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A NOTE ON HYPERBOLIC POLYNOMIALS

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We shall show that the space of polynomials of a given degree which are hyperbolic with respect to a given vector has trivial connectivity properties. We shall also show that every such polynomial is the limit of strictly hyperbolic polynomials. Precise formulations are given in the theorem below. The proofs, which are very simple, use a splitting operator

$$f(t) \to f(t)+sf'(t)$$

that reduces the multiplicity of the multiple zeros of a polynomial $f(t)$. Except for two simple references to Hörmanders book [1], the paper is self-contained.

A polynomial P of degree m in n variables $\xi=(\xi_1,\ldots,\xi_n)$ and with principal part P_m is said to be hyperbolic with respect to a real vector N if $P_m(N)\neq 0$ and $P(\xi+tN)=0$, ξ real, implies $\operatorname{Im} t<t_0$ where t_0 does not depend on ξ. For homogeneous polynomials this implies that $P(\xi+tN)$ considered as a polynomial in t has only real zeros, when ξ is real. If a polynomial is hyperbolic so is its principal part and $P_m(\xi)/P_m(N)$ is real [1, p. 133]. A hyperbolic polynomial P is said to be strictly hyperbolic if the zeros of $P_m(\xi+tN)$ are simple for every real ξ which is not proportional to N. We define a topology in a space of polynomials of given degree by using the euclidean norm for the coefficients.

We assume for the sake of simple notation that the vector N points in the direction of the first coordinate and we define operators $T_{k,s}$ by

$$T_{k,s}P(\xi) = P(\xi)+s\xi_k\partial P(\xi)/\partial\xi_1 ,$$

where $k>1$ and s is real. The following lemma shows that $T_{k,s}P$ is also hyperbolic.

LEMMA. *If the zeros z_j of a polynomial p in one variable satisfy* $\operatorname{Im} z_j<a$

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(or $\operatorname{Im} z_j > a$*) then the zeros of* $p+sp'$ *obey the same relation for every real number* s.

PROOF. We write $p(z) = A\prod_{j=1}^{k}(z-z_j)^{m_j}$ and get

$$p(z)+sp'(z) = p(z)\left(1+s\sum_{j=1}^{k} m_j(z-z_j)^{-1}\right).$$

If now $\operatorname{Im} z \geqq a$ then the imaginary part of each term in the sum is positive and therefore $p(z)+sp'(z)$ cannot be zero. In the case of real zeros it follows in particular that $p+sp'$ has only real zeros as well, and multiple zeros of p give rise to zeros with a multiplicity that is one less (if s is not zero), while zeros of $p+sp'$ which are not shared by p, are simple, as is seen by a simple argument using change of sign.

The following theorem summarizes what we shall prove. It is understood that the polynomials have a fixed degree m and are hyperbolic with respect to a fixed vector N. We say that P is normalized if $P_m(N) = 1$.

THEOREM. a) *The space of strictly hyperbolic* (*homogeneous*) *polynomials is open.*

b) *Every hyperbolic* (*homogeneous*) *polynomial is the limit of strictly hyperbolic* (*homogeneous*) *polynomials.*

c) *The space of* (*strictly*) *hyperbolic polynomials is contractible to the space of* (*strictly*) *hyperbolic homogeneous polynomials.*

d) *The space of normalized* (*strictly*) *hyperbolic* (*homogeneous*) *polynomials is connected and simply connected.*

Note: Words within parentheses may be left out. When such a word occurs in two places in the same statement it should be suppressed or included in both.

PROOF. We start the proof by collecting some properties of the operators $T_{k,s}$. First we remark that if P is hyperbolic with respect to $(1,0,\ldots,0)$ then $T_{k,s}P$ has the same property and if P is homogeneous then $T_{k,s}P$ is homogeneous. In that case all zeros of $P(\xi+tN)$ are real and according to the lemma, $T_{k,s}$ reduces the multiplicity of the zeros except when $s\xi_k = 0$. Therefore

$$F_sP(\xi+tN) = T_{2,s}^m \ldots T_{n,s}^m P(\xi+tN)$$

has simple zeros if $s \neq 0$ except when $\xi_2 = \ldots = \xi_n = 0$, which means that F_sP is strictly hyperbolic. Because of the linearity of F_s this is the case also if P is inhomogeneous. It is clear that F_sP converges to P if s approaches zero. This proves b).

To prove c), put $H_tP(\xi)=t^mP(t^{-1}\xi)$ when $0<t\leqq 1$ and $H_0P(\xi)=P_m(\xi)$. If $P(\xi+tN)=0$ implies $\operatorname{Im}t<A$, it follows that $H_sP(\xi+tN)=0$ implies $\operatorname{Im}t<sA$. Further, the principal parts of H_tP and P are the same. Hence H_t is a contraction with the properties required to establich c). To prove that the set of strictly hyperbolic homogeneous polynomials is open we note that the number

$$\inf\{|t_j(\xi')-t_k(\xi')|\,;j\neq k,\,|\xi'|=1\}$$

where $\xi'=(\xi_2,\ldots,\xi_n)$ and the $t_j(\xi')$ denote the zeros of $P(t,\xi')$, is greater than zero, and depends continuously on the coefficients of P. Now openness follows for inhomogeneous polynomials as well. In fact, corollary 5.5.2 of [1] shows that one may add an arbitrary lower degree part to a polynomial that is strictly hyperbolic without destroying this property.

In order to prove d) we first define an operator G_t by $G_tP(\xi_1,\xi')=P(\xi_1,t\xi')$ acting on homogeneous polynomials, and remark that G_tP is strictly hyperbolic if P is and if $t\neq 0$. Also $G_1P=P$ and $G_0P=P_m(N)\xi_1{}^m$. We connect a normalized hyperbolic homogeneous polynomial P with the polynomial F_1G_0P via the polynomials $F_{1-s}G_sP$, $0\leqq s\leqq 1$. They are normalized and homogeneous and strictly hyperbolic when $s<1$, and also when $s=1$ if P is strictly hyperbolic. This connecting operation is an equicontinuous function of s on every bounded set of polynomials, so it follows that an arbitrary closed curve in the set of normalized (strictly) hyperbolic homogeneous polynomials can be contracted to a single point F_1G_0P, which proves d) in the case of homogeneous polynomials. The general case follows by using the contraction H_t defined in connection with c).

Remark. In Theorem 5.6.1 of [1] it is proved that the operator $P(D)$ has one and only one fundamental solution with support in the halfspace $\{x;\langle x,N\rangle\geqq 0\}$ if P is hyperbolic with respect to N. Here $\langle x,N\rangle$ stands for $x_1N_1+\ldots+x_nN_n$. It is easy to prove that this fundamental solution E_s of $F_sP(D)$, of $G_sP(D)$, and of $H_sP(D)$ depends continuously on s. In fact, E_s is given by the formula

$$E_s(u) \;=\; (2\pi)^{-n}\int\frac{\hat{u}(\xi+tiN)}{F_sP(\xi+tiN)}\,d\xi\,,$$

where $t<t_0$ and $\hat{u}$ is the Fouriertransform of u. From the lemma it follows that

$$|F_s P(\xi + itN)| \geqq |P_m(N)| \, |t - t_0|^m$$

if $t < t_0$, whatever be s. The same inequality is valid for $G_s P$, and for $H_s P$ if $|s| < 1$. Let B be a bounded set in $\mathscr{D}$. Then there exists a constant C such that $|\hat{u}(\xi + itN)| \leqq C(1 + |\xi|)^{-n-1}$ for all u in B. Now if $\varepsilon > 0$ there is a constant M such that for all s

$$\left| \int_{|\xi| > M} \frac{\hat{u}(\xi + itN)}{F_s P(\xi + itN)} \, d\xi \right| < \varepsilon .$$

On the set $|\xi| \leqq M$, $F_s P(\xi + itN)$ converges uniformly, which proves that E_s converges strongly in $\mathscr{D}'$.

REFERENCE

1. L. Hörmander, *Linear partial differential operators* (Grundlehren math. Wissensch. 116), Springer-Verlag, Berlin · Göttingen · Heidelberg, 1963.

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