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POLYGONS OF ORDER n IN L_n WITH $n+2$ VERTICES

DOUGLAS DERRY

Dedicated to Otto Haupt on his eightieth birthday.

A polygon of order n in real projective n-space L_n, $n>1$, with $n+2$ vertices is not uniquely determined by its vertices. This is a special property of this class of polygon for if a polygon of order n has more than $n+2$ vertices then it is the only polygon of order n with these vertices. In as much as every polygon of order n with more than $n+2$ vertices is an extension of one with $n+2$ vertices this special class is of importance in the general theory. An application along this line is given in § 4 where compatible sets R are defined by means of the polygons of order n with $n+2$ vertices. This notion is local in the sense that it is defined by the subsets of R which contain $n+3$ points. It is shown that R is the vertex set of a polygon of order n if and only if it is compatible. Polygons of order n in an affine subspace E_n of L_n, n even, have applications in the theory of neighborly sets as the vertex sets of such polygons are neighborly. (A set R of m points in E_n, n even, is defined to be neighborly if, for every subset of $\frac{1}{2}n$ points of R, a hyperplane exists which contains the subset and supports R.) Gale [5] showed that the convex hulls of neighborly sets R with m points, $n+2 \leqq m \leqq n+3$, are combinatorially equivalent to the convex hull of m points on the moment curve in E_n. By use of polygons of order n with $n+2$ vertices it is shown in § 5 that all neighborly sets of m points in E_n are the vertex sets of polygons of order n in E_n, $n+2 \leqq m \leqq n+3$. As the polygons of order n in E_n can be constructed [1], this gives a direct construction for all the neighborly sets in E_n of m points, $n+2 \leqq m \leqq n+3$. Grünbaum [6, pp. 124–125] constructed a neighborly set of eight points in E_4 the convex hull of which is not combinatorially equivalent to the convex hull of eight points on the moment curve in E_4. This example not only shows that Gale's result is the best possible but also indicates the complexity of the problem of constructing all the neighborly sets. As the convex hulls of polygons of order n with m vertices are all combinatorially equivalent, Grünbaum's example also shows that neighborly sets are not necessarily the vertex sets of polygons of order n. A second example

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of a neighborly set of eight points in E_4 is given in the present paper which is not the set of vertices of a polygon of order 4. Its interest lies in the fact that its construction depends on polygons of order 4 with 6 vertices.

All the polygons of order n with $n+2$ vertices in L_n are constructed in § 2. In § 3 R is a set of $n+2$ points in general position in an affine subspace E_n of L_n. An invariant of R is defined by means of convexity and its relationship to the polygons of order n with the vertex set R is studied in this section.

The section which now follows contains definitions and known or easily proved results needed in the proofs.

1. Preliminaries.

1.1. The symbol $[A,B,\ldots]$ denotes the subspace of the real projective n-space L_n spanned by the points or point sets $A,B,\ldots$.

1.2. If $A_1,A_2,\ldots,A_m$ are points of L_n in general position, $\pi: A_1A_2\ldots A_m$ denotes a *closed* polygon with the sides A_iA_{i+1}, $1 \leqq i \leqq m$, where the subscripts are computed modulo m. The points A_i are called the *vertices* of π. A segment $A_jA_{j+1}\ldots A_{j+k}$, $1 \leqq k \leqq m-1$, of π with the endpoints A_j, A_{j+k} is called an *arc*.

If, for a given polygon π, L_{n-1} is any hyperplane of L_n for which $A_i \notin L_{n-1}$, $1 \leqq i \leqq m$, then the number of points in the intersection $L_{n-1} \cap \pi$ is either even for all such L_{n-1} or odd for all such L_{n-1}. A polygon π is defined to be *even* or *odd* according as the number of points of an intersection $L_{n-1} \cap \pi$ is even or odd.

For a given polygon π: $A_1A_2\ldots A_m$ the maximum number of points of $L_{n-1} \cap \pi$ for all L_{n-1} with $A_i \notin L_{n-1}$, $1 \leqq i \leqq m$, is defined to be the *order* of π.

If $m \geqq n+1$, the order of a polygon is at least n. The symbol π_n will be used to represent polygons with exact order n.

1.3. If $A_iA_{i+1}\ldots A_{i+k-1}$, $1 \leqq k \leqq n-1$, is an arc of a polygon π_n: $A_1A_2\ldots A_m$ in L_n then the arc $A_{i-1}A_i\ldots A_{i+k}$ of π_n can be closed by exactly one of the two (projective) segments $\overline{A_{i-1}A_{i+k}}$ so that it becomes a polygon π_{k+1}. A space L_k for which

$$[A_i, A_{i+1}, \ldots, A_{i+k-1}] \subseteq L_k \subseteq [A_{i-1}, A_i, \ldots, A_{i+k}], \quad L_k \cap \overline{A_{i-1}A_{i+k}} = \emptyset,$$

is defined to be an *osculating k-space* of π_n and will be represented by the symbol $L(A_iA_{i+1}\ldots A_{i+k-1})$. The spaces $L(A_i)$ are also called tangents of π_n.

1.4. If $n>1$, the projection from A_i into a hyperplane of an arc $A_{i+1}A_{i+2}\ldots A_{i+m-1}$ of a polygon π_n: $A_1A_2\ldots A_m$ with the endpoints $A_{i+1}, A_{i-1}(=A_{i+m-1})$ together with that of all the tangents $L(A_i)$ is a polygon π_{n-1}. We describe π_{n-1} simply as the projection of π_n from the vertex A_i.

π_{n-1} is even or odd according as n is odd or even. If A_{i+1}', A_{i+m-1}' be the projections of A_{i+1}, A_{i+m-1}, respectively, from A_i then π_{n-1} is the projection of the arc $A_{i+1}A_{i+2}\ldots A_{i+m-1}$ from A_i closed by the segment with endpoints A_{i+1}', A_{i+m-1}' chosen so as to make the resulting polygon even or odd according as n is odd or even.

1.5. The number of vertices of a polygon π_n: $A_1A_2\ldots A_m$ in a hyperplane L_{n-1} together with the number of points of $L_{n-1}\cap\pi_n$ which are on sides A_iA_{i+1} of π_n for which $A_i, A_{i+1} \notin L_{n-1}$ is at most n. This follows by induction with the use of 1.4.

In [1], [2], [3] π_n was defined as a polygon for which the above number was at most n for all spaces L_{n-1}. Consequently if the order of π_n is defined, as in 1.2, to conform to weak order of Haupt the results of the above three papers will be valid.

1.6. The symbol S will be used to denote a simplex or the set of its interior points.

1.7. For a polygon π_n: $A_1A_2\ldots A_m$, $n>1$, the set of all points $\bigcap_{i=m-n+2}^{m+1} L(A_iA_{i+1}\ldots A_{i+n-2})$ is an n-simplex which will be represented by the symbol $S(A_mA_1)$. Its vertices are

$$\begin{gathered} A_1,\quad [A_1, A_2]\cap[A_m, A_{m-1},\ldots, A_{m-n+1}]\,, \\ [A_1, A_2, A_3]\cap[A_m, A_{m-1},\ldots, A_{m-n+2}]\,, \\ \ldots, [A_1, A_2,\ldots, A_n]\cap[A_m, A_{m-1}],\quad A_m\,. \end{gathered}$$

The side A_mA_1 is a 1-face of $S(A_mA_1)$ [2, 3.2].

1.8. For a polygon π_n: $A_1A_2\ldots A_m$, $n>1$, a polygon $A_1A_2\ldots A_mA_{m+1}$ composed of the arc $A_1A_2\ldots A_m$ of π_n with the endpoints A_1, A_m closed by an arc $A_mA_{m+1}A_1$ with the endpoints A_m, A_1 has order n if and only if all the interior points of $A_mA_{m+1}A_1$ are within $S(A_mA_1)$ [3, 4.5].

1.9. If, for a polygon π_n: $A_1A_2\ldots A_m$, P is a point of L_n for which $P, A_1, A_2,\ldots, A_m$ are in general position in L_n then P is contained within at most n osculating hyperplanes $L(A_iA_{i+1}\ldots A_{i+n-2})$ of π_n [1, 5.6].

1.10. For a set R: $A_1, A_2,\ldots, A_m$, $m \geqq n+2$, of points in general position in L_n a segment A_iA_j is said to be R-admissible if, for every hyperplane L_{n-1} spanned by n points of R different from A_i, A_j, $L_{n-1} \cap A_i A_j = \emptyset$.

1.11. If, for $m \geqq n+3$, π_n is a polygon with the vertices $A_1, A_2, \ldots, A_m$ then π_n is the only polygon of order n with these vertices [4, 3.2].

1.12. *Any set R of $n+3$ points $A_1, A_2, \ldots, A_{n+3}$ in general position in L_n is the set of vertices of a polygon π_n of order n. The sides of π_n coincide with the set of R-admissible segments with endpoints in R.*

PROOF. A norm curve exists in L_n which contains the points $A_1, A_2, \ldots, A_{n+3}$. The polygon inscribed in this norm curve with the vertices $A_1, A_2, \ldots, A_{n+3}$ has order n. Thus a polygon π_n with the given vertices exists. By 1.5 a necessary condition that a segment $A_h A_k$ be a side of π_n is that it be R-admissible. By [4, 3.1] this condition is sufficient.

2. A construction for the polygons.

2.1. *If R is a set of three points A_1, A_2, A_3 in general position in L_1 then the sides of the polygon π_1: $A_1 A_2 A_3$ are the three R-admissible segments with endpoints in R.*

PROOF. Because π_1 has order 1 and is closed, it coincides with the projective line L_1. Each of its sides $A_i A_{i+1}$, $1 \leqq i \leqq 3$, is the segment with the endpoints A_i, A_{i+1} which does not contain the point A_{i+2}. Hence $A_i A_{i+1}$ is R-admissible in accordance with 1.10. As there is only one R-admissible segment with endpoints A_i, A_{i+1} the result is clear.

2.2. *If, for a fixed point A_i of a set R: $A_1, A_2, \ldots, A_{n+2}$ of $n+2$ points in general position in L_n, $n > 1$, A_j' is the projection of A_j from A_i, $i \neq j$, and R' is the set $A_1', A_2', \ldots, A_{i-1}', A_{i+1}', \ldots, A_{n+2}'$, then the projection $A_h' A_k'$ of a segment $A_h A_k$, $h \neq i$, $k \neq i$, from A_i is R'-admissible if and only if $A_h A_k$ is R-admissible.*

PROOF. The points of R' are in general position in the projected space for otherwise the set R would not be in general position in L_n.

Let L_{n-1} be the hyperplane spanned by the n points of R different from A_h, A_k. Now $A_i \in L_{n-1}$ as $A_i \neq A_h$, $A_i \neq A_k$. Hence the projection of L_{n-1} from A_i is a hyperplane L_{n-2} of the projected space spanned by the $n-1$ points of R' different from A_h' and A_k'. Because the points of R are in general position the projection of $[A_h, A_k]$ is the line $[A_h', A_k']$ and that of the point $L_{n-1} \cap [A_h, A_k]$ is the point $L_{n-2} \cap [A_h', A_k']$. Consequently $L_{n-1} \cap A_h A_k = \emptyset$ if and only if $L_{n-2} \cap A_h' A_k' = \emptyset$ where $A_h' A_k'$ is the projection of $A_h A_k$.

Thus the result is proved.

2.3. *If A_h, A_j, A_k are distinct points of a set R: $A_1, A_2, \ldots, A_{n+2}$ of $n+2$ points in general position in L_n, $n \geqq 1$, the three R-admissible segments $A_h A_j, A_j A_k, A_k A_h$ form an odd triangle.*

Proof. If $n=1$, the three points A_h, A_j, A_k are the set R. By 2.1 the three R-admissible segments are the sides of a polygon π_1. As this has order 1 the result follows for $n=1$.

For $n>1$ the set R contains at least one point A_i different from each of A_h, A_j, A_k. Let A_p' be the projection of A_p from A_i, $p \neq i$, and $A_h' A_j', A_j' A_k', A_k' A_h'$ those of $A_h A_j, A_j A_k, A_k A_h$, respectively. If R' be the set $A_1', A_2', \ldots, A_{i-1}', A_{i+1}', \ldots, A_{n+2}'$ then, by 2.2, $A_h' A_j', A_j' A_k', A_k' A_h'$ are R'-admissible. If $n=2$ these segments are the sides of a polygon π_1 by 2.1 and so form the projective line. This means that every line through A_i intersects the union of the three segments $A_h A_j, A_j A_k, A_k A_h$ in exactly one point. Hence they build an odd triangle and the result is proved if $n=2$. We now assume it is true for spaces L_{n-1}, $n>2$, and proceed by induction. By the induction assumption the R'-admissible segments $A_h' A_j', A_j' A_k', A_k' A_h'$ form an odd triangle. This means that the triangle composed of $A_h A_j, A_j A_k, A_k A_h$ is also odd because the projection of an even triangle from a point outside its plane is even. The proof now follows by induction.

2.4. *If $B_1, B_2, \ldots, B_{n+2}$ is any permutation of a set $R: A_1, A_2, \ldots, A_{n+2}$ of $n+2$ points in general position in L_n then a polygon π: $B_1 B_2 \ldots B_{n+2}$ has order n if and only if each of its sides $B_i B_{i+1}$, $1 \leqq i \leqq n+2$, is R-admissible.*

Proof. If π: $B_1 B_2 \ldots B_{n+2}$ has order n then each side $B_i B_{i+1}$, $1 \leqq i \leqq n+2$, is R-admissible for otherwise the hyperplane spanned by the n vertices of π different from B_i, B_{i+1} would contain exactly one interior point of $B_i B_{i+1}$, in contradiction to 1.5.

We now prove the converse that π has order n if each of its sides is R-admissible. If $n=1$, the three points B_1, B_2, B_3 are the vertices of a polygon π_1: $B_1 B_2 B_3$ the sides $B_1 B_2, B_2 B_3, B_3 B_1$ of which are R-admissible by 2.1. As there is exactly one R-admissible segment with given endpoints, π_1 coincides with π and the result is clear if $n=1$. We now assume it to be true in spaces of dimension $n-1$, $n>1$, and proceed by induction. A hyperplane H_{n-1} which contains no vertex of π can be displaced continuously to a position L_{n-1} which contains exactly one vertex B_j of π in such a way that none of its intermediate positions contains any vertex of π. The set $L_{n-1} \cap \pi$ contains at least as many points as $H_{n-1} \cap \pi$ except possibly in the case in which L_{n-1} contains a tangent $L(B_j)$ in which case $L_{n-1} \cap \pi$ may contain one point less than $H_{n-1} \cap \pi$. Let $B_{j-1}B_{j+1}$ be the segment which together with the sides $B_{j-1}B_j, B_jB_{j+1}$ of π forms an odd triangle. As $B_{j-1}B_j, B_jB_{j+1}$ are, by the hypothesis, R-admissible it follows from 2.3 that $B_{j-1}B_{j+1}$ is also R-admissible. If B_p' be the projection of B_p from B_j, $p \neq j$, $B_p'B_{p+1}'$ that of the side B_pB_{p+1} of $\pi, p \neq j, p+1 \neq j$, and $B_{j-1}'B_{j+1}'$ that of $B_{j-1}B_{j+1}$ then, by 2.2, the sides of the polygon π': $B_1'B_2' \ldots B_{j-1}'B_{j+1}' \ldots B_{n+2}'$ are R'-admissible where R' is the set $B_1', B_2', \ldots, B_{j-1}', B_{j+1}', \ldots, B_{n+2}'$. The points of R' are in general position otherwise the points of R would not be in general position. Hence R' satisfies the hypothesis and so by the induction assumption has order $n-1$. If L_{n-2} be the projection of L_{n-1} from B_j then, as B_j is the only vertex of π in L_{n-1}, L_{n-2} contains no vertex of π'. As π' has order $n-1$, L_{n-2} contains at most $n-1$ points of π'. If $B_{j+1}B_{j+2} \ldots B_{j+m-1}$ be an arc of π with the endpoints B_{j+1}, B_{j+m-1} its projection will be the arc $B_{j+1}'B_{j+2}' \ldots B_{j+m-1}'$ with the endpoints B_{j+1}', B_{j+m-1}'. As the projection defines a $1-1$ correspondence between the two arcs both $L_{n-1} \cap B_{j+1}B_{j+2} \ldots B_{j+m-1}$ and $L_{n-2} \cap B_{j+1}'B_{j+2}' \ldots B_{j+m-1}'$ contain the same number of points.

We now distinguish two cases. In the first of these $L_{n-2} \cap B_{j-1}'B_{j+1}'$ is assumed not to be empty. This implies that $L_{n-2} \cap B_{j+1}'B_{j+2}' \ldots B_{j+m-1}'$ contains at most $n-2$ points. Consequently $L_{n-1} \cap B_{j+1}B_{j+2} \ldots B_{j+m-1}$ also contains at most $n-2$ points. It follows from the definition 1.3 that any line spanned by B_j and an interior point of $B_{j-1}B_{j+1}$ is a tangent $L(B_j)$ of π. Therefore, as $L_{n-2} \cap B_{j-1}'B_{j+1}' \neq \emptyset$, L_{n-1} contains a tangent $L(B_j)$. As $L_{n-1} \cap \pi$ contains at most $n-1$ points of π including B_j it now follows that $H_{n-1} \cap \pi$ contains at most n points of π because L_{n-1} supports π at B_j.

In the remaining case, where $L_{n-2} \cap B_{j-1}'B_{j+1}' = \emptyset$, L_{n-1} does not contain any tangent $L(B_j)$ but intersects π in at most n points including B_j. As L_{n-1} does not support π at B_j, $H_{n-1} \cap \pi$ and $L_{n-1} \cap \pi$ both contain the same number of points. Hence $H_{n-1} \cap \pi$ contains at most n points in both cases. It now follows by induction that π has order n which completes the proof.

This theorem gives a construction for all the polygons of order n which have the vertex set R.

3. Convex hulls.

3.1. The following notation is used throughout this section. $R\colon A_1, A_2, \ldots, A_{n+2}$ is a set of $n+2$ points in general position in an affine subspace E_n of L_n. The space E_n will remain fixed. $C(R)$ is the convex hull of R defined with reference to E_n. At most one of the points A_i of R can be an interior point of $C(R)$ in which case $C(R)$ is the n-simplex of E_n with the vertices $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n+2}$. In this case $S_0(R)$ is defined to be the 0-simplex of the single point A_i. If all the points of R are on the boundary of $C(R)$ then $S_k(R)$ is defined to be a simplex with vertices belonging to R and of minimum dimension k such that every interior point of it is an interior point of $C(R)$.

3.2. *If, for a given $S_k(R)$, $S'_{n-k}(R)$ is the $n-k$-simplex the vertices of which are the points of R which are not vertices of $S_k(R)$ then $S_k(R) \cap S'_{n-k}(R)$ is a single point interior to both simplexes.*

Proof. If $k=0$, $S_0(R)$ is an interior point of $S'_n(R)$ and the result is clear. (The single vertex of a 0-simplex is to be regarded as an interior point of the simplex.) If $k>0$ then every point of R is on the boundary of $C(R)$. This in turn implies $n>1$. Let L_k be the space spanned by the vertices of $S_k(R)$ and L'_{n-k} that spanned by those of $S'_{n-k}(R)$. If X is defined to be $L_k \cap L'_{n-k}$, then X is a single point because the points of R are in general position.

We now prove (A) that X is not a boundary point of $S_k(R)$. If this were false, X would be an interior point of an h-face S_h of $S_k(R)$, $h<k$. This would imply that $X \in L_{k-1} \cap L'_{n-k}$ where L_{k-1} is the space spanned by the vertices of a $k-1$-face of $S_k(R)$ which contains S_h. This is impossible as $L_{k-1} \cap L'_{n-k} = \emptyset$ because the points of R are in general position. (A) is now clear.

We next prove (B) that $X \in S'_{n-k}(R)$. If $k=n$ this is trivial as X would be the 0-simplex $S'_0(R)$. If, for $k<n$, (B) were false then $L_k \cap S'_{n-k}(R) = \emptyset$ as

$$L_k \cap S'_{n-k}(R) \subseteq L_k \cap L'_{n-k} = X .$$

Then, by the separation theorem, a hyperplane H_{n-1} would exist for which $L_k \subseteq H_{n-1}$ and which would support $S'_{n-k}(R)$ and consequently R itself. As $S_k(R) \subseteq L_k \subseteq H_{n-1}$ this contradicts the fact that $S_k(R)$ contains at least one interior point of $C(R)$. Thus (B) is established.

We now suppose the theorem is false. It then follows from (B) that $X \notin S_k(R)$ and also that $X \in C(R)$ as $X \in S'_{n-k}(R) \subseteq C(R)$. By its definition $S_k(R)$ contains a point Y interior to $C(R)$. Hence every interior point of XY is an interior point of $C(R)$. As $X, Y \in L_k$, $[X, Y] \subseteq L_k$.

Hence XY contains a boundary point of $S_k(R)$ which by (A) cannot be X. Therefore this boundary point of $S_k(R)$ must be an interior point of $C(R)$. Such a boundary point would be an interior point of a proper face S_p of $S_k(R)$. In this event every interior point of S_p would be an interior point of $C(R)$. As $p<k$, k would not be a minimum in contradiction to the definition of $S_k(R)$ in 3.1. This contradiction establishes the result and so completes the proof.

3.3. *If k is the dimension of a simplex $S_k(R)$, then $0 \leqq k \leqq [\frac{1}{2}n]$.*

Proof. If the result is false and $k > [\frac{1}{2}n]$ then $n-k \leqq [\frac{1}{2}n] < k$. By 3.2 $S'_{n-k}(R)$ contains an interior point of $C(R)$. As $S'_{n-k}(R) \subseteq C(R)$, every interior point of $S'_{n-k}(R)$ is an interior point of $C(R)$. Thus $S'_{n-k}(R)$ is a simplex of dimension lower than k every interior point of which is an interior point of $C(R)$ in contradiction to the definition of $S_k(R)$ in 3.1. Therefore $k > [\frac{1}{2}n]$ is impossible and the proof is complete.

3.4. *A hyperplane spanned by n points of R supports R if and only if exactly k of these n points are vertices of $S_k(R)$.*

Proof. If a hyperplane spanned by n points of R contains exactly k vertices of $S_k(R)$ it must contain exactly $n-k$ vertices of $S'_{n-k}(R)$. Such a hyperplane cannot contain either $S_k(R)$ or $S'_{n-k}(R)$ but must support both simplexes. Both simplexes are on the same side of the hyperplane as they have a common point by 3.2. Hence the hyperplane supports R.

A hyperplane which contains more than k vertices of $S_k(R)$ must contain $S_k(R)$ and so must contain at least one interior point of $C(R)$. If a hyperplane spanned by n points of R contains less than k vertices of $S_k(R)$, it must contain more than $n-k$ vertices of $S'_{n-k}(R)$ and hence contain $S'_{n-k}(R)$ and so by 3.2 contain at least one interior point of $C(R)$. Thus any hyperplane spanned by n points of R cannot support R unless it contains exactly k vertices of $S_k(R)$. This completes the proof.

3.5. *If the vertices of an m-simplex S_m, $0 \leqq m \leqq n$, are points of R, then S_m contains an interior point of $C(R)$ if, and only if, one of $S_k(R), S'_{n-k}(R)$ is a face of S_m.*

Proof. It follows by induction that if a face of S_m contains an interior point of $C(R)$ then every interior point of S_m is an interior point of $C(R)$. By 3.1 and 3.2 both $S_k(R), S'_{n-k}(R)$ contain an interior point of $C(R)$. Consequently if either of these simplexes is a face of S_m then S_m contains at least one interior point of $C(R)$.

If neither $S_k(R)$ nor $S'_{n-k}(R)$ is a face of S_m then the vertices of S_m are a subset of n points of R of which at most k points are vertices of $S_k(R)$ and at most $n-k$ are vertices of $S'_{n-k}(R)$. By 3.4 the hyperplane spanned by these n points supports R. As it contains S_m, this simplex cannot contain an interior point of $C(R)$. This completes the proof.

3.6. *If $k<\frac{1}{2}n$, $S_k(R)$ is uniquely determined by R. If $k=\frac{1}{2}n$ then $S_k(R)$ and $S'_{n-k}(R)$ are the only k-simplexes each interior point of which is an interior point of $C(R)$.*

Proof. k is uniquely determined by the definition of $S_k(R)$ in 3.1. Let S_k be a k-simplex with its vertices in R each interior point of which is an interior point of $C(R)$. By 3.5 one of $S_k(R)$, $S'_{n-k}(R)$ must be a face of S_k. If $k<\frac{1}{2}n$ only $S_k(R)$ can be a face of S_k in which case $S_k=S_k(R)$. If $k=\frac{1}{2}n$ then either $S_k(R)$ or $S'_{n-k}(R)$ is a face of S_k which means $S_k=S_k(R)$ or $S_k=S'_{n-k}(R)$. Thus the result is established.

3.7. *A segment A_iA_j is R-admissible if and only if* (1) *exactly one of A_i,A_j is a vertex of $S_k(R)$ and $A_iA_j\subseteq E_n$ or* (2) *neither or both A_i,A_j are vertices of $S_k(R)$ and $A_iA_j\nsubseteq E_n$.*

Proof. Let L_{n-1} be the hyperplane spanned by the n points of R different from A_i,A_j. In the case (1) L_{n-1} contains k vertices of $S_k(R)$ and $n-k$ vertices of $S'_{n-k}(R)$. By 3.4 L_{n-1} supports R and so $L_{n-1}\cap A_iA_j=\emptyset$ if and only if $A_iA_j\subseteq E_n$. Again by 3.4 L_{n-1} does not support R in case (2). Hence L_{n-1} must separate A_i and A_j which means $L_{n-1}\cap A_iA_j=\emptyset$ if and only if $A_iA_j\nsubseteq E_n$. The proof is now complete.

3.8. We now suppose the vertices of $S_k(R)$ to be colored black and those of $S'_{n-k}(R)$ to be colored red. If $B_1,B_2,\ldots,B_{n+2}$ is any permutation of the points of R let $\pi\colon B_1B_2\ldots B_{n+2}$ be the polygon constructed so that a side B_iB_{i+1} is within E_n if B_i,B_{i+1} are points of different color while $B_iB_{i+1}\nsubseteq E_n$ if B_i,B_{i+1} both have the same color. By 3.7 each side of this polygon is R-admissible. By 2.4 π has order n. It follows from 2.4 and 3.7 that all the polygons of order n with the vertex set R can be constructed in this way.

4. Vertex sets of polygons of order n in L_n.

4.1. If, for a polygon $\pi\colon A_1A_2\ldots A_m$, $m>3$, in L_n, $A_{i-1}A_{i+1}$ is the segment which together with the sides $A_{i-1}A_i,A_iA_{i+1}$ of π forms an even triangle then the polygon $A_1A_2\ldots A_{i-1}A_{i+1}\ldots A_m$ obtained by closing the arc $A_{i+1}A_{i+2}\ldots A_{i+m-1}$ of π by $A_{i-1}A_{i+1}$ is defined to be the *contraction* of π with respect to the vertex A_i.

4.2. The contraction of any polygon π_n: $A_1A_2\ldots A_m$ in L_n, $m>n+1$, with respect to any of its vertices is also a polygon of order n.

4.3. *If, for a polygon* π_n: $A_1A_2\ldots A_m$ *in* L_n, $n>1$, L_{n-1} *is a hyperplane in* L_n *for which* $A_i, A_{i+1},\ldots,A_{i+n-2}\in L_{n-1}$, *then* L_{n-1} *is an osculating hyperplane* $L(A_iA_{i+1}\ldots A_{i+n-2})$ *of* π_n *if* $L_{n-1}\cap\pi_n$ *is the arc* $A_iA_{i+1}\ldots A_{i+n-2}$.

Proof. If $m=n+1$, this is an immediate consequence of the definition 1.3. We assume the result to be true for polygons of order n with $m-1$ vertices, $m>n+1$, and proceed by induction. Let

$$\pi(A_{i+n}):\ A_iA_{i+1}\ldots A_{i+n-1}A_{i+n+1}\ldots A_{i+m-1}$$

be the contraction of π_n with respect to the vertex A_{i+n}. By 4.2 $\pi(A_{i+n})$ has order n. By the definition 4.1 the sides $A_{i+n-1}A_{i+n}$, $A_{i+n}A_{i+n+1}$ of π_n together with the side $A_{i+n-1}A_{i+n+1}$ of $\pi(A_{i+n})$ form an even triangle. If $L_{n-1}\cap\pi_n$ is the arc $A_iA_{i+1}\ldots A_{i+n-2}$ it follows that $L_{n-1}\cap\pi(A_{i+n})$ is also the arc $A_iA_{i+1}\ldots A_{i+n-2}$ for if a hyperplane intersects one side of an even triangle it must intersect two of its sides. Therefore by the induction assumption L_{n-1} is an osculating hyperplane $L(A_iA_{i+1}\ldots A_{i+n-2})$ of $\pi(A_{i+n})$. It follows then from 1.3 that it is also an osculating hyperplane $L(A_iA_{i+1}\ldots A_{i+n-2})$ of π_n. The result now follows by induction.

4.4. *If, for a polygon* π_n: $A_1A_2\ldots A_m$ *in* L_n, $n>1$, P *is a point for which the points of the set* $P, A_1, A_2,\ldots,A_m$ *are in general position in* L_n *then* $n-1$ *vertices* $A_{i_1}, A_{i_2},\ldots,A_{i_{n-1}}$ *of* π_n *exist so that, for the hyperplane* $L_{n-1}=[P, A_{i_1},\ldots,A_{i_{n-1}}]$, *a side* A_jA_{j+1} *of* π_n *exists for which* $L_{n-1}\cap A_jA_{j+1}\neq\emptyset$, $A_j, A_{j+1}\notin L_{n-1}$.

Proof. As $P, A_1, A_2,\ldots,A_m$ are in general position, each space $[P, A_i, A_{i+1},\ldots,A_{i+n-2}]$, $1\leqq i\leqq m$, is a hyperplane. By 1.9 at most n of these spaces can be an osculating space $L(A_iA_{i+1}\ldots A_{i+n-2})$. As $m\geqq n+1$, at least one of these spaces $L_{n-1}=[P, A_i, A_{i+1},\ldots,A_{i+n-2}]$ is not an osculating space $L(A_iA_{i+1}\ldots A_{i+n-2})$. By 4.3 this hyperplane must contain a point of π_n not in the arc $A_iA_{i+1}\ldots A_{i+n-2}$. This point must be an interior point of a side A_jA_{j+1} as $P, A_1, A_2,\ldots,A_m$ are in general position. Hence $L_{n-1}\cap A_jA_{j+1}\neq\emptyset$, $A_j, A_{j+1}\notin L_{n-1}$ which proves the result.

4.5. If Q: $B_1, B_2,\ldots,B_{n+2}$ is a subset of $n+2$ distinct points of a set R: $A_1, A_2,\ldots,A_m$, $m\geqq n+3$, of points in general position in L_n, $n>1$, then by 1.12 there is a unique polygon $\pi(A,Q)$ of order n with the vertices $A,B_1,B_2,\ldots,B_{n+2}$, $A\in R$, $A\notin Q$. If, for each Q, the contraction of $\pi(A,Q)$ with respect to A is the same for all A then the set R is defined to be *compatible.*

4.6. HYPOTHESIS: *R: $A_1,A_2,\ldots,A_m$ is a set of m points in general position in L_n, $n>1$, $m\geqq n+4$, with the properties*

(1) *every subset of $m-1$ points $A_1,A_2,\ldots,A_{p-1},A_{p+1},\ldots,A_m$, $1\leqq p\leqq m$, is the vertex set of a polygon of order n which will be written as $\pi(p)$;*

(2) *the contraction of $\pi(p)$ with respect to A_q is the same as the contraction of $\pi(q)$ with respect to A_p, $p\neq q$.*

CONCLUSION: *If the subscripts of the points of R are adjusted so that $\pi(m)$ is $A_1A_2\ldots A_{m-1}$, a side A_jA_{j+1} of $\pi(m)$ exists so that every hyperplane spanned by A_m and $n-1$ other points of R different from A_j,A_{j+1} intersects A_jA_{j+1}.*

PROOF. As the points $A_1,A_2,\ldots,A_m$ are in general position, the point A_m and the polygon $\pi(m)$: $A_1A_2\ldots A_{m-1}$ satisfy the hypothesis of 4.4. Consequently $n-1$ vertices $A_{i_1},A_{i_2},\ldots,A_{i_{n-1}}$ and a side A_jA_{j+1}, where j is to be computed modulo $m-1$, of $\pi(m)$ exist so that if

$$L_{n-1} = [A_m,A_{i_1},A_{i_2},\ldots,A_{i_{n-1}}]$$

then

$$L_{n-1}\cap A_jA_{j+1} \neq \emptyset,\quad A_j,A_{j+1}\notin L_{n-1}\,.$$

Let A_p be a point different from $A_m,A_j,A_{j+1},A_{i_1},A_{i_2},\ldots,A_{i_{n-1}}$. By (2) the contraction of $\pi(p)$ with respect to A_m is the contraction $A_1A_2\ldots A_{p-1}A_{p+1}\ldots A_{m-1}$ of $\pi(m)$ with respect to A_p. Now by 1.5 A_jA_{j+1} cannot be a side of $\pi(p)$ because $L_{n-1}\cap A_jA_{j+1}\neq\emptyset$ as L_{n-1} is spanned by n vertices of $\pi(p)$. Consequently $\pi(p)$ must have the form $A_1A_2\ldots A_jA_mA_{j+1}\ldots A_{p-1}A_{p+1}\ldots A_{m-1}$. Hence its sides A_jA_m,A_mA_{j+1} and the side A_jA_{j+1} of its contraction with respect to A_m define an even triangle. Any hyperplane spanned by n vertices of $\pi(p)$ including A_m but not A_j or A_{j+1} intersects A_jA_{j+1} otherwise it would support the even triangle at A_m and, if suitably displaced, would intersect the sides A_jA_m,A_mA_{j+1} of $\pi(p)$ and contain $n-1$ of its vertices in contradiction to 1.5. Thus the result is established for all the hyperplanes spanned by A_m and $n-1$ vertices of $\pi(m)$ different from A_j,A_{j+1},A_p. If A_q be any point of R different from A_m,A_j,A_{j+1},A_p then as $m\geqq n+4$, a hyperplane exists spanned by A_m and $n-1$ other points different from A_j,A_{j+1},A_p,A_q. Such a hyperplane intersects A_jA_{j+1}. Exactly as above it follows that any hyperplane spanned by A_m and $n-1$ other vertices of $\pi(q)$ different from A_j, A_{j+1} intersects $A_j A_{j+1}$. If H_{n-1} be any hyperplane spanned by A_m and $n-1$ other points of R different from A_j, A_{j+1} then, as R contains at least $n+4$ points, a point A_q of R exists for which $A_q \notin H_{n-1}$, $A_q \neq A_j$, $A_q \neq A_{j+1}$, $A_q \neq A_p$. As H_{n-1} is spanned by n vertices of $\pi(q)$ different from A_j, A_{j+1}, $H_{n-1} \cap A_j A_{j+1} \neq \emptyset$ and the proof is thus completed.

4.7. *A set of points* R: $A_1, A_2, \ldots, A_m$, $m \geqq n+3$, *in general position in* L_n, $n > 1$, *is the set of vertices of a polygon of order* n *if and only if it is compatible.*

PROOF. The result is true for $m = n+3$ for in this case it is trivial that R is compatible while R is always the set of vertices of a polygon of order n by 1.12. We assume that compatible sets of $m-1$, $m > n+3$, points are always the vertices of a polygon of order n and proceed by induction.

The first step in the proof is to show that R satisfies the hypothesis of 4.6. As every subset of a compatible set is likewise compatible, it follows from the induction assumption that for every point A_p of R a polygon $\pi(p)$ of order n exists with the vertices $A_1, A_2, \ldots, A_{p-1}$, $A_{p+1}, \ldots, A_m$. Thus R satisfies the condition (1) of 4.6. As R is compatible, the contraction of $\pi(p)$ with respect to A_q, $p \neq q$, is the same as the contraction of $\pi(q)$ with respect to A_p if $m = n+4$. If $m > n+4$ then by 1.11 there is at most one polygon of order n with $m-2$ vertices. Hence in this case the contraction of $\pi(p)$ with respect to A_q coincides with the contraction of $\pi(q)$ with respect to A_p as both of these contractions are polygons of order n with the $m-2$ vertices $A_1, A_2, \ldots$, $A_{p-1}, A_{p+1}, \ldots, A_{q-1}, A_{q+1}, \ldots, A_m$ and $m-2 \geqq n+3$. Thus R also satisfies condition (2) of 4.6.

A class of polygons $\pi(p)$, $p < m$, can now be constructed. It follows from 4.6, applied to R, that a side $A_j A_{j+1}$ of $\pi(m)$ exists so that every hyperplane spanned by A_m and $n-1$ vertices of $\pi(m)$ different from A_j, A_{j+1} intersects $A_j A_{j+1}$. The subscripts may be adjusted, if necessary, so that $A_j A_{j+1}$ becomes $A_{m-1} A_1$. Let $\pi(p)$ be a polygon for which $1 < p < m-1$. Then $A_{m-1} A_1$ cannot be a side of $\pi(p)$ by 1.5 as a hyperplane spanned by A_m and $n-1$ vertices of $\pi(p)$ different from A_{m-1}, A_1 intersects $A_{m-1} A_1$. As R satisfies condition (2) the contraction of $\pi(p)$ with respect to A_m is the contraction $A_1 A_2 \ldots A_{p-1} A_{p+1} \ldots A_{m-1}$ of $\pi(m)$ with respect to A_p. Thus $\pi(p)$ has the form $A_1 A_2 \ldots A_{p-1} A_{p+1} \ldots A_{m-1} A_m$ as this is the only possible form which does not contain the side $A_{m-1} A_1$ of $\pi(m)$.

A polygon of order n with the vertices R can now be obtained. As

R satisfies the condition (2), the contraction of $\pi(p)$ with respect to A_q coincides with the contraction of $\pi(q)$ with respect to A_p. Hence the arc $A_{m-1}A_mA_1$ is common to both $\pi(p),\pi(q)$ if $1<p<m-1$, $1<q<m-1$. Let π be the polygon consisting of $A_{m-1}A_mA_1$ and the arc $A_1A_2\ldots A_{m-1}$ of $\pi(m)$ with the endpoints A_1,A_{m-1}. For $1<p<m-1$, the contraction of π with respect to A_p consists of the arc $A_{m-1}A_mA_1$ and the arc $A_1A_2\ldots A_{p-1}A_{p+1}\ldots A_{m-1}$ of the contraction of $\pi(m)$ with respect to A_p. But this contraction is by (2) the contraction of $\pi(p)$ with respect to A_m. Hence the contraction of π with respect to A_p is $\pi(p)$. To prove π has order n, let L_{n-1} be any hyperplane for which $A_i \notin L_{n-1}$, $1 \leqq i \leqq m$. As $\pi(m)$ has order n, $L_{n-1}\cap\pi(m)$ contains at most n points. The arc $A_1A_2\ldots A_{m-1}$ of $\pi(m)$ contains at least $n+1$ sides as $m \geqq n+4$. Hence at least one side of this arc does not contain any point of L_{n-1}. If A_p be an endpoint of such a side different from A_1 and A_{m-1}, it follows that $L_{n-1}\cap\pi(p)$ contains the same number of points as $L_{n-1}\cap\pi$ since $\pi(p)$ is the contraction of π with respect to A_p. This means that $L_{n-1}\cap\pi$ contains at most n points. Hence π has order n and it now follows by induction that if R is compatible then R is the vertex set of a polygon of order n.

To prove the converse that the set of vertices R of a polygon π_n: $A_1A_2\ldots A_m$, $n>1$, $m \geqq n+4$, is compatible, let $B_1,B_2,\ldots,B_{n+2}$, A be $n+3$ distinct points of R. By 1.12 these points are the vertices of a unique polygon of order n which, after an adjustment of the subscripts if necessary, can be written as $B_1B_2\ldots B_{q-1}AB_q\ldots B_{n+2}$. This polygon can be constructed by contracting π_n with respect to the vertices not in the set of the $n+3$ points in which case it will have the form $A_{p_1}A_{p_2}\ldots A_{p_{n+3}}$, $1 \leqq p_1 < p_2 < \ldots < p_{n+3} \leqq m$. As this polygon is unique, the cyclic order of the points $B_1,B_2,\ldots,B_{n+2}$ is that defined by the vertices of π_n. By 2.4 there is exactly one polygon of order n with the $n+2$ vertices $B_1,B_2,\ldots,B_{n+2}$ for which the order of the vertices is prescribed. Therefore the contraction of the polygon $B_1B_2\ldots B_{q-1}AB_q\ldots B_{n+2}$ with respect to A is the same for all points A. Thus R is compatible and the proof is complete.

4.8. *If every subset of $n+4$ points of a set of points R: $A_1,A_2,\ldots,A_m$ in general position in L_n, $n>1$, $m \geqq n+4$, is the set of vertices of a polygon of order n then R is compatible.*

PROOF. Let $B_1,B_2,\ldots,B_{n+2},A,B$ be any $n+4$ points of R. By the hypothesis these points are the vertices of a polygon σ of order n which after an adjustment of the subscripts, if necessary, can be written as $B_1B_2\ldots B_{p-1}AB_p\ldots B_{q-1}BB_q\ldots B_{n+2}$. By 1.11 the contraction

$B_1B_2 \ldots B_{p-1}AB_p \ldots B_{n+2}$ of σ with respect to B is uniquely determined by its vertices as it has order n by 4.2. Consequently the cyclic order in which the points $B_1, B_2, \ldots, B_{n+2}$ occur in σ is independent of B. Again by 4.2 the contraction $B_1B_2 \ldots B_{q-1}BB_q \ldots B_{n+2}$ of σ with respect to A has order n and in turn its contraction with respect to B is the polygon $B_1B_2 \ldots B_{n+2}$ of order n. This polygon, by 2.4, is uniquely determined by the vertices $B_1, B_2, \ldots, B_{n+2}$ because their order is independent of B. Therefore the contraction of the polygon with the $n+3$ vertices $B_1, B_2, \ldots, B_{n+2}, B$ is the same for all points B, $B \neq A$, as that of the contraction of the polygon with the vertices $B_1, B_2, \ldots, B_{n+2}, A$ with respect to A. Thus R is compatible and the proof is complete.

It follows from 4.7 that R is the set of vertices of a polygon of order n. An elementary proof of this result was given in [4]. The present proof depends on the duality theorem for polygons of order n [1] used in 4.4.

5. Neighborly sets.

5.1. A set of m points R in general position in real affine n-space, E_n, $n \geqq 2$, n even, $m \geqq n+1$, is said to be *neighborly*, if, for each subset of $\frac{1}{2}n$ points of R, a hyperplane exists which contains this subset and supports R.

5.2. *A hyperplane L_{n-1} which contains n vertices of a polygon π_n: $A_1A_2 \ldots A_m$ in E_n, n even, supports π_n if each vertex in $L_{n-1} \cap \pi_n$ is within an arc $A_iA_{i+1} \ldots A_{i+k-1}$ of π_n for which k is even and $A_i, A_{i+1}, \ldots, A_{i+k-1} \in L_{n-1}$, but $A_{i-1}, A_{i+k} \notin L_{n-1}$.*

Proof. Let H be the set of all arcs $A_{i-1}A_i \ldots A_{i+k-1}A_{i+k}$ of π_n for which $A_{i-1}, A_{i+k} \notin L_{n-1}$ but $A_i, A_{i+1}, \ldots, A_{i+k-1} \in L_{n-1}$. Let $A_{p_1}, A_{p_2}, \ldots, A_{p_n}$ be the n distinct vertices of π_n in L_{n-1}. By [4, 2.11] sequences $X_1, X_2, \ldots, X_n$ of points of π_n exist for which $X_i \to A_{p_i}$, $1 \leqq i \leqq n$, and

$$A_j \notin L'_{n-1} = [X_1, X_2, \ldots, X_n], \qquad 1 \leqq j \leqq m .$$

Provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$, L'_{n-1} will intersect π_n only in points interior to the arcs of H and moreover L'_{n-1} will intersect each arc $A_{i-1}A_i \ldots A_{i+k}$ of H in at least k points. It follows from the order of π_n and the fact that L_{n-1} contains n vertices of π_n that L'_{n-1} intersects each such arc $A_{i-1}A_i \ldots A_{i+k}$ in exactly k points. As k is even, A_{i-1}, A_{i+k} are on the same side of L'_{n-1} provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. This implies that A_{i-1}, A_{i+k} are on the same side of L_{n-1}. If $A_{i-1}A_i \ldots A_{i+k}$, $A_{j-1}A_j \ldots A_{j+h}$

be two successive arcs of H, that is, arcs of H with the property that no vertex of the arc $A_{i+k}A_{i+k+1}\dots A_{j-1}$ is within L_{n-1}, then as L'_{n-1} contains no point of this latter arc, all its vertices will be on the same side of L'_{n-1} and consequently on the same side of L_{n-1}. Thus the vertices of *any* two consecutive arcs of H as well as all the vertices of π_n between these arcs are either within L_{n-1} or are all on the same side of L_{n-1}. This implies that all the vertices of π_n are either within L_{n-1} or on the same side of L_{n-1}. As $\pi_n \subseteq E_n$, L_{n-1} supports π_n. Thus the proof is completed.

5.3. *The set R of the vertices of a polygon π_n: $A_1A_2\dots A_m$ in E_n, n even, is neighborly.*

PROOF. If Q is any set of at most $\frac{1}{2}n$ vertices of π_n then a hyperplane L_{n-1} exists which satisfies the hypothesis of 5.2 and contains Q. This is clear if $n=2$. We assume it true for all polygons π_{n-2}, $n \geqq 4$, n even, and proceed by induction. If Q is empty then L_{n-1} may be taken to be any hyperplane spanned by n consecutive vertices. If Q contains at least one vertex A_i of π_n, let A_j' be the projection of A_j from the line $[A_i, A_{i+1}]$, $j \neq i$, $j \neq i+1$. Then it follows, by the use of 1.4, that the projection of π_n from the line $[A_i, A_{i+1}]$ will be a polygon π_{n-2}: $A_1'A_2'\dots A_{i-1}'A_{i+2}'\dots A_m'$. If Q' be the set of vertices of π_{n-2} which are projections of vertices of Q, then Q' contains at most $\frac{1}{2}(n-2)$ points. Hence, by the induction assumption, a hyperplane L_{n-3} of the projected space exists which contains Q' and satisfies the hypothesis of 5.2, that is, L_{n-3} is spanned by $n-2$ vertices of π_{n-2} and each arc of maximum length in $L_{n-3} \cap \pi_{n-2}$ contains an even number of vertices of π_{n-2}. Let L_{n-1} be the hyperplane of L_n which is projected into L_{n-3}. Then $Q \subseteq L_{n-1}$ and $A_i, A_{i+1} \in L_{n-1}$. Let $A_pA_{p+1}\dots A_{p+k-1}$ be an arc of π_n of maximum length in $L_{n-1} \cap \pi_n$. If at least one of A_i, A_{i+1} is included in the arc $A_{p-1}A_p\dots A_{p+k}$ then both of A_i, A_{i+1} are within $A_pA_{p+1}\dots A_{p+k-1}$ as they are within L_{n-1} while $A_{p-1}, A_{p+k} \notin L_{n-1}$. The projection of the arc $A_pA_{p+1}\dots A_{p+k-1}$ is an arc of maximum length with $k-2$ vertices in $L_{n-3} \cap \pi_{n-2}$. Such arcs are assumed to contain an even number of vertices. Therefore k is even. If, on the other hand, neither of A_i, A_{i+1} is within $A_{p-1}A_p\dots A_{p+k}$, the projection of $A_pA_{p+1}\dots A_{p+k-1}$ is an arc of maximum length in $L_{n-3} \cap \pi_{n-2}$ which contains k vertices. Again k is even. Hence it follows by induction that Q is contained in a hyperplane L_{n-1} spanned by n vertices of π_n with the property that each arc of maximum length in $L_{n-1} \cap \pi_n$ contains an even number of vertices. Consequently, by 5.2, L_{n-1} supports R. This completes the proof that R is neighborly.

5.4. *A set of $n+2$ points R in general position in L_n, n even, is neighborly if and only if the dimension k of a simplex $S_k(R)$ is $\frac{1}{2}n$.*

Proof. If $k<\frac{1}{2}n$ then, by the definition 3.1, $S_k(R)$ has at most $\frac{1}{2}n$ vertices. Hence the vertices of $S_k(R)$ are contained in a subset of $\frac{1}{2}n$ points of R. As $S_k(R)$, by its definition 3.1, contains at least one interior point of $C(R)$, no hyperplane which contains this subset of the $\frac{1}{2}n$ points can support R. Hence, by 5.1, R is not neighborly. By 3.3, $k \leqq \frac{1}{2}n$. Therefore if R is neighborly $k=\frac{1}{2}n$.

If $k=\frac{1}{2}n$ then any subset of $\frac{1}{2}n$ points of R is within a subset of n points of R of which exactly $\frac{1}{2}n$ points are vertices of $S_k(R)$. By 3.4 the hyperplane spanned by this set of n points of R supports R which shows R is neighborly and completes the proof.

5.5. *A set R of $n+2$ points in general position in E_n, n even, is neighborly if and only if R is the set of vertices of a polygon π_n in E_n.*

Proof. The set R of the vertices of a polygon π_n is neighborly by 5.3. If a set R of $n+2$ points is neighborly, the dimension k of the simplex $S_k(R)$ is $\frac{1}{2}n$ by 5.4. It follows from 3.8 that R is the vertex set of a polygon π_n in E_n and so the result is proved.

5.6. **Hypothesis:** *R is a neighborly set of $n+3$ points $A_1,A_2,\ldots,A_{h+1}$, $B_1,B_2,\ldots,B_{h+2}$ in general position in E_n, $n=2h$ even, $n \geqq 2$.*

S_h, S_{h+1} are the simplexes with the vertices $A_1,A_2,\ldots,A_{h+1}$; $B_1,B_2,\ldots,B_{h+2}$, respectively.

Every interior point of S_h is an interior point of $C(R)$.

Conclusion:

(1) *$S_h \cap S_{h+1}$ is an open segment AB for which $A,B \in S_h$.*

(2) *A,B are interior points of different h-faces of S_{h+1} with the vertices $C_1,C_2,\ldots,C_h,X$; $C_1,C_2,\ldots,C_h,Y$.*

(3) *XY is R-admissible.*

Proof. Let L_n be a projective space which contains E_n. Because the points of R are in general position,

$$[A_1,A_2,\ldots,A_{h+1}] \cap [B_1,B_2,\ldots,B_{h+2}]$$

is a line L_1 of L_n while

$$[A_1,A_2,\ldots,A_{i-1},A_{i+1},\ldots,A_{h+1}] \cap [B_1,B_1,\ldots,B_{j-1},B_{j+1},\ldots,B_{h+2}] = \emptyset$$

for $1 \leqq i \leqq h+1$, $1 \leqq j \leqq h+2$. This latter relation implies that the boundary of S_h has no point in common with that of S_{h+1}.

If $\bar{S}_h$ is the closure of S_h, we now show

$$[A_1,A_2,\ldots,A_{h+1}] \cap C(R) = \bar{S}_h .$$

Clearly

$$\bar{S}_h \subseteq [A_1,A_2,\ldots,A_{h+1}] \cap C(R) .$$

It is therefore sufficient to show that

$$[A_1,A_2,\ldots,A_{h+1}] \cap C(R) \subseteq \bar{S}_h .$$

If this were false, a point U of the intersection would exist for which $U \notin \bar{S}_h$. By the hypothesis an interior point V of $C(R)$ exists in S_h. Hence every interior point of the segment UV of E_n would be an interior point of $C(R)$. But as $U \notin \bar{S}_h$, $V \in S_h$ and $[U,V]$ is in the space spanned by the vertices of S_h, UV would contain a boundary point of S_h. Consequently a proper face of S_h would contain an interior point of $C(R)$ in contradiction to the assumption that R is neighborly. This shows that no point U exists and proves that

$$[A_1,A_2,\ldots,A_{h+1}] \cap C(R) = \bar{S}_h .$$

We have

$$[A_1,A_2,\ldots,A_{h+1}] \cap S_{h+1} \neq \emptyset ,$$

for otherwise, by the separation theorem, a hyperplane L_{n-1} containing $[A_1,A_2,\ldots,A_{h+1}]$ would exist which supported S_{h+1}. As each vertex B_i is a limit point of interior points of S_{h+1}, this would imply that L_{n-1} supported R in contradiction to the assumption that S_h contains at least one interior point of $C(R)$.

By the two previous paragraphs

$$\emptyset \neq [A_1,A_2,\ldots,A_{h+1}] \cap S_{h+1} \subseteq [A_1,A_2,\ldots,A_{h+1}] \cap C(R) \cap S_{h+1}$$
$$= \bar{S}_h \cap S_{h+1} \subseteq L_1 .$$

Hence $L_1 \cap S_{h+1}$ is an open segment AB. Moreover

$$AB \subseteq [A_1,A_2,\ldots,A_{h+1}] \cap C(R) = \bar{S}_h .$$

It was proved in the first paragraph that no boundary point of S_{h+1} can be a boundary point of S_h. As A,B are boundary points of S_{h+1} it follows that $A,B \in S_h$. Hence

$$AB \subseteq S_h \cap S_{h+1} \subseteq L_1 \cap S_{h+1} = AB .$$

Thus (1) is proved. As

$$[A_1,A_2,\ldots,A_{h+1}] \cap [B_1,B_2,\ldots,B_{j-1},B_{j+1},\ldots,B_{k-1},B_{k+1},\ldots,B_{h+2}] = \emptyset$$

for $j \neq k$, A,B must be interior points of h-faces of S_{h+1}. They cannot be on the same h-face for then $L_1 \cap S_{h+1} = \emptyset$. This proves (2). Accordingly let $C_1, C_2, \ldots, C_h, X$ be the vertices of the h-face of S_{h+1} which contains A and $C_1, C_2, \ldots, C_h, Y$ those of the h-face of S_{h+1} which contains B. To show that XY is R-admissible let H_{n-1} be any hyperplane spanned by n points of R different from X, Y. Suppose first that $S_h \subseteq H_{n-1}$. Therefore $A, B \in H_{n-1}$. Now, H_{n-1} can only contain $h-1$ of the vertices $C_1, C_2, \ldots, C_h$ of S_{h+1} and so C_i exists with $C_i \notin H_{n-1}$. Further, H_{n-1} cannot support the simplex with the vertices $C_1, C_2, \ldots, C_h, X$ as it contains the interior point A of this simplex. As $C_1, C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_h \in H_{n-1}$, it follows that H_{n-1} separates C_i and X, that is, $H_{n-1} \cap C_i X \neq \emptyset$. Similarly $H_{n-1} \cap C_i Y \neq \emptyset$. Consequently $H_{n-1} \cap XY = \emptyset$ as $C_i X, C_i Y, XY$ form an even triangle. In the remaining case H_{n-1} contains $C_1, C_2, \ldots, C_h$ and all but one of the vertices $A_1, A_2, \ldots, A_{h+1}$ of S_h. Hence H_{n-1} supports S_h and the simplex $C_1, C_2, \ldots, C_h, X$. These simplexes are on the same side of H_{n-1} as they have the common interior point A. Likewise S_h and $C_1, C_2, \ldots, C_h, Y$ are on the same side of H_{n-1} as H_{n-1} supports both of these simplexes and B is a common interior point. Hence R is on one side of H_{n-1} and $H_{n-1} \cap XY = \emptyset$. This completes the proof that XY is R-admissible and so the result (3) is established.

5.7. *A set R of $n+3$ points in general position in E_n, $n = 2h$ even, $n \geqq 2$, is neighborly if and only if it is the set of vertices of a polygon of order n.*

Proof. The set of vertices R of a polygon π_n is neighborly by 5.3. To prove the converse let L_n be a projective space which contains E_n. As $R \subseteq E_n \subseteq L_n$, by 1.12, R is the set of vertices of a polygon π_n the sides of which are the R-admissible segments with endpoints in R. Consequently to show that $\pi_n \subseteq E_n$ it is sufficient to show, for an arbitrary point X of R, that the two sides of π_n which have the endpoint X are within E_n. The subset R' of the $n+2$ points of R different from X is neighborly as R is neighborly. As, by 5.4, a simplex $S_k(R')$ has dimension $\frac{1}{2}n = h$, such a simplex can be written as S_h. By its definition each interior point of S_h is an interior point of $C(R')$ and consequently also of $C(R)$. Let S_{h+1} be the simplex the vertices of which are points of R which are not vertices of S_h. Then S_h, S_{h+1} satisfy the hypothesis of 5.6. It follows from that result that $S_h \cap S_{h+1}$ is an open segment AB where A,B are interior points of different h-faces of S_{h+1}. The points of R' which are not vertices of S_h are the vertices of a simplex $S'_h(R')$ which is an h-face of S_{h+1}. By 3.2 $S_h \cap S'_h(R')$ is an interior point of S_h. This intersection must be one of A, B, say B, as these are the only points of S_h within an h-face of S_{h+1}. If $C_1, C_2, \ldots, C_h, X$ be the vertices of the h-face of S_{h+1} which contains A, then a vertex Y of S_{h+1} exists so that the vertices of $S'_h(R')$ are $C_1, C_2, \ldots, C_h, Y$. It follows then, from 5.6, that the segment XY of E_n is R-admissible and consequently, by 1.12, a side of π_n. If S_h is defined to be $S'_h(R')$ a similar procedure leads to a second side XZ of π_n in E_n. As $Y \in \bar{S}'_h(R')$, $Z \in \bar{S}_h(R')$, it follows that $Z \neq Y$, as $S_h(R'), S'_h(R')$ have no common vertex. Thus the two sides of π_n with the endpoint X are within E_n. As X is arbitrary, $\pi_n \subseteq E_n$. The proof is now complete.

5.8. *C_1, C_2, C_3, C_4, X, Y are points in general position in L_2.*

(1) *If π^1: $C_1XC_2C_3C_4$, π^2: $C_1C_2C_3YC_4$ both have order 2, then the polygon σ: $C_1XC_2C_3YC_4$ composed of the arc $C_4C_1XC_2C_3$ of π^1 and the arc C_3YC_4 of π^2 has order 2.*

(2) *If π^1: $C_1XC_2C_3C_4$, π^3: $C_1C_2ZC_3C_4$ both have order 2 and $[C_2,Z] \cap \pi^1 = C_2$, then the polygon τ: $C_1XC_2ZC_3C_4$ composed of the arc $C_3C_4C_1XC_2$ of π^1 and the arc C_2ZC_3 of π^3 has order 2.*

PROOF. To prove σ and τ have order 2 it is sufficient to show that they are boundaries of convex regions. By 2.8 a unique polygon π_2: $C_1C_2C_3C_4$ exists with the vertices taken in this order. Let E_2 be an affine subspace of L_2 in which π_2 is a parallelogram. The strip of E_2 bounded by the lines $[C_1, C_4]$, $[C_2, C_3]$ is subdivided by the sides C_1C_2, C_3C_4 of π_2 into two triangles and the interior P of π_2. Let T be the triangle of the strip with the side C_1C_2. As π_2 is uniquely defined by its vertices and their order, the contraction of π^1 with respect to X is π_2. Consequently the arc $C_2C_3C_4$ is common to π_2 and π^1. As the line $[C_1, X]$ cannot intersect $C_2C_3C_4$, by 1.5, this line supports P. Hence T intercepts $[C_1, X]$ in a segment C_1D of E_2. The line $[X, C_2]$ likewise supports P and so separates C_1 and D. Thus $X \in T$. Moreover the segment C_1X of E_2 is a side of π^1 as its complement in the projective line contains the point D of $[C_2, C_3]$. Similarly it follows that XC_2 is a side of π^1 and that C_2Y, YC_4 are sides of π^2 all three of these segments being in E_2. As the six lines spanned by a side of σ: $C_1XC_2C_3YC_4$ all simultaneously support π^1 and π^3, σ is the boundary of a convex region and so has order 2.

As above it follows that π^3 is the polygon $C_1C_2ZC_3C_4$ of E_2. Any line spanned by one of the sides $ZC_3, C_3C_4, C_4C_1, C_1X$ of τ: $C_1XC_2ZC_3C_4$ supports τ as it supports π^1 and π^3 simultaneously. The same is true for the line $[C_2, Z]$ as, by the hypothesis, $[C_2, Z] \cap \pi^1 = C_2$. If the remaining line $[X, C_2]$ were not to support τ it would separate Z and C_3 in which case the line $[Z, C_2]$ would separate C_1 and X contrary to the hypothesis that $[Z,C_2] \cap \pi^1 = C_2$. Hence τ has order 2 as it is the boundary of the convex hull of its six vertices. Thus (1) and (2) are now both established.

5.9. *If the two hyperplanes* $[B_1,A_2,B_2,Y]$, $[A_1,B_2,A_3,Y]$ *both support a set* R *of eight points* $A_1,A_2,A_3,B_1,B_2,B_3,X,Y$ *in general position in* E_4 *for which the two polygons* σ: $A_1XB_1A_2B_2A_3B_3$, τ: $A_2B_1A_1B_2YA_3B_3$ *of* E_4 *both have order* 4, *then* R *is neighborly but is not the vertex set of a polygon of order* 4 *in* E_4.

PROOF. The projections $A_1',B_1',A_2',B_2',X',Y'$ of the points A_1,B_1,A_2,B_2,X,Y from $[A_3,B_3]$, respectively, are in general position in the projected space as the points of R are in general position in E_4. Consequently, as the polygon $A_1XB_1A_2B_2$ of E_4 is even, its projection $A_1'X'B_1'A_2'B_2'$ is likewise even. Hence this projection may be regarded as the projection of the arc $A_1XB_1A_2B_2$ closed by the segment with endpoints A_1',B_2' chosen so as to make the resulting polygon even. It follows from 1.4 that the projection $A_1'X'B_1'A_2'B_2'$ has order 2 as it coincides with the projection σ' of σ: $A_1XB_1A_2B_2A_3B_3$ from $[A_3,B_3]$.

If the vertices of τ are written in the reverse order it becomes $B_3A_3YB_2A_1B_1A_2$. As in the previous paragraph its projection τ': $A_1'B_1'A_2'Y'B_2'$ from $[A_3,B_3]$ coincides with that of the polygon $A_1B_1A_2YB_2$ of E_4. We check that the polygons σ',τ' satisfy the hypothesis of 5.8 (1). If C_1,C_2,C_3,C_4,X,Y are replaced by $A_1',B_1',A_2',B_2',X',Y'$, respectively, then π^1,π^2 become the polygons σ',τ' of order 2. It follows then, by applying 5.8 (1), that the polygon $A_1'X'B_1'A_2'Y'B_2'$ composed of the arc $B_2'A_1'X'B_1'A_2'$ of σ' and the arc $A_2'Y'B_2'$ of τ' has order 2. But as these arcs are the projections of the arcs $B_2A_1XB_1A_2$, A_2YB_2 of E_4 from $[A_3,B_3]$ it follows that the projection of the polygon φ_1: $A_1XB_1A_2YB_2$ from $[A_3,B_3]$ has order 2.

If σ',τ' now denote the projections of σ,τ respectively, from $[B_1,A_2]$, it follows, as in the first paragraph, that these projections coincide with the projections of the polygons $A_1XB_2A_3B_3$, $A_1B_2YA_3B_3$ of E_4 from the same line. The method of the previous paragraph can now be used to show that the projection φ_2': $A_1'X'B_2'Y'A_3'B_3'$ of the polygon φ_2: $A_1XB_2YA_3B_3$ of E_4 from $[B_1,A_2]$ has order 2. To show that σ',τ', satisfy the hypothesis of 5.8 (2), C_1,C_2,C_3,C_4,X,Z are now replaced by $A_1',B_2',A_3',B_3',X',Y'$, respectively. π^1,π^3 thus become the polygons σ',τ' of order 2. The additional condition of 5.8 (2) that $[C_2,Z] \cap \pi^1 = C_2$ which becomes $[B_2',Y'] \cap \sigma' = B_2'$ is equivalent to

$$[B_1,A_2,B_2,Y] \cap A_1XB_2A_3B_3 = B_2 .$$

By the hypothesis $[B_1,A_2,B_2,Y]$ supports R and consequently φ_2. Hence

$$[B_1,A_2,B_2,Y] \cap \varphi_2 = B_2Y$$

which implies that

$$[B_1,A_2,B_2,Y] \cap A_1XB_2A_3B_3 = B_2 .$$

Thus σ', τ' satisfy the hypothesis of 5.8 (2). It follows from that result that the polygon $A_1'X'B_2'Y'A_3'B_3'$ composed of the arc $A_3'B_3'A_1'X'B_2'$ of σ' and the arc $B_2'Y'A_3'$ of τ' has order 2. But this polygon is the projection of φ_2: $A_1XB_2YA_3B_3$. Thus its projection φ_2' from $[B_1,A_2]$ has order 2.

To prove R neighborly it is necessary to show that any line spanned by two points of R is within a hyperplane which supports R. As the projection of φ_1: $A_1XB_1A_2YB_2$ from $[A_3,B_3]$ has order 2, any hyperplane spanned by $[A_3,B_3]$ and a side of φ_1 supports R. Likewise any line spanned by $[B_1,A_2]$ and a side of φ_2 supports R. Consequently any line spanned by two points of R of which at least one is one of the four points A_3,B_3,B_1,A_2 is within a hyperplane which supports R. The three lines which contain the vertex B_2 which remain to be considered are $[B_2,A_1]$, $[B_2,X]$, $[B_2,Y]$. As A_1B_2 is a side of φ_1 while B_2X, B_2Y are sides of φ_2 each of these three lines is within a hyperplane which supports R. Of the lines which contain the vertex A_1 only $[A_1,X]$, $[A_1,Y]$ have not already been considered. As A_1X is a side of φ_1 and $[A_1,Y] \subseteq [A_1,B_2,A_3,Y]$ which by the hypothesis supports R, both of these lines are within a hyperplane which supports R.

If an interior point of XY is on the boundary of $C(R)$ then $[X,Y]$ is within a hyperplane which supports $C(R)$ and consequently R itself. Accordingly we assume that every interior point of XY is an interior point of $C(R)$ and obtain a contradiction. As the contraction $A_1B_1A_2B_2A_3B_3$ of σ with respect to X is a polygon of order 4 in E_4, it follows from 3.7 that $A_1A_2A_3$ is a simplex $S_2(R')$ where R' is the set A_1,A_2,A_3,B_1,B_2,B_3. By 3.4 $L_3=[A_1,A_3,B_1,B_2]$ supports R'. It separates the segment XY from R'. If this were false at least one of X,Y would be on the same side of L_3 as R'. If L_3 did not separate X and R' then it would contain a tangent $L(A_1)$ of σ: $A_1XB_1A_2B_2A_3B_3$. A suitable displacement of L_3 would then intersect the sides B_3A_1, A_1X of σ and contain its vertices A_3,B_1,B_2. This is impossible by 1.5 as σ has order 4. If R' and Y were on the same side of L_3 then L_3 would contain a tangent $L(A_3)$ of τ: $A_2B_1A_1B_2YA_3B_3$. A similar displacement of L_3 would intersect the sides YA_3, A_3B_3 of τ and contain its three vertices A_1,B_1,B_2 in contradiction to the order of τ. These contradictions prove that L_3

separates XY from R'. As every interior point of XY is assumed to be an interior point of $C(R)$, $[X,Y] \cap C(R) = XY$ because X,Y are boundary points of $C(R)$ by the previous paragraph. Hence

$$[X,Y] \cap C(R') \subseteq [X,Y] \cap C(R) = XY .$$

Consequently

$$[X,Y] \cap C(R') \subseteq XY \cap C(R') = \emptyset$$

as L_3 separates XY from R' and does not contain X or Y because the points of R are in general position. Hence, by the separation theorem, a supporting hyperplane of R' exists which contains $[X,Y]$. This contradicts the assumption that XY contains an interior point of $C(R)$. Hence $[X,Y]$ is contained in a hyperplane which supports R. Thus the proof that R is neighborly is complete.

The points of R are not compatible in the sense of 4.3 as the contraction $A_1B_1A_2B_2A_3B_3$ of σ with respect to X is different from the contraction $A_2B_1A_1B_2A_3B_3$ of τ with respect to Y. Hence by 4.6 the points of R are not the vertices of a polygon of order 4. The proof is now complete.

5.10. To check that eight points exist in E_4 which satisfy the conditions of 5.9, let R' be the set of the vertices of two 2-simplexes $A_1A_2A_3$, $B_1B_2B_3$ which intersect in a single point interior to each of them.

A hyperplane which contains this interior point separates the vertices of at least one of the simplexes. Hence this interior point is an interior point of the convex hull of R'. Any hyperplane spanned by a 1-face of one simplex and a 1-face of the other supports R'. Hence the simplexes are the simplexes $S_k(R'), S'_{n-k}(R')$. By 3.8 the polygons $A_1B_1A_2B_2A_3B_3$, $A_2B_1A_1B_2A_3B_3$ in E_4 both have order 4. By 1.8, if the interior points of the arc A_1XB_1 are within the simplex $S(A_1B_1)$ defined for the polygon $A_1B_1A_2B_2A_3B_3$, then the polygon σ: $A_1XB_1A_2B_2A_3B_3$ has order 4. By 1.7 the side A_1B_1 is a 1-face of $S(A_1B_1)$. Hence σ will be in E_4 if the arc A_1XB_1 is chosen sufficiently close to A_1B_1. Similarly a polygon τ: $A_2B_1A_1B_2YA_3B_3$ of order 4 can be constructed in E_4. By 5.2 the hyperplane $[B_1,A_2,B_2,Y]$ supports τ. If this hyperplane does not support σ it intersects A_1X. If σ is modified by taking X sufficiently close to A_1 in A_1X, it will still be of order 4 and the hyperplane will support the simplex A_1XB_1 and so support σ. Again by 5.2 the hyperplane $[A_1,B_2,A_3,Y]$ supports τ. If it does not support σ (modified) it intersects XB_1. If X is chosen sufficiently close to B_1 in the side XB_1, the hyperplane will support the resulting simplex A_1XB_1 and consequently will support σ. The set $A_1,A_2,A_3,B_1,B_2,B_3,X,Y$ now satisfies the conditions of 5.9.

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UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, CANADA
AND
ISTITUTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI, TRIESTE, ITALY