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ON A CONCEPT OF SUMMABILITY IN AMENABLE SEMIGROUPS

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1. Introduction.

Banach's generalized Limit [1, page 33] gives rise to a notion of almost convergence of a sequence $\{x_n\}$ to s — namely, that $\mathrm{Lim}_n x_n = s$ for each generalized Limit. Lorentz [7] obtained the following interesting characterization:

THEOREM (Lorentz). *A sequence $\{x_n\}$ almost converges to s if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} x_{m+k} = s$$

uniformly in m.

The purpose of the present note is to extend the notion of an arithmetic mean to amenable semigroups, following the lead of Følner [5], Day [2] and Namioka [8], and to obtain the same characterization of almost convergence in the more general setting. We also obtain an analogous result for vector-valued functions.

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2. Preliminaries.

Let G denote a discrete semigroup, $m(G)$ the real Banach space of all bounded, real-valued functions on G, endowed with the sup norm $\|f\|_\infty = \sup_{g\in G} |f(g)|$, and $l_1(G)$ the collection of all $f \in m(G)$ satisfying $\|f\|_1 = \sum_{g\in G} |f(g)| < \infty$. Endowed with the convolution

$$(f_1 * f_2)(g) = \sum_{hh'=g} f_1(h) f_2(h') ,$$

$l_1(G)$ is a Banach algebra.

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A weight, or a finite mean, on G is a non negative function $\varphi \in l_1(G)$, having finite support such that $\|\varphi\|_1 = 1$; a simple weight is a weight which is constant on its support. We denote by Φ the collection of all weights on G.

A mean on $m(G)$ is a real, linear functional λ on $m(G)$ such that

$$\inf\{f(g) :\ g \in G\} \leqq \lambda(f) \leqq \sup\{f(g) :\ g \in G\}$$

for all $f \in m(G)$. Clearly, $\lambda \geqq 0$ and $\lambda(1) = 1$ where 1 denotes the function identically unity on G. A mean λ is said to be left invariant if $\lambda({}^g f) = \lambda(f)$ for all $g \in G$ and all $f \in m(G)$, where ${}^g f(h) = f(gh)$. There is the obvious, analogous definition of right invariance: $\lambda(f^g) = \lambda(f)$ where $f^g(h) = f(hg)$. The semigroup G is said to be left (right) amenable if there exists a left (right) invariant mean on $m(G)$, and G is amenable if it is both left and right amenable. Let G be amenable. A function $f \in m(G)$ almost converges to s if $\lambda(f) = s$ for every left and every right invariant mean λ.

3. Summing sequences.

Throughout this paper G will denote a discrete, countable amenable semigroup with identity e in which both the right and the left cancellation laws hold. Namioka [8] has shown that G is amenable if and only if, for each finite subset F of G and each $\varepsilon > 0$, there exists a finite subset S of G such that

$$|Sg \cap S| > (1-\varepsilon)|S| \quad \text{and} \quad |gS \cap S| > (1-\varepsilon)|S|$$

for all $g \in F$. Here $|A|$ denotes the number of elements in the finite set A. Namioka showed also that, if G is a countable, amenable group, then there exists a sequence $\{S_n\}$ of finite subsets of G such that

(1) $\quad G = \bigcup_1^\infty S_n$,

(2) $\quad S_n \subset S_{n+1}, \qquad n = 1, 2, \ldots$,

(3) $\quad \lim_n |S_n|^{-1} |S_n g \cap S_n| = 1, \quad \lim_n |S_n|^{-1} |g S_n \cap S_n| = 1 \quad$ for all $g \in G$.

An appeal to Day's theorem in the 2-sided case (see [2]) and a simple modification of Namioka's proof of the Følner–Frey theorem (see [8, Theorem 3.5]) lead to a similar proof of the existence of such a sequence in case G is a countable, amenable semigroup in which both cancellation laws hold.

Definition 3.1. Any sequence of finite subsets of G satisfying (1), (2) and (3) is called a summing sequence for G.

Let $\{S_n\}$ be a summing sequence for G. Denote by γ_n the simple weight $\gamma_n(g) = |S_n|^{-1} \chi_n(g)$ where χ_n is the characteristic function of S_n.

The significance of a summing sequence for G is that the sequence $\{\gamma_n\}$ of simple weights approximates, in a sense, any weight on G.

LEMMA 3.1. *For any* $\varphi \in \Phi$,

$$\lim_n \|\gamma_n * \varphi - \gamma_n\|_1 = 0 \quad \textit{and} \quad \lim_n \|\varphi * \gamma_n - \gamma_n\|_1 = 0 .$$

PROOF. Let F be the (finite) support of φ. Given $\varepsilon > 0$, property (3) of Definition 3.1 insures the existence of n_0 such that $n \geqq n_0$ implies

$$(*) \qquad |S_n \cap \bigcap_{h\in F} S_n h| > (1 - \tfrac{1}{2}|F|^{-1}\varepsilon)|S_n| .$$

For convenience, put $R = S \cap \bigcap_{h\in F} S_n h$ and put $G_l(h,g) = \{h' \in G : h'h = g\}$. By the right cancellation law $G_l(h,g)$ is either empty or consists of a single point. Now

$$\|\gamma_n * \varphi - \gamma_n\|_1 = \sum_{g\in G}\Big|\sum_{h'h=g} \gamma_n(h')\varphi(h) - \gamma_n(g)\Big|$$
$$= \Big(\sum_{g\in R} + \sum_{g\in S_n\setminus R} + \sum_{g\in G\setminus S_n}\Big)\Big|\sum_{h\in F}\varphi(h)\Big\{\sum_{h'\in G_l(h,g)}\gamma_n(h') - \gamma_n(g)\Big\}\Big| .$$

If $g \in R \subset S_n$, then for every $h \in F$ there is a unique $h' \in S_n$ such that $g = h'h$; so $\gamma_n(h') = \gamma_n(g)$. Therefore

$$\sum_{g\in R}\Big|\sum_{h\in F}\varphi(h)\{\gamma_n(h') - \gamma_n(g)\}\Big| = 0 .$$

Consider the second sum:

$$\sum_{g\in S_n\setminus R}\Big|\sum_{h\in F}\varphi(h)\Big\{\sum_{h'\in G_l(h,g)}\gamma_n(h') - \gamma_n(g)\Big\}\Big|$$
$$\leqq \sum_{h\in F}\varphi(h)\sum_{g\in S_n\setminus R}\Big|\sum_{h'\in G_l(h,g)}\gamma_n(h') - \gamma_n(g)\Big|$$
$$\leqq \sum_{h\in F}\varphi(h)|S_n|^{-1}|S_n\setminus R| = |S_n|^{-1}|S_n \setminus R| < \tfrac{1}{2}\varepsilon$$

by (*).

In disposing of the third sum we argue as follows. Fix $g \in G \setminus S_n$, so $\gamma_n(g) = 0$. Suppose $h'h = g$. Then $\varphi(h)\gamma_n(h') = 0$ unless $h \in F$ and $h' \in S_n$, that is, unless $g = h'h \in S_n h \setminus S_n$. Observe that $|S_n h \setminus S_n| \leqq |S_n \setminus R|$ for all $h \in F$. The number of $g \in G \setminus S_n$ for which these conditions hold is

$$\leqq \sum_{h\in F}|S_n h \setminus S_n| \leqq |F|\,|S_n \setminus R| < \tfrac{1}{2}|S_n|\,\varepsilon$$

by (*). We conclude, since $\varphi(h) \leqq 1$ and $\gamma_n(h') = |S_n|^{-1}$ for $h' \in S_n$, that

$$\sum_{g\in G\setminus S_n}\sum_{h'h=g}\varphi(h)\gamma_n(h') < \tfrac{1}{2}\varepsilon .$$

Therefore $\|\gamma_n * \varphi - \gamma_n\|_1 < \varepsilon$ and the first assertion is proved. The second is proved by taking $R = S_n \cap \bigcap_{h' \in F} h' S_n$ and invoking the left cancellation law.

4. Almost convergence.

This section is devoted to the proof of the promised generalization of Lorentz' theorem.

THEOREM 4.1. *Let G be a countable, amenable semigroup with identity e in which both cancellation laws hold. A necessary and sufficient condition that $f \in m(G)$ almost converge to s is that, for any summing sequence $\{S_n\}$ for G,*

$$\lim_n |S_n|^{-1} \Sigma_{g \in S_n} f(gh) = s \quad \text{and} \quad \lim_n |S_n|^{-1} \Sigma_{g \in S_n} f(hg) = s$$

uniformly in h.

In order to prove the theorem we introduce the following functions:

$$\bar{v}f = \inf_{\varphi \in \Phi} \sup_{\eta \in \Phi} \Sigma_g \Sigma_h \varphi(g) \eta(h) f(gh) ,$$
$$\underline{v}f = \sup_{\varphi \in \Phi} \inf_{\eta \in \Phi} \Sigma_g \Sigma_h \varphi(g) \eta(h) f(gh) ,$$
$$\bar{w}f = \inf_{\varphi \in \Phi} \sup_{\eta \in \Phi} \Sigma_g \Sigma_h \varphi(g) \eta(h) f(hg) ,$$
$$\underline{w}f = \sup_{\varphi \in \Phi} \inf_{\eta \in \Phi} \Sigma_g \Sigma_h \varphi(g) \eta(h) f(hg) .$$

It is easy to show that

$$\bar{v}f = \inf_{\varphi \in \Phi} \sup_{h \in G} \Sigma_g \varphi(g) f(gh), \qquad \underline{v}f = \sup_{\varphi \in \Phi} \inf_{h \in G} \Sigma_g \varphi(g) f(gh)$$

with similar equations holding for $\bar{w}$ and $\underline{w}$. Dye [4] has proved that $f \in m(G)$ almost converges to s if and only if $\bar{v}f = \underline{v}f = \bar{w}f = \underline{w}f = s$. We shall utilize this fact in proving Theorem 4.1.

LEMMA 4.1. *For $f \in m(G)$ and $\varepsilon > 0$, there exists an n_0 such that, if $n \geqq n_0$, then*

1) $\sup_{h \in G} \Sigma_g \gamma_n(g) f(gh) < \bar{v}f + \varepsilon, \qquad \sup_{h \in G} \Sigma_g \gamma_n(g) f(hg) < \bar{w}f + \varepsilon ,$
2) $\inf_{h \in G} \Sigma_g \gamma_n(g) f(gh) > \underline{v}f - \varepsilon, \qquad \inf_{h \in G} \Sigma_g \gamma_n(g) f(hg) > \underline{w}f - \varepsilon .$

PROOF. Choose $\varphi \in \Phi$ such that

$$(*) \qquad \sup_{h \in G} \Sigma_g \varphi(g) f(gh) < \bar{v}f + \tfrac{1}{2}\varepsilon .$$

By Lemma 3.1, choosing n sufficiently large, we have

$$(**) \qquad \|\varphi * \gamma_n - \gamma_n\|_1 < \frac{\varepsilon}{2\|f\|_\infty} .$$

Now

$$\sup_{h\in G}\sum_g (\varphi*\gamma_n)(g)f(gh) = \sup_{h\in G}\sum_g\sum_{h'h''=g}\varphi(h')\gamma_n(h'')f(h'h''h)$$
$$\leqq \sum_{h''\in G}\gamma_n(h'')\sup_{h\in G}\sum_{g\in G}\sum_{h'\in G_l(h'',g)}\varphi(h')f(h'h) < \bar{v}f+\tfrac{1}{2}\varepsilon$$

by (*). Consequently

$$\sup_{h\in G}\sum\gamma_n(g)f(gh) < \bar{v}f+\tfrac{1}{2}\varepsilon+\|\gamma_n-\varphi*\gamma_n\|_1\,\|f\|_\infty < \bar{v}f+\varepsilon$$

by (**). The remaining statements are proved similarly, using the fact that $\|\gamma_n*\varphi-\gamma_n\|_1\to 0$ in connection with the formulas involving $\bar{w}$ and $\underline{w}$.

PROOF OF THEOREM 4.1.

(i) If $|S_n|^{-1}\sum_{g\in S_n}f(gh)$ converges to s uniformly in h, then for any left mean λ,

$$\lambda(f) = |S_n|^{-1}\sum_{g\in S_n}\lambda({}^g f)\to\lambda(s) = s\,.$$

Therefore $\lambda(f)=s$. Likewise for any right mean, $\lambda(f)=s$. Consequently f almost converges to s.

(ii) Given $\varepsilon>0$ and $f\in m(G)$, for n sufficiently large, we have

$$\underline{v}f-\varepsilon < \inf_{h\in G}\sum_g\gamma_n(g)f(gh) \leqq \sup_{h\in G}\sum_g\gamma_n(g)f(gh) < \bar{v}f+\varepsilon$$

and

$$\underline{w}f-\varepsilon < \inf_{h\in G}\sum_g\gamma_n(g)f(hg) \leqq \sup_{h\in G}\sum_g\gamma_n(g)f(hg) < \bar{w}f+\varepsilon\,.$$

Since the almost convergence of f to s entails $\bar{v}f=\underline{v}f=\bar{w}f=\underline{w}f=s$, the conditions of the theorem follow immediately.

5. Vector-valued functions.

Let X be a real Banach space and denote by $m_X(G)$ the collection of all norm-bounded X-valued functions on G and by $\tilde{m}_X(G)$ the collection of all $F\in m_X(G)$ such that ${}^gF$ and F^g are in $\tilde{m}_X(G)$ whenever F is and such that $\overline{\text{co}}\{F(g):g\in G\}$ is weakly compact. Here $\overline{\text{co}}$ denotes the norm closure of the convex hull.

An X-mean Λ on $\tilde{m}_X(G)$ is a continuous, linear map of $\tilde{m}_X(G)$ into X such that $\Lambda(F)\in\overline{\text{co}}\{F(g):g\in G\}$ for all $F\in\tilde{m}_X(G)$ (see Dixmier [3]). Using the same notions of left and right invariance, Dixmier has shown that every left (right) invariant mean λ on $m(G)$ induces a left (right) invariant X-mean Λ on $\tilde{m}_X(G)$ via the relation

$$\lambda((F(\cdot),u)) = (\Lambda(F),u)\,,$$

for $F\in\tilde{m}_X(G)$ and $u\in X^*$, the dual of X.

THEOREM 5.1. *Let G be a countable, amenable semigroup with identity e in which both cancellation laws hold. A necessary and sufficient condition that $F \in \tilde{m}_X(G)$ almost converge to $\xi \in X$ is that, for any summing sequence $\{S_n\}$ for G,*

$$\left(|S_n|^{-1}\textstyle\sum_{g\in S_n} F(gh), u\right) \to (\xi, u)$$

and

$$\left(|S_n|^{-1}\textstyle\sum_{g\in S_n} F(hg), u\right) \to (\xi, u)$$

uniformly in h, for all $u \in X^*$.

PROOF. (i) Suppose F almost converges to ξ, that is $\Lambda(F)=\xi$ for each left invariant and each right invariant X-mean Λ. Let λ be any left (right) mean on $m(G)$ and let $u \in X^*$. We have

$$\lambda((F(\cdot),u)) = (\Lambda(F),u) = (\xi,u)$$

where Λ is the left (right) X-mean induced by λ. In other words, the function $(F(\cdot),u)$, qua function in $m(G)$, almost converges to (ξ,u). By Theorem 4.1, the conditions of the theorem follow.

(ii) Suppose, on the other hand, that $|S_n|^{-1}\sum_{g\in S_n} {}^gF(h)$ converges to ξ weakly, uniformly in h. For each $g \in G$ let $X_g = X$, endowed with the weak topology, and let $Y = \prod_{g\in G} X_g$. The weak topology on Y is the product of the weak topologies on X_g (see Kelley et al. [6, page 160]). Any element of $\tilde{m}_X(G)$ can be considered as a member of Y and, given any left X-mean Λ and any $u \in X^*$, the linear functional $\Lambda_u = (\Lambda(\cdot),u)$ can be extended to an element $\tilde{\Lambda}_u$ of Y^*. Of course, an extension of Λ_u will not possess, generally, the left invariance property except on elements of $\tilde{m}_X(G)$. Observe that ξ can be considered as a member of Y, and that ξ is in the weak closure of $\mathrm{co}\{{}^gF : g \in G\}$ in Y. This follows immediately from the definition of the weak topology in Y and from the hypothesis. Since Λ is left invariant on elements of $\tilde{m}_X(G)$,

$$\Lambda(F) = \Lambda(|S_n|^{-1}\textstyle\sum_{g\in S_n} {}^gF)\,.$$

Hence, by the weak continuity of $\tilde{\Lambda}_u$,

$$(\Lambda(F),u) = \tilde{\Lambda}_u(F) \to \tilde{\Lambda}_u(\xi) = \Lambda_u(\xi) = (\xi,u) \qquad \text{for all } u \in X^*\,.$$

So $\Lambda(F)=\xi$. Similarly $\Lambda(F)=\xi$ for any right-invariant X-mean. That is, F almost converges to ξ.

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