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MEASURES AND PSEUDOMEASURES ON COMPACT SUBSETS OF THE LINE

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1. Introduction.

Suppose that f is a continuous function defined on a compact subset E of the real line R, such that

$$\left|\int f(x)\,d\mu(x)\right| \leqq c\ \sup_{y\in R}\left|\int e^{-ixy}\,d\mu(x)\right| \quad \text{for every } \mu\in M(E)\ ,$$

where c is a constant and $M(E)$ is the class of finite, complex-valued Borel measures whose support is contained in E. Does it follow that f is the restriction to E of a Fourier transform? That is, must there exist an element $\hat{f}$ of $L^1(R)$ such that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y)\,e^{-ixy}\,dy \quad \text{for } x\in E\,?$$

In Section 2 we construct a set E, consisting of a convergent sequence and its limit point, such that the answer is No. In Section 3 we show that the answer is No for every set E of this description except when E is a Helson set. In Section 4 we construct a perfect set F for which the answer is No, and which also has the following property: every pseudomeasure S supported by F decomposes uniquely into a sum $S=S_d+S_c$, where $\hat{S}_d$ is almost periodic and S_c is a continuous measure (i.e., one which annihilates countable sets); and such that the total variation of S_c is bounded by a constant times ess $\sup_{y\in R}|\hat{S}(y)|$. In Section 5 we list some open questions.

Before further discussion, we need to introduce some notation. Let A denote the Banach algebra of Fourier transforms g of functions $\hat{g}$ in $L^1=L^1(R)$. In A we have pointwise multiplication and the norm induced by L^1. Let PM denote the conjugate space of A. Its elements are called *pseudomeasures*, S: for each $\hat{S}\in L^\infty=(L^1)^*$, S is the functional on A defined by

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$$(g, S) = \int_{-\infty}^{\infty} \hat{g}(y)\overline{\hat{S}(y)}\,dy \quad \text{for } g \in A\ .$$

Thus

$$\|S\|_{PM} = \|\hat{S}\|_{L^\infty} = \operatorname*{ess\,sup}_{y \in R} |\hat{S}(y)|\ .$$

Let C denote the Banach algebra of continuous functions on R which vanish at infinity, with pointwise multiplication and the supremum norm, $\|f\|_C$. Then $M = M(R)$, with the total variation norm $\|\mu\|_M$, is the conjugate space of C; let

$$(f, \mu) = \int_{-\infty}^{\infty} f(x)\,\overline{d\mu(x)} = \lim_{Y \to \infty} \frac{1}{2Y} \int_{-Y}^{Y} \hat{f}(y)\overline{\hat{\mu}(y)}\,dy \quad \text{for } f \in C, \qquad \mu \in M;$$

$$\hat{\mu}(y) = \int_{-\infty}^{\infty} e^{ixy}\,d\mu(x) \quad \text{for } y \in R\ .$$

We have

$$A \subset C, \quad \text{and} \quad \|g\|_C \leqq \|g\|_A \quad \text{for } g \in A;$$
$$M \subset PM, \quad \text{and} \quad \|\mu\|_{PM} \leqq \|\mu\|_M \quad \text{for } \mu \in M\ .$$

We denote by AP the closed subspace of L^∞ consisting of the continuous almost periodic functions. It contains $\hat{\mu}$ for every $\mu \in M$ with countable support. It contains $\hat{S}$ for every $S \in PM$ with countable and compact support ([12]; or cf. [10, Chapter 6]).

Let $I(E)$ denote the closed ideal in A consisting of those functions in A which vanish on E; let $A(E)$ denote the quotient algebra $A/I(E)$. An element of $A(E)$ may be viewed as the restriction to E of a function in A. When we say, "$f \in A(E)$", we mean that $f|_E \in A(E)$. The norm in $A(E)$ is given by

$$\|f\|_{A(E)} = \inf\{\|g\|_A : g \in A \text{ and } g|_E = f|_E\}\ . \tag{1.1}$$

Let $N(E)$ denote $I(E)^\perp$, the subspace of PM which is the conjugate space of $A(E)$. Similarly, $M(E) = C(E)^*$. The set E is a *Helson set* if $A(E) = C(E)$; or, equivalently, if $M(E) = N(E)$; or, if the quantity

(1.2)

$$h(E) = \inf\left\{\frac{\|\mu\|_{PM}}{\|\mu\|_M} : \mu \in M(E),\ \mu \neq 0\right\} = \inf\left\{\frac{\|f\|_{C(E)}}{\|f\|_{A(E)}} : f \in A(E), f \neq 0\right\}$$

is positive.

For $f \in C(E)$, let

$$\|f\|_{B(E)} = \sup\left\{\frac{|(f,\mu)|}{\|\mu\|_{PM}} : \mu \in M(E),\ \mu \neq 0\right\}.$$

Let $B(E)$ be the class of those functions $f \in C(E)$ for which $\|f\|_{B(E)}$ is finite. Clearly,

$$A(E) \subset B(E), \quad \text{and} \quad \|f\|_{B(E)} \leqq \|f\|_{A(E)} \quad \text{for } f \in A(E);$$
$$B(E) \subset C(E), \quad \text{and} \quad \|f\|_{C(E)} \leqq \|f\|_{B(E)} \quad \text{for } f \in C(E).$$

It is easy to show that $B(E)$ forms a Banach space.

Let us restate the question we asked at the outset: Given the compact set E, does $A(E) = B(E)$? The characterization of sets E for which $A(E) = B(E)$ remains a problem. This paper gives examples of sets E for which $A(E) \neq B(E)$; in the other direction, Helson proved that if every portion of E (i.e., every non-void intersection of E with an open interval) supports a measure μ such that $\lim_{|y|\to\infty} \hat{\mu}(y) = 0$, then $A(E) = B(E)$. It is true, furthermore, that if $\eta > 0$ and if for every $f \in C(E)$,

$$\|f\|_{B(E)} = \sup\left\{\frac{|(f,\mu)|}{\|\mu\|_{PM}} : \mu \in M(E),\ \mu \neq 0, \text{ and } \frac{\limsup_{|y|\to\infty} |\hat{\mu}(y)|}{\|\mu\|_{PM}} \leqq 1-\eta\right\},$$

then $A(E) = B(E)$. For proofs of these assertions see [13, Section 10]; see also the methods of [5].

Another question of interest is the characterization of the sets E such that the $A(E)$ and $B(E)$ norms are equivalent in $A(E)$, that is, for which the quantity

$$b(E) = \inf_{f \in A(E)} \frac{\|f\|_{B(E)}}{\|f\|_{A(E)}}$$

is positive. In the terminology of Dixmier [6], $b(E)$ is the *characteristic* of $M(E)$ in $N(E)$; it is positive if and only if the weak* limits of sequences in the unit ball of $M(E)$ fill a ball of positive radius in $N(E)$ (cf. [6]). So far as we know, $b(E)$ always equals one.

We have different questions if, in the definition of $B(E)$ and $b(E)$, we replace $M(E)$ by $M'(E)$, the space of measures supported by finite subsets of E. For this case let us write $B'(E)$ and $b'(E)$. Rudin constructed a set E for which $C(E) = B'(E) \neq A(E)$ and $b'(E) = 0$ ([16]; or [9, p. 103]; cf. also [8], [11], [20]). In the other direction, Bochner [2] proved the fundamental result that every element of $B'(R)$ is a Fourier–Stieltjes transform; for the generalization to locally compact abelian groups see [17, 1.9.1] (cf. also [14] and [7]). Kreĭn proved that $B'(E) = A(E)$ if E is a compact interval (see [15], or [1, Section 77]); and Rosenthal [15] has shown that if every portion of a compact set E has positive measure, then $B'(E) = A(E)$.

2. An elementary example.

One way to construct a set E with $A(E) \neq B(E)$ is as follows. Let $E = \bigcup_{j=0}^{\infty} F_j$, where $F_0 = \{0\}$ and, for $j > 0$, F_j is an arithmetic progression of length 4^j:

$$F_j = \{r_j + ks_j : k = 1, 2, \ldots, 4^j\},$$

where the r_j's and s_j's are chosen so that F_j is contained, say, in the interval $((j+1)^{-1}, j^{-1})$; and so that the set

$$\{r_j : j = 1, 2, \ldots\} \cup \{s_j : j = 1, 2, \ldots\}$$

is linearly independent over the rationals. From a well-known result about arithmetic progressions (cf. [21, V. 4.7]; or [9, p. 134, Lemma 2]), we know that $h(F_j) < c2^{-j}$ for all j, for some constant c (h is defined by (1.2)). Therefore there is a function $f_j \in A$ such that $\|f_j\|_{A(F_j)} = 1$, $\|f_j\|_{C(F_j)} < c2^{-j}$. Let f be the function in $C(E)$ defined by $f|_{F_j} = f_j$ for $j > 0$ and $f(0) = 0$. We shall show that $f \notin A(E)$ and that $f \in B(E)$.

If f were in $A(E)$, then for each $\varepsilon > 0$ we would have, taking $j > \varepsilon^{-1}$,

$$\|f\|_{A(E \cap [-\varepsilon, \varepsilon])} \geqq \|f_j\|_{A(F_j)} = 1$$

(cf. (1.1)). But this situation is impossible, by a classical theorem of Wiener (cf. [17, 2.6.4 and 7.2]): if $g \in A$ and $g(0) = 0$, then

$$\lim_{\varepsilon \to 0} \|g\|_{A([-\varepsilon, \varepsilon])} = 0 .$$

Therefore $f \notin A(E)$.

Now we shall prove that $f \in B(E)$. Consider an arbitrary $\mu \in M(E)$, written in the form $\mu = \sum_{j=0}^{\infty} \mu_j$, where $\mu_j \in M(F_j)$; and look at the functions $\hat{\mu}_j(y)$. It may be shown from Kronecker's Theorem [4, p. 53 or 99] and our independence condition that

$$\|\mu\|_{PM} = \sum_{j=0}^{\infty} \|\mu_j\|_{PM} \quad \text{for } \mu \in M(E) .$$

Then for $g \in C(E)$, we have

$$\begin{aligned}\left|\int g\,d\mu\right| &= \left|\sum_{j=0}^{\infty}\int g\,d\mu_j\right| \\ &\leqq \sum_{j=0}^{\infty}\|g\|_{A(F_j)}\,\|\mu_j\|_{PM} \\ &\leqq (\sup_{0\leqq j<\infty}\|g\|_{A(F_j)})\,\|\mu\|_{PM} \quad \text{for all } \mu\in M(E)\,.\end{aligned}$$

It follows that

$$\|g\|_{B(E)} = \sup_{0\leqq j<\infty}\|g\|_{A(F_j)} \quad \text{for } g\in C(E)\,.$$

In particular, $\|f\|_{B(E)}=1$ and $f\in B(E)$.

3. Countable sets.

THEOREM I. *Let E be a countable set with a finite number of accumulation points. If E is not a Helson set, then $A(E)\neq B(E)$.*

NOTATION. For $\lambda>0$ let $K_\lambda(x)$ denote the function whose graph is an isosceles triangle, centered at 0 with height 1 and base 2λ. Then the family $\{\hat{K}_\lambda(y)\colon \lambda>0\}$ is the familiar Fejér kernel.

LEMMA 1. *Given $0<\varepsilon<1$ and a finite set G of nonzero real numbers, there exists a positive measure ν, with finite support contained in the real subgroup generated by G, such that the following conditions hold:*

(i) *$\nu(\{0\})=1$ and $\nu(\{x\})=1$ for $x\in G$.*
(ii) *$\|\nu * K_\lambda\|_A\leqq 1+\varepsilon$ for small enough $\lambda>0$.*
(iii) *The support of ν is contained in $\{x\colon |x|\leqq a(1+4\varepsilon^{-1})\}$, where $a=\max\{|x|: x\in G\}$.*

PROOF. The measure ν will be a modification of a Bochner–Fejér measure ϱ (cf. [3, p. 80–88]). Let $t_1,\ldots,t_N$ be real numbers which are linearly independent over the rationals, such that each point of G can be written as a linear combination of the t_n's, using integer coefficients. Let ϱ be the measure defined by the equation

$$\hat{\varrho}(y) = \prod_{n=1}^{N}\left(\sum_{p=-P}^{P}\left(1-\frac{|p|}{P}\right)e^{iypt_n}\right).$$

Then if P is sufficiently large, there exists a positive measure $\sigma\in M(G)$ such that $\|\sigma\|_M\leqq\frac{1}{4}\varepsilon$ and $(\varrho+\sigma)(\{x\})=1$ for $x\in G$. Note that $\nu=\varrho+\sigma$ would satisfy (i) and (ii); but to obtain (iii) also, let $b>1$ and let

$$\nu = \left(\frac{b}{b-1}K_{ab}-\frac{1}{b-1}K_a\right)(\varrho+\sigma)\,.$$

Then ν agrees with $\varrho+\sigma$ on $[-a,a]$ and vanishes outside $[-ab,ab]$; also,

$$\|\nu * K_\lambda\|_A \leqq \frac{b+1}{b-1}\,(1+\tfrac{1}{4}\varepsilon) \quad \text{for small enough } \lambda>0\,.$$

If we set $b=1+4\varepsilon^{-1}$, then the measure ν satisfies (i), (ii), and (iii). The lemma is proved.

PROOF OF THEOREM I. It suffices to deal with the case in which E has only one accumulation point, equal to zero. We shall select two subsets of E:

$$F=\bigcup_{k=0}^{\infty} F_k, \quad G=\bigcup_{k=0}^{\infty} G_k, \quad \text{with } F_0=G_0=\{0\} \text{ and } G_k\subset F_k;$$

where the F_k are finite, pairwise disjoint sets. We shall select also functions $g_k\in A(E)$ for $k=1,2,\ldots$. The following conditions will hold.

(3.1) $\|g_k\|_{C(E)} < 2^{-k}; \quad \|g_k\|_{A(E)} = 1; \quad g_k(x) = 0$ for $x\in E\setminus G_k$.

(3.2) $$\|g_k\|_{A(F_k)} \geqq \tfrac{1}{3}\,.$$

(3.3) If $\mu\in M(F)$ and $\mu_k = \mu|_{F_k}$, then $\sum_{k=0}^{\infty}\|\mu_k\|_{PM} \leqq 13\|\mu\|_{PM}$.

Condition (3.2) and the methods of Section 2 show that although the function

$$g=\sum_{k=1}^{\infty} g_k$$

belongs to $C(E)$, nevertheless $g\notin A(E)$. We shall first describe the selection process inductively, and then prove that $g\in B(E)$.

Let $\varepsilon_j>0$, $\prod_{j=1}^{\infty}(1+\varepsilon_j)<2$. It is easy to show that we may choose a finite set $G_1\subset E\setminus\{0\}$ and a function $g_1\in A(E)$ satisfying (3.1) for $k=1$. Let ν_1 be a measure selected as in Lemma 1 where $\varepsilon=\varepsilon_1$ and $G=G_1$. Let F_1 be the intersection of $E\setminus\{0\}$ with the support of ν_1.

Let $k\geqq 2$ and assume that G_j, g_j, ν_j, F_j have been selected for $j=1,2,\ldots,k-1$. Let $\eta>0$. Since F_j is finite, there exists a number T_j such that if $\mu\in M(F_j)$ and $z\in R$, then in every interval of length T_j, $\hat\mu$ takes on a value differing from $\hat\mu(z)$ by at most $2^{-j}\eta\|\mu\|_{PM}$ (cf. [13, Lemma 2]); that is, given $\eta>0$ and F_j, there exists T_j such that

(3.4) $$\hat\mu([y,y+T_j]) \text{ is } (2^{-j}\eta\|\mu\|_{PM})\text{-dense in } \hat\mu(R)$$
$$\text{for every } y\in R \text{ and for all } \mu\in M(F_j)\,.$$

Now we choose a small enough $\delta_k > 0$ so that

$$2\delta_k \leqq \min\{|x-y|: x \in \bigcup_{j=1}^{k-1} F_j,\ y \in E,\ x \neq y\}; \tag{3.5}$$

and so that for $\mu \in M([-\delta_k, \delta_k])$, the value of $\hat{\mu}$ is "almost constant" on every interval of length $\sum_{j=1}^{k-1} T_j$; precisely,

$$\begin{cases} \hat{\mu}|(y_1) - \hat{\mu}(y_2)| \leqq 2^{-k}\eta \|\mu\|_{PM} \\ \text{if} \\ \mu \in M([-\delta_k, \delta_k]) \quad \text{and} \quad |y_1 - y_2| \leqq \sum_{j=1}^{k-1} T_j . \end{cases} \tag{3.6}$$

This choice is possible because if $\delta > 0$, $\mu \in M([-\delta, \delta])$, and $y_1, y_2 \in R$, then

$$\begin{aligned} |\hat{\mu}(y_1) - \hat{\mu}(y_2)| &\leqq \|(e^{ixy_1} - e^{ixy_2})\|_{A([-\delta,\delta])} \|\mu\|_{PM} \\ &= \|(e^{ix(y_1-y_2)} - 1)\|_{A([-\delta,\delta])} \|\mu\|_{PM} \leqq 4\delta |y_1 - y_2| \, \|\mu\|_{PM} . \end{aligned}$$

Now let $a_k > 0$ be so small that

$$\delta_k \geqq a_k(1 + 4/\varepsilon_k) . \tag{3.7}$$

Let $G_k \subset \{x \in E: 0 < |x| \leqq a_k\}$ be a finite set, and let $g_k \in A$, such that (3.1) is satisfied. Let ν_k be a measure selected as in Lemma 1 where $\varepsilon = \varepsilon_k$ and $G = G_k$. Let F_k be the intersection of $E \setminus \{0\}$ and the support of ν_k; by (3.7) we know that

$$F_k \subset [-\delta_k, \delta_k] . \tag{3.8}$$

Our selection process is now completely described.

To prove (3.2) it suffices to prove that for $k = 1, 2, \ldots,$

$$\|g_k\|_{A(F_k)} \geqq 1/(2 + \varepsilon_k) .$$

For $\mu \in M(E)$, let $\mu' = ((\nu_k * K_\lambda) - K_\lambda)\mu$. For small enough $\lambda > 0$, $\|\nu_k * K_\lambda\|_A \leqq 1 + \varepsilon_k$ and $\mu'|_{G_k} = \mu|_{G_k}$; so that

$$\|\mu'\|_{PM} \leqq (2 + \varepsilon_k)\|\mu\|_{PM} ,$$

and

$$\begin{aligned} \|g_k\|_{A(F_k)} = \sup_{\mu \in M(F_k)} \frac{|(g_k, \mu)|}{\|\mu\|_{PM}} &= \sup_{\mu \in M(E)} \frac{|(g_k, \mu')|}{\|\mu'\|_{PM}} \\ &\geqq \sup_{\mu \in M(E)} \frac{|(g_k, \mu)|}{(2 + \varepsilon_k)\|\mu\|_{PM}} = \frac{1}{2 + \varepsilon_k}. \end{aligned}$$

Now to prove (3.3). We know that (3.4), (3.6), and (3.8) hold for every $j \geqq 1$ or $k \geqq 2$. It is an easy exercise to show from these facts that for every $n \geqq 2$, if $\mu = \sum_{k=1}^{n} \mu_k$, where $\mu_k \in M(F_k)$ for $1 \leqq k \leqq n-1$ and $\mu_n \in M([-\delta_n, \delta_n])$, then

$$\hat{\mu}(R) \text{ is } \left(2\eta \sum_{k=1}^{n} \|\mu_k\|_{PM}\right)\text{-dense in } \sum_{k=1}^{n} \hat{\mu}_k(R) .$$

It is another easy exercise to show that

$$\sum_{k=1}^{n} \|\mu_k\|_{PM} \leqq 12 \sup \left| \sum_{k=1}^{n} \hat{\mu}_k(R) \right| .$$

It follows that

$$(\tfrac{1}{12} - 2\eta) \sum_{k=1}^{n} \|\mu_k\|_{PM} \leqq \|\mu\|_{PM} = \sup |\hat{\mu}(R)| ,$$

for every n. Condition (3.3) follows if η is small enough.

It remains to show that $g \in B(E)$. First we define a map $\mu \to \tilde{\mu}$ from $M(E)$ into $M(F)$ such that

$$\text{(3.9)} \qquad \|\tilde{\mu}\|_{PM} \leqq 2\|\mu\|_{PM}, \qquad \tilde{\mu}|_{G_j} = \mu|_{G_j} \quad \text{for } j = 0, 1, \ldots .$$

By (3.5), for each $k \geqq 2$,

$$\{x : (\nu_1 * \ldots * \nu_{k-1} * K_{\delta_k})(x) \neq 0\} \cap E \subset (-\delta_k, \delta_k) \cup \bigcup_{j=1}^{k-1} F_j .$$

Therefore if $\mu \in M(E)$, the measure

$$\text{(3.10)} \qquad (\nu_1 * \ldots * \nu_{k-1} * K_{\delta_k})\mu$$

is supported in the set $E \cap ((-\delta_k, \delta_k) \cup \bigcup_{j=1}^{k} F_j)$ and thus has the form

$$K_{\delta_k}\mu + \sum_{j=1}^{k} \tilde{\mu}_j ,$$

where $\tilde{\mu}_j \in M(F_j)$. But (3.5) insures that for $j < k$, $\tilde{\mu}_j$ is independent of k; that the norm of (3.10) is no greater than $\|\mu\|_{PM} \prod_{j=1}^{k-1}(1+\varepsilon_j) < 2\|\mu\|_{PM}$; and that $\tilde{\mu}_j|_{G_j} = \mu|_{G_j}$. Let $\tilde{\mu}_0 = \mu|_{\{0\}} = \lim_{k\to\infty} K_{\delta_k}\mu$ and let $\tilde{\mu} = \sum_{j=0}^{\infty} \tilde{\mu}_j$; (3.9) is immediate. Then

$$\begin{aligned}\left|\int g\,d\mu\right| = \left|\int g\,d\tilde{\mu}\right| &= \left|\sum_{j=1}^{\infty} \int g\,d\tilde{\mu}_j\right| \\ &\leqq (\sup_j \|g_j\|_{A(E)}) \sum_{j=1}^{\infty} \|\tilde{\mu}_j\|_{PM} \\ &\leqq 13\,\|\tilde{\mu}\|_{PM} \leqq 26\,\|\mu\|_{PM} \quad \text{for all } \mu \in M(E) .\end{aligned}$$

Therefore $\|g\|_{B(E)} \leqq 26$ and $g \in B(E)$. Theorem I is proved.

REMARK. The support of $S \in N(E)$ is compact and countable, so that $\hat{S} \in AP$, and thus the restrictions of S to finite subsets of E are well-defined. Slightly modified, the above proof shows that for all $S \in N(E)$, (g,S) may be defined in the natural manner, $(g,S)=\sum_{j=1}^{\infty}(g_j, S|_{G_j})$, and that $|(g,S)| \leqq 26\|S\|_{PM}$.

4. Perfect sets.

THEOREM II. *There exists a perfect set E such that for every portion G of E, $A(G) \neq B(G)$.*

PROOF. The set E will be the closure of the union of a sequence of arithmetic progressions

$$F_j = \{r_j + ms_j : m = 1,\ldots,4^j\}, \quad j = 1,2,\ldots ,$$

which we define inductively. The set $\{r_j : j=1,2,\ldots\} \cup \{s_j : j=1,2,\ldots\}$ will be chosen to be linearly independent over the rationals.

Select F_1, subject only to the condition that r_1 and s_1 be independent. Suppose that $k \geqq 1$ and that $F_1,\ldots,F_k$ have been selected. Consider an arbitrary partition P of this class of sets into two classes:

$$P: \{F_1,\ldots,F_k\} = \{F_{j_1},\ldots,F_{j_n}\} \cup \{F_{j_{n+1}},\ldots,F_{j_k}\} .$$

Let $V_\lambda(x) = 2K_{2\lambda}(x) - K_\lambda(x)$, so that $\{\hat{V}_\lambda(y) : \lambda > 0\}$ is the familiar de la Vallée Poussin kernel:

$$\|V_\lambda\|_A \leqq 3; \qquad \begin{aligned} V_\lambda(x) &= 0 \quad \text{for } |x| \geqq 2\lambda ,\\ V_\lambda(x) &= 1 \quad \text{for } |x| \leqq \lambda . \end{aligned}$$

By independence of $\{r_1,\ldots,r_k,s_1,\ldots,s_k\}$ and by Lemma 1 (parts (i) and (ii), using $\varepsilon < \frac{1}{3}$), we may select a measure σ_P such that

$$\begin{aligned} \sigma_P(\{x\}) &= 1 \quad \text{if } x \in \textstyle\bigcup_{i=1}^{n} F_{j_i} ,\\ \sigma_P(\{x\}) &= 0 \quad \text{if } x \in \textstyle\bigcup_{i=n+1}^{k} F_{j_i}; \end{aligned}$$

and such that for a small enough $\lambda_P > 0$ we have, whenever $\lambda \leqq \lambda_P$,

$$\begin{aligned} \|\sigma_P * V_\lambda\|_A &\leqq 4 ,\\ (\sigma_P * V_\lambda)(x) &= 0 \quad \text{if} \quad \text{distance } (x, \textstyle\bigcup_{i=n+1}^{k} F_{j_i}) \leqq \lambda ,\\ (\sigma_P * V_\lambda)(x) &= 1 \quad \text{if} \quad \text{distance } (x, \textstyle\bigcup_{i=1}^{n} F_{j_i}) \leqq \lambda . \end{aligned}$$

Let d_k be the minimum value of λ_P, considering all the possible partitions P. We now select F_{k+1}, subject to two conditions: first, $F_{k+1} \subset (x_0 - \frac{1}{2}d_k,\, x_0 + \frac{1}{2}d_k)$, where x_0 is a point of $\bigcup_{j=1}^{k} F_j$ chosen so that the distance from x_0 to the rest of the set $\bigcup_{j=1}^{k} F_j$ is maximal; second, r_{k+1} and s_{k+1} are chosen so that the set $\{r_1,\ldots,r_{k+1},s_1,\ldots,s_{k+1}\}$ is linearly independent over the rationals. The first stipulation insures that the set E, which is the closure of $\bigcup_{j=1}^{\infty} F_j$, is a perfect set. Clearly $E \subset \bigcup_{j=1}^{k} F_j + (-d_k, d_k)$ for every k, and hence E is a totally disconnected set of Lebesgue measure zero.

Let $S \in N(E)$. We shall show that

$$(4.1) \qquad S = \mu + \sum_{j=1}^{\infty} S_j, \quad \text{where } S_j \in M(F_j)\,,$$
$$\mu \in M(E), \quad \text{and} \quad \|\mu\|_M \leqq 16\|S\|_{PM}\,.$$

For an arbitrary k, let ν_k be a measure which assigns mass 1 to each point of $\bigcup_{j=1}^{k} F_j$ and annihilates each point of $\bigcup_{j=k+1}^{\infty} F_j$, and such that

$$\|\nu_k * V_{d_p}\|_A \leqq 4 \text{ for } p = k, k+1, \ldots\,.$$

Then for each k, the sequence $\{(\nu_k * V_{d_p})S \colon p = k, k+1, \ldots\}$ is bounded in norm by $4\|S\|_{PM}$ and hence includes a subsequence which converges weak* to an element of $M(\bigcup_{j=1}^{k} F_j)$. Therefore by a diagonal process we may find $S_j \in M(F_j)$ for $j = 1, 2, \ldots$ and a sequence $\{p(m) \colon m = 1, 2, \ldots\}$, such that for each k,

$$\text{weak*}\lim_{m\to\infty} (\nu_k * V_{d_{p(m)}})S = \sum_{j=1}^{k} S_j\,.$$

Since by Kronecker's theorem

$$\sum_{j=1}^{k} \|S_j\|_{PM} = \Big\|\sum_{j=1}^{k} S_j\Big\|_{PM} \leqq 4\|S\|_{PM} \quad \text{for all } k\,,$$

the series $\sum_{j=1}^{\infty} S_j$ converges in norm to a pseudomeasure whose transform is in AP. To show that the remainder,

$$\mu = S - \sum_{j=1}^{\infty} S_j\,,$$

is a measure with $\|\mu\|_M \leqq 16\|S\|_{PM}$, it suffices to prove that

$$(4.2) \qquad |(g, \mu)| \leqq 16\, \|S\|_{PM}\, \|g\|_{C(E)}$$

for all the step functions g in $C(E)$, since E is totally disconnected. So we consider first a function $g \in C(E)$ with range $\{0, 1\}$. Let $\varepsilon > 0$. Fix k large enough so that

$$\Big(\sum_{j=k+1}^{\infty} \|S_j\|_{PM}\Big) \|g\|_{A(E)} < \varepsilon \quad \text{and hence} \quad \Big|\Big(g, \mu - \Big(S - \sum_{j=1}^{k} S_j\Big)\Big)\Big| < \varepsilon\,.$$

Now fix $p = p(m) > k$ large enough so that

$$\left|\left(g, \sum_{j=1}^{k} S_j - S(\nu_k * V_{d_p})\right)\right| < \varepsilon .$$

Then

$$|(g, \mu - S(1 - \nu_k * V_{d_p}))| < 2\varepsilon .$$

Since g is constant (0 or 1) on each set $F_{k+1}, \ldots, F_p$, there exists a measure ν such that $\nu * V_{d_p}$ agrees with g on $\bigcup_{j=k+1}^{p} F_j + (-d_p, d_p)$ and equals zero on $\bigcup_{j=1}^{k} F_j + (-d_p, d_p)$, on which $(1 - \nu_k * V_{d_p})$ also vanishes; and such that $\|\nu * V_{d_p}\|_A \leqq 4$. Thus

$$|(g, S(1 - \nu_k * V_{d_p}))| = |(\nu * V_{d_p}, S)| \leqq 4\|S\|_{PM} ,$$

and hence

$$|(g, \mu)| \leqq 4\|S\|_{PM} + 2\varepsilon .$$

Consequently $|(g, \mu)| \leqq 4\|S\|_{PM}$; and therefore if g is an arbitrary step function in $C(E)$, the inequality (4.2) holds. Our decomposition property (4.1) is proved.

For each j, let $f_j \in A$ be constant on each of the 4^j sets $\{x + [-d_j, d_j] \colon x \in F_j\}$ and zero on every other portion of E, such that

$$\|f_j\|_{A(F_j)} \geqq 1 ; \qquad \|f_j\|_{C(E)} \leqq c 2^{-j} .$$

Let $f = \sum_{j=1}^{\infty} f_j$. Then clearly $f \in C(E)$. If G is any portion of E, then by the methods of Section 2 it follows that $f \notin A(G)$; and it is easy to show, using (4.1), that $f \in B(G)$; thus $A(G) \neq B(G)$. Theorem II is proved.

Remark. Rudin ([16]; or [8, p. 103]) constructed a perfect set of multiplicity E whose points are independent over the rationals. Then E is not a Helson set, even though $\|\mu\|_M = \|\mu\|_{PM}$ for all μ with finite support; so the "mischief" which makes $h(E) = 0$ all occurs among the continuous measures. But on the set F of Theorem II, the mischief is due to the discrete measures.

5. Some questions.

(1) Does $M(E)$ always have characteristic 1 in $N(E)$ (cf. Section 1)? Can it ever have characteristic 0 in $N(E)$, or is $A(E)$ always closed in the $B(E)$ norm?

(2) We say that x is a *non-Helson point of* E if for every $\varepsilon > 0$, $E \cap (x - \varepsilon, x + \varepsilon)$ is a non-Helson set; if, for some $\varepsilon > 0$, x is the only non-Helson point of E in $(x - \varepsilon, x + \varepsilon)$, then we call x an *isolated* non-Helson point of E. Is $A(E) \neq B(E)$ whenever E possesses an isolated non-Helson point?

(3) Is $A(E) \neq B(E)$ whenever E is a countable non-Helson set?

(4) For $\theta > 2$, let

$$E_\theta = \left\{ \sum_{j=1}^{\infty} \varepsilon_j \theta^{-j} : \ \varepsilon_j = 0 \text{ or } 1 \text{ for each } j \right\}.$$

For which θ is it the case that $A(E_\theta) = B(E_\theta)$? (It is known that equality holds if θ is an integer or if θ is not a Pisot–Vijaraghavan number (cf. [18], or [9, Chapter VI]).

(5) What structural properties characterize the sets E which have $A(E) = B(E)$ (or $= B'(E)$)?

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