,\ldots,1) + 1/j$ and $|x| > r + 1/j$. Considering V_j as a distribution we obtain

$$V_j(\psi) = (2\pi)^{-n} \int_{\mathbf{R}^{n-1}} d\xi' \int_{\Gamma(\xi')} \frac{\hat{\mu}(\xi',\tau)\, \hat{\varphi}_j(\xi',\tau)\, \hat{\psi}(-\xi',-\tau)\, d\tau}{P(\xi',\tau)}$$

provided $\operatorname{supp}\psi$ is contained in the domain of definition of V_j. Now let ψ be a fixed and let j tend to $+\infty$. Using estimates analogous to (4.15) we find that the integral converges to

$$V(\psi) = (2\pi)^{-n} \int_{\mathbf{R}^{n-1}} d\xi' \int_{\Gamma(\xi')} \frac{\hat{\mu}(\xi',\tau)\,\hat{\psi}(-\xi',-\tau)}{P(\xi',\tau)}\,d\tau\,.$$

In view of the Banach–Steinhaus theorem the limit $V(\psi)$ defines a distribution V in $\{x; x_n > H_K(0,\ldots,1)\}$ and it is obvious that $P(D)V=0$. (That V is a distribution follows also directly from similar estimates of the integral defining it.) It is also clear that V and U_μ agree in the open set $\{x; x_n > H_K(0,\ldots,1), |x|>r\}$. The rest of the proof is analogous to that for functionals on $\mathscr{D}'_P$. Theorem 4.1 is proved.

We note two simple consequences of the theorem.

COROLLARY 4.3. *A functional on $\mathscr{D}'_P$ and its restriction to $\mathscr{E}_P$ have the same convex carriers.*

PROOF. Theorem 4.1 shows that the convex carriers of a functional are completely determined by its Fourier transform. But the latter is an element in $A(\mathbf{C}^n)/P\cdot A(\mathbf{C}^n)$ which is the same for a functional on $\mathscr{D}'_P$ and its restriction to $\mathscr{E}_P$.

COROLLARY 4.4. *Let $P=QR$ where Q and R are without common factor. Then a functional on $\mathscr{D}'_P$ is carried by a convex compact set K if and only if its restrictions to $\mathscr{D}'_Q$ and $\mathscr{D}'_R$ are carried by K. Similarly for $(\mathscr{E}_P)'$.*

This is also an immediate consequence of Theorem 4.1.

REFERENCES

1. G. Bengel, *Das Weylsche Lemma in der Theorie der Hyperfunktionen,* Math. Z. 96 (1967), 373–392.
2. A. Friedman, *Generalized functions and partial differential equations,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.
3. I. M. Gel'fand and G. E. Šilov, *Generalized functions* II, Fizmatgiz, Moscow, 1958. (Russian; English and German translations.)
4. A. Grothendieck, *Sur les espaces de solutions d'une classe générale d'équations aux dérivées partielles,* J. Analyse Math. 2 (1953), 243–280.
5. L. Hörmander, *Linear partial differential operators,* Springer-Verlag, Berlin, 1963.
6. V. K. Ivanov, *The distribution of singularities of a potential and a three-dimensional analogue of Pólya's theorem,* Mat. Sb. 40 (82) (1956), 319–338. (Russian.)
7. A. A. Kozmanova and A. V. Mesis, *On the singularities of metaharmonic functions,* Matematičeskie Zapiski 5 (1965), no. 2, 66–70. (Russian.)
8. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution,* Ann. Inst. Fourier (Grenoble) 6 (1955/56), 271–355.

9. G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Math. Z. 29 (1929), 549–640.
10. L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
11. F. Trèves, *Solution élémentaire d'équations aux dérivées partielles dépendant d'un paramètre*, C. R. Acad. Sci. Paris 242 (1956), 1250–1252.
12. B. L. van der Waerden, *Algebra* I, 7th edition. Springer-Verlag, Berlin, 1966.

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