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WEIGHTED MEAN APPROXIMATION IN CARATHÉODORY REGIONS

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A region D in the complex plane is called Carathéodory if it is simply connected, bounded, and if its boundary, ∂D, coincides with the boundary of the infinite component of the complement of the closure of D.

If D is a region and $a(z)$ a continuous, positive function in D, we denote by $H^p(a; D)$, $1 \leqq p < \infty$, the Banach space of all functions $h(z)$ which are analytic in D and satisfy

$$\|h\|_a{}^p = \int_D |h(z)|^p \, a(z) \, dA < \infty ,$$

where dA denotes plane Lebesgue measure.

The purpose of this note is to give conditions on the weight a for the polynomials to be dense in $H^p(a; D)$, D Carathéodory. For a survey of earlier results, see Mergeljan [3]. See also Hedberg [1] and H. S. Shapiro [4], [5]. We give general, sufficient conditions both in the case when a is the modulus of an analytic function (Theorems 2, 3, 4) and in the general case (Theorem 5). Theorem 5 is a considerable sharpening of Theorem 1 in [1].

In [1] we also gave a result on generators in the Banach algebra $l^1(0, \infty)$. This result is improved in Corollary 2.

Acknowledgements. In a personal communication F. S. Lisin has informed me that he has proved Theorem 5 by a method different from the present one. I am grateful to him for communicating this result to me. See [6].

I am indebted to K.-O. Widman for many stimulating discussions on the topics treated here.

In what follows D is always a Carathéodory region. We denote by $P^p(a; D)$ the closure of the polynomials in $H^p(a; D)$. If $f(z)$ is any function which is defined in D, we extend its domain of definition to the whole plane by defining $f(z) = 0$, $z \notin D$.

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We shall start by proving the following general theorem, from which our other results will be deduced.

THEOREM 1. *Let $a(z) > 0$ be continuous and integrable in D. Then every function h in $H^p(a; D)$, $p > 1$, which satisfies*

$$\sup_{r>0} \frac{1}{r^2} \int_D |h(z)|^p \, dA_z \int_{|w| \leq r} a(z+w)\, dA_w < \infty , \tag{1}$$

belongs to $P^p(a; D)$, and every function h in $H^1(a; D)$, which satisfies

$$\int_D |h(z)| \, dA_z \left\{ \sup_{r>0} \frac{1}{r^2} \int_{|w| \leq r} a(z+w)\, dA_w \right\} < \infty , \tag{1'}$$

belongs to $P^1(a; D)$.

The proof depends on the following well-known lemma of Mergeljan [3].

LEMMA 1. *Let $\delta(z)$ denote the distance from $z \in D$ to ∂D. Then for every $z \in D$ there is a polynomial Q_z and there are absolute constants C_1 and C_2 such that for all $\zeta \in D$*

$$\left| \frac{1}{\zeta - z} - Q_z(\zeta) \right| \leq \frac{C_1 \delta^2(z)}{|\zeta - z|^3}$$

and

$$|Q_z(\zeta)| \leq C_2/\delta(z) .$$

We also need the following elementary lemma, which is proved by an integration by parts.

LEMMA 2. *If $f \geq 0$ is a function such that for some $C > 0$*

$$\int_0^r t f(t)\, dt \leq Cr^2, \quad 0 < r \leq r_0 ,$$

then

$$\int_0^r f(t)\, dt \leq 2Cr, \quad 0 < r \leq r_0 ,$$

and

$$\int_r^{r_0} \frac{f(t)}{t^2}\, dt \leq \frac{4C}{r}, \quad 0 < r \leq r_0 .$$

PROOF OF THEOREM 1. Suppose $h \in H^p(a; D)$ satisfies (1) or (1′). We are required to prove that if $g \in L^{p'}(a; D)$, $1/p + 1/p' = 1$, is such that

$$\int_D Q(z)\, \overline{g(z)}\, a(z)\, dA = 0$$

for all polynomials Q, then also

$$\int_D h(z)\, \overline{g(z)}\, a(z)\, dA = 0 .$$

Let $D_q = \{z \in D;\, q \leqq \delta(z) \leqq 2q\}$, $q > 0$, and put $\mu(z) = \overline{g(z)}\, a(z)$. It follows from Hölder's inequality that μ is integrable.

We proved in [1], p. 544, that it is sufficient to show that

$$\lim_{q \to 0} \frac{1}{q} \int_{D_q} h(z)\, dA_z \int_D \frac{\mu(\zeta)}{\zeta - z}\, dA_\zeta = 0 .$$

This was proved under the assumption that μ is smooth, but there is no difficulty in extending it to the general case when μ is only integrable.

Now assume that $\int_D Q\mu\, dA = 0$ for all polynomials Q. Then, for $z \in D_q$, Lemma 1 gives

$$\left| \int_D \frac{\mu(\zeta)}{\zeta - z}\, dA \right| = \left| \int_D \mu(\zeta) \left(\frac{1}{\zeta - z} - Q_z(\zeta) \right) dA_\zeta \right| \leqq$$

$$\leqq 4C_1 q^2 \int_{|\zeta - z| \geqq q} \frac{|\mu(\zeta)|}{|\zeta - z|^3}\, dA_\zeta + (1 + C_2) \int_{|\zeta - z| \leqq q} \frac{|\mu(\zeta)|}{|\zeta - z|}\, dA_\zeta .$$

It is therefore enough to show that

$$\lim_{q \to 0} \frac{1}{q} \int_{D_q} |h(z)|\, dA_z \int_{|w| \leqq q} \frac{|\mu(z + w)|}{|w|}\, dA_w = 0 , \tag{2}$$

and

$$\lim_{q \to 0} q \int_{D_q} |h(z)|\, dA_z \int_{|w| \geqq q} \frac{|\mu(z + w)|}{|w|^3}\, dA_w = 0 . \tag{3}$$

Suppose $p > 1$. Choose $\varepsilon > 0$ arbitrarily, and then choose $q_0 > 0$ so that

$$\int_{\delta(z) \leqq 3q_0} |g(z)|^{p'}\, a(z)\, dA < \varepsilon .$$

Consider (2). By changing the order of integration and introducing polar coordinates for w, we obtain by Lemma 2 for $q < q_0$

$$\frac{1}{q}\int_{D_q} |h(z)|\, dA_z \int_{|w|\leq q} \frac{|\mu(z+w)|}{|w|}\, dA_w$$

$$\leq \frac{1}{q}\int_{|w|\leq q} \frac{dA_w}{|w|} \int_{\delta(z)\leq 2q_0} |h(z)\mu(z+w)|\, dA_z$$

$$\leq 2\sup_{r\leq q} \frac{1}{r^2}\int_{|w|\leq r} dA_w \int_{\delta(z)\leq 2q_0} |h(z)\mu(z+w)|\, dA_z\,.$$

By the Hölder inequality this is less than

$$2\left\{\sup_{r\leq q}\frac{1}{r^2}\int_{|w|\leq r} dA_w \int_{\delta(z)\leq 2q_0} |h(z)|^p\, a(z+w)\, dA_z\right\}^{1/p}\cdot$$

$$\cdot\left\{\sup_{r\leq q}\frac{1}{r^2}\int_{|w|\leq r} dA_w \int_{\delta(z)\leq 2q_0} |g(z+w)|^{p'}\, a(z+w)\, dA_z\right\}^{1/p'}$$

$$\leq 2\left\{\sup_{r\leq q}\frac{1}{r^2}\int_{D} |h(z)|^p\, dA_z \int_{|w|\leq r} a(z+w)\, dA_w\right\}^{1/p}\cdot$$

$$\cdot\left\{\sup_{r\leq q}\frac{1}{r^2}\int_{|w|\leq r} dA_w \int_{\delta(z)\leq 3q_0} |g(z)|^{p'}\, a(z)\, dA_z\right\}^{1/p'} \leq C\varepsilon^{1/p'},$$

where the last inequality follows from our assumptions. Since ε is arbitrary, this proves (2).

If we consider (3) we similarly obtain for $q < q_0$

$$q\int_{D_q} |h(z)|\, dA_z \int_{q\leq |w|\leq q_0} \frac{|\mu(z+w)|}{|w|^3}\, dA_w \leq C\varepsilon^{1/p'},$$

and since

$$\lim_{q\to 0} q\int_{D_q} |h(z)|\, dA_z \int_{|w|\geq q_0} \frac{|\mu(z+w)|}{|w|^3}\, dA_w = 0\,,$$

and ε is arbitrary, this proves Theorem 1 for $p > 1$.

If $p = 1$ Hölder's inequality is replaced by simpler estimates.

COROLLARY 1. *Let a be as in Theorem 1. If, in addition, $a \in L^s(D)$ for some s, $1 < s \leqq \infty$, then every h in $H^p(a; D) \cap L^{ps'}(D)$, $1 \leqq p < \infty$, $1/s + 1/s' = 1$, belongs to $P^p(a; D)$.*

PROOF. The corollary is obvious if $s = \infty$. Assume $1 < s < \infty$, and let

$$a^*(z) = \sup_{r>0} \frac{1}{\pi r^2} \int_{|w| \leqq r} a(z+w)\, dA_w, \quad z \in D .$$

Then, by Hardy's maximal theorem, $a^* \in L^s(D)$ if $a \in L^s(D)$. Thus, by Hölder's inequality, (1) or (1′) is satisfied by all $h \in L^{ps'}(D)$.

REMARK. Corollary 1 contains in particular the classical theorem of Farrell and Markuševič that $P^p(1; D) = H^p(1; D)$. See [2, p. 112].

THEOREM 2. *Let $\alpha(z) \neq 0$ be analytic in D. Then*

$$P^p(|\alpha|; D) = H^p(|\alpha|; D), \quad 1 \leqq p < \infty ,$$

if for some $\delta > 0$

$$\int_D (|\alpha|^{-\delta} + |\alpha|^{1+\delta})\, dA < \infty . \tag{4}$$

This theorem is a consequence of the following more general result.

THEOREM 3. *Let $\alpha(z) \neq 0$ be analytic and integrable in D. Then*

$$P^p(|\alpha|; D) = H^p(|\alpha|; D), \quad 1 < p < \infty ,$$

if for some $\varepsilon > 0$

$$\sup_{r>0} \frac{1}{r^2} \int_D \frac{1}{|\alpha(z)|^\varepsilon}\, dA_z \int_{|w| \leqq r} |\alpha(z+w)|\, dA_w < \infty , \tag{5}$$

and

$$P^1(|\alpha|; D) = H^1(|\alpha|; D)$$

if for some $\varepsilon > 0$

$$\int_D \frac{1}{|\alpha(z)|^\varepsilon}\, dA_z \left\{ \sup_{r>0} \frac{1}{r^2} \int_{|w| \leqq r} |\alpha(z+w)|\, dA_w \right\} < \infty . \tag{5′}$$

PROOF OF THEOREM 3. Since α has no zeros, we can define a regular branch of α^λ for all λ. To prove the theorem it is enough to show that $\alpha^{-1/p} \in P^p(|\alpha|; D)$. (See [2, p. 132]). In fact, if $\alpha^{-1/p} \in P^p(|\alpha|; D)$, then so does $Q\alpha^{-1/p}$ for every polynomial Q, and the approximation of h

in $H^p(|\alpha|; D)$ by functions $Q\alpha^{-1/p}$ is equivalent to the approximation of $h\alpha^{1/p}$ in $H^p(1; D)$ by polynomials.

We use an idea due to H. S. Shapiro [4, p. 327]. Suppose (5) or (5′) is satisfied for some $\varepsilon > 0$. Then, by Theorem 1, $\alpha^{-\varepsilon/p} \in P^p(|\alpha|; D)$.

We shall show that if $\alpha^{-(1-\lambda)/p} \in P^p(|\alpha|; D)$ for some λ, $0 < \lambda < 1$, then also $\alpha^{-(1-\lambda(1-\varepsilon))/p} \in P^p(|\alpha|; D)$. It is clearly sufficient for this to show that there are polynomials Q such that

$$\int_D |\alpha^{-(1-\lambda(1-\varepsilon))/p} - Q\alpha^{-(1-\lambda)/p}|^p \, |\alpha| \, dA = \int_D |\alpha^{-\lambda\varepsilon/p} - Q|^p \, |\alpha|^\lambda \, dA$$

is arbitrarily small. But this follows from Theorem 1 applied to the weight $|\alpha|^\lambda$, for by Hölder's inequality

$$\begin{aligned} &\int_D dA_z \int_{|w|\leq r} |\alpha(z)|^{-\lambda\varepsilon} \, |\alpha(z+w)|^\lambda \, dA_w \\ &\leqq \left\{\int_D dA_z \int_{|w|\leq r} |\alpha(z)|^{-\varepsilon} \, |\alpha(z+w)| \, dA_w\right\}^\lambda \left\{\int_D dA_z \int_{|w|\leq r} dA_w\right\}^{1-\lambda} \\ &\leqq \text{Const.}\, r^{2\lambda} r^{2-2\lambda}, \end{aligned}$$

by (5), and similarly for $p = 1$.

It follows by induction that $\alpha^{-(1-(1-\varepsilon)^n)/p} \in P^p(|\alpha|; D)$ for all positive integers n.

But by Lebesgue's theorem on dominated convergence

$$\lim_{n\to\infty} \int_D |\alpha^{-1/p} - \alpha^{-(1-(1-\varepsilon)^n)/p}|^p \, |\alpha| \, dA = 0,$$

for the integrand tends to zero pointwise, and

$$|\alpha|^{-(1-(1-\varepsilon)^n)} \, |\alpha| = |\alpha|^{(1-\varepsilon)^n} \leqq 1 + |\alpha| .$$

This proves the theorem.

Remark. A similar argument shows that in H. S. Shapiro [4], Theorem 1, his condition (2) is redundant.

Proof of Theorem 2. Let

$$\alpha^*(z) = \sup_{r>0} \frac{1}{\pi r^2} \int_{|w|\leq r} |\alpha(z+w)| \, dA_w,$$

and assume that (4) is satisfied for some $\delta > 0$. Then (5) or (5′) is satisfied for $\varepsilon \leqq \delta^2/(1+\delta)$, by Hardy's maximal theorem and Hölder's inequality.

If α is bounded we can also prove the following theorem, which depends only on the Farrell–Markuševič theorem. Since $\alpha(z) \neq 0$ we can write $\alpha = e^{-\beta}$, where β is an analytic function.

THEOREM 4. *Let β be analytic in D, and assume $|\alpha(z)| = |e^{-\beta(z)}| < 1$ in D. Then*

$$P^p(|\alpha|; D) = H^p(|\alpha|; D), \quad 1 \leqq p < \infty,$$

if there are positive constants C_1 and C_2 so that

$$|\mathrm{Im}\beta(z)| \leqq C_1 \mathrm{Re}\beta(z) + C_2, \quad z \in D. \tag{6}$$

PROOF. It is clear from (6) that $\beta^n \in H^p(|\alpha|; D)$ for all positive integers n. We claim that also $\beta^n \in P^p(|\alpha|; D)$. We show this first for $n=1$.

The assumption implies that $\mathrm{Re}\beta > 0$ in D. Thus the function $\gamma = (\beta-1)/(\beta+1)$ is bounded by 1 in D, and hence, by the Farrell–Markuševič theorem, $\gamma^\nu \in P^p(|\alpha|; D)$ for all positive integers ν. But

$$\beta = (1+\gamma)/(1-\gamma) = 1 + 2\sum_1^\infty \gamma^\nu,$$

and therefore it is enough to show that

$$\lim_{m\to\infty} \int_D \left|\beta - 1 - 2\sum_1^m \gamma^\nu\right|^p |\alpha|\, dA = 0.$$

This follows, however, from Lebesgue's theorem on dominated convergence, for

$$\left|\sum_1^m \gamma^\nu\right|^p |\alpha| \leqq 2|\alpha|/|1-\gamma|^p = |1+\beta|^p \exp(-\mathrm{Re}\beta),$$

which is a bounded function, by (6).

Now assume that $\beta^n \in P^p(|\alpha|; D)$ for some positive integer n. Then also $\beta^{n+1} \in P^p(|\alpha|; D)$, for if Q is any polynomial,

$$\int_D |\beta^{n+1} - Q\beta^n|^p |\alpha|\, dA = \int_D |\beta - Q|^p |\beta|^{np} |\alpha|\, dA,$$

which can be made arbitrarily small, by the above argument applied to the bounded weight $|\beta|^{np}|\alpha|$. It follows by induction that $\beta^n \in P^p(|\alpha|; D)$ for all positive integers n, and hence also that the sum

$$Q_N = \sum_0^N \frac{t^n \beta^n}{n!}$$

belongs to $P^p(|\alpha|; D)$ for all N and t. We claim that

$$\lim_{N\to\infty} \int |\alpha^{-t} - Q_N|^p \, |\alpha| \, dA = 0 \quad \text{for} \quad 0 \leqq t \leqq 1/(C_1+1)p\,.$$

Again, this follows from Lebesgue's theorem, for $\lim_{N\to\infty} Q_N(z) = \alpha(z)^{-t}$ pointwise, and by (6)

$$|Q_N| \leqq \sum_0^N \frac{t^n}{n!} \left((C_1+1)\operatorname{Re}\beta + C_2\right)^n$$

$$\leqq e^{C_2 t} |\alpha|^{-(C_1+1)t} \leqq e^{C_2/(C_1+1)p}(1 + |\alpha|^{-1/p})\,.$$

Thus,

$$\alpha^{-1/(C_1+1)p} \in P^p(|\alpha|; D)\,.$$

To complete the proof we now only have to repeat the induction in Theorem 3 with $\varepsilon = 1/(C_1+1)$, applying the above argument to the weight $|\alpha|^\lambda$.

Example. Let D be the unit disc and

$$\alpha(z) = \exp\left(\frac{z+1}{z-1}\right)^t.$$

Then the above theorem shows that $P^p(|\alpha|; D) = H^p(|\alpha|; D)$ for $0 < t < 1$. On the other hand, Keldyš has shown (see [2, p. 134]) that $P^2(|\alpha|; D) \neq H^2(|\alpha|; D)$ for $t = 1$, and the proof extends to $p \neq 2$.

We shall now consider general weights. We denote by f a Riemann mapping function which maps D onto the unit disc U, and we denote by φ the inverse to f.

Theorem 5. *Let $a(z) > 0$ be continuous in D. Then*

$$P^p(a; D) = H^p(a; D), \quad 1 \leqq p < \infty\,,$$

if $a \in L^s(D)$ for some $s > 1$, and if $a \circ \varphi \in L^1(U)$ is such that $P^p(a \circ \varphi; U) = H^p(a \circ \varphi; U)$.

Proof. The proof is similar to that of Theorem 3. Suppose a satisfies the above conditions. Since f is univalent, $f'(z) \neq 0$, and we can therefore define regular branches of $(f')^\lambda$ and $(\varphi')^\lambda$ for all λ.

It is enough to show that for every integer $n \geqq 0$ we have

$f^n(f')^{2/p} \in P^p(a; D)$. (See [2, p. 136]). Indeed, if $h \in H^p(a; D)$, and Q is a polynomial,

$$\int_D |h - Q(f)(f')^{2/p}|^p \, a \, dA = \int_D |h(f')^{-2/p} - Q(f)|^p \, a|f'|^2 \, dA$$

$$= \int_U |(h \circ \varphi)(\varphi')^{2/p} - Q|^p \, (a \circ \varphi) \, dA ,$$

which can be made arbitrarily small by the assumptions, since

$$\int_U |(h \circ \varphi)(\varphi')^{2/p}|^p \, (a \circ \varphi) \, dA = \int_D |h|^p \, a \, dA < \infty .$$

Then it is clearly also enough to prove that $(1+f)^n(f')^{2/p} \in P^p(a; D)$, $n \geqq 0$. We fix n and put $(1+f)^n(f')^{2/p} = g$. Then $g(z) \neq 0$, and g^λ is analytic for all λ. We know that

$$\int_D |g|^p \, dA = \int_U |1+w|^{np} \, dA_w < \infty ,$$

so, by Corollary 1, $g^{1-1/s} \in P^p(a; D)$.

We claim that if $g^{1-\lambda} \in P^p(a; D)$ for some λ, $0 < \lambda < 1$, then also $g^{1-\lambda/s} \in P^p(a; D)$. Put $\delta = \lambda(s-1)/s$. Then, for every polynomial Q we find

$$\int_D |g^{1-\lambda/s} - Q g^{1-\lambda}|^p \, a \, dA = \int_D |g^\delta - Q|^p \, |g|^{p(1-\lambda)} \, a \, dA .$$

Here $g^\delta \in L^{p/\delta}(D)$, so the assertion follows from Corollary 1, if we show that the weight $|g|^{p(1-\lambda)} a \in L^{1/(1-\delta)}$.

We obtain by Hölder's inequality

$$\int_D |g^{p(1-\lambda)} a|^{1/(1-\delta)} \, dA = \int_D |g^p a|^{(1-\lambda)/(1-\delta)} \, a^{\lambda/(1-\delta)} \, dA$$

$$\leqq \left\{ \int_D |g^p a| \, dA \right\}^{(1-\lambda)/(1-\delta)} \left\{ \int_D a^{\lambda/(\lambda-\delta)} \, dA \right\}^{(\lambda-\delta)/(1-\delta)} ,$$

which is finite, since $\lambda/(\lambda-\delta) = s$. It follows by induction that $g^{1-1/s^n} \in P^p(a; D)$ for all positive integers n.

Now Lebesgue's theorem on dominated convergence shows that

$$\lim_{n \to \infty} \int_D |g - g^{1-1/s^n}|^p \, a \, dA = 0 ,$$

for the integrand converges to 0 pointwise, and $|g|^{p-p/s^n} \leqq 1+|g|^p$. This proves the theorem.

In [1] we also studied the problem of finding generators of the Banach algebra $l^1(0,\infty)$, or, equivalently, the algebra A of all analytic functions, $g(w)=\sum_0^\infty g_n w^n$, in the unit disc, such that $\|g\|=\sum_0^\infty |g_n|$ is finite. See [1] for references. We can now improve the result given there.

COROLLARY 2. *A function $\varphi(w)=\sum_0^\infty \varphi_n w^n$ is a generator for A if it is univalent in $|w|\leqq 1$, and if, for some $\varepsilon>0$,*

$$\sum_2^\infty n(\log n)^{1+\varepsilon}\,|\varphi_n|^2 < \infty .$$

The proof is the same as in [1].

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