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ON PARTIAL ORDERINGS OF NORMED SPACES

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1. Introduction.

We discuss some ways in which an arbitrary normed space can be partially ordered so that the norm, or at least the topology is determined by the ordering. The orderings which we discuss induce either an order unit norm topology or a base norm topology or both. In this connection we characterise vertices of unit balls of normed spaces, and also show that the unique decomposition property for base normed spaces does not imply the same property for the second dual space. We study finally the determining factors for the space of normal linear functionals on an order unit normed Banach dual space.

2. O.U.B.-Spaces.

By a *partially ordered normed space* we mean a real normed space partially ordered by a proper cone containing non-zero elements. We use the notation and terminology of [7].

We call a partially ordered normed space an *O.U.B.-space* if its ordering induces both an order unit norm and a base norm, both of which are equivalent to the original norm. Such spaces are very numerous.

In fact let X be any real normed space, let $e \in X$ with $\|e\| = 1$, and let λ be a real number with $\lambda \geqq 1$. Choose $f \in X^*$ such that $f(e) = \|f\| = 1$, and define

$$K(f,\lambda) = \{y \in X : \|y\| \leqq \lambda f(y)\} .$$

Then (cf. [1]) $K(f,\lambda)$ is a closed non-empty cone generated by the base $B(f,\lambda) = K(f,\lambda) \cap f^{-1}(1)$, and if $\lambda > 1$ then e is an interior point of $K(f,\lambda)$ and is thus an order unit for the partial ordering of X induced by this cone. In the case $\lambda > 1$ it is easily verified that the sets $[-e,e]$ and $\mathrm{co}(B(f,\lambda) \cup -B(f,\lambda))$ are both bounded and have non-empty interiors; hence for this ordering X is an O.U.B.-space.

We now list some facts concerning O.U.B.-spaces, most of which are readily obtained from known results. We will assume that both the dual space X^* of X and the space $B(X)$ of bounded linear operators

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in X have the natural partial orderings induced by the ordering of X. An element x of a convex subset C of X is said to be *strongly exposed* in C if there exists an $f \in X^*$ such that $f(x) < f(y)$ for all $y \neq x$ in C and such that whenever $\{x_n\}$ is a sequence in C with $f(x_n) \to f(x)$ then $\|x_n - x\| \to 0$.

THEOREM 1. *Let X be a partially ordered Banach space with closed positive cone K. Then the following statements are equivalent:*

(i) *X is an O.U.B.-space;*
(ii) *K is normal, has non-empty interior and 0 is a strongly exposed point of K;*
(iii) *X^* is an O.U.B.-space with a w^*-locally compact positive cone;*
(iv) *$B(X)$ is an O.U.B.-space whose positive cone possesses a base which is closed in the strong operator topology.*

PROOF. The equivalence of statements (i), (iii) and (iv) follows readily from [7], [8]. We prove the equivalence of (i) and (ii).

If statement (i) holds then certainly the first two conditions of (ii) are satisfied. Moreover, if f is a strictly positive linear functional on X then f defines a base norm $\|\cdot\|_f$ in X equivalent to the original norm and such that $f(x) = \|x\|_f$ for all $x \in K$. Consequently 0 is a strongly exposed point in K supported by f, and so statement (ii) holds.

Conversely suppose that statement (ii) holds. Then the first two conditions immediately imply that X possesses an equivalent order unit norm. Since 0 is a strongly exposed point of K let $f \in X^*$ be the corresponding functional. It is evident that f is strictly positive, and so the set $B = K \cap f^{-1}(1)$ is a base for K.

Define $\alpha = \inf\{f(x) : x \in K, \ \|x\| = 1\}$ and $\beta = \sup\{f(x) : x \in K, \ \|x\| = 1\}$. If $\alpha = 0$ then there exists a sequence $\{x_n\}$ in K, with $\|x_n\| = 1$ for each n, and such that $f(x_n) \to 0$, and this clearly contradicts the properties of f. If $x \in B$ it now follows that $\beta^{-1} \leqq \|x\| \leqq \alpha^{-1}$. The second of these inequalities shows that $S = \mathrm{co}(B \cup -B)$ is bounded while the first inequality, together with the fact that the positive cone in an order unit normed space is strictly generating, shows that S has non-empty interior. Therefore X has an equivalent base norm, and so (i) holds.

The positive cone in $C[0,1]$ is an example where 0 is an exposed point but not a strongly exposed point.

It is not difficult to see that a lattice ordered O.U.B.-space must be finite dimensional. In fact such a space would be an AL-space whose positive cone has non-empty interior (cf. [4]). Conversely, it is well known that a finite dimensional vector lattice has the properties of an O.U.B.-space.

3. Vertices.

Let X be a real Banach space with unit ball S, and let e be a boundary point of S. We consider the possibility of defining a partial ordering in X for which e is an order unit inducing a norm in X dominating, or dominated by, the original norm. We define

$$\begin{aligned} K_1 &= \{y=\lambda(3e+x) \in X : \lambda \geqq 0, \|x\| \leqq 1\}, \\ K_2 &= \{y=\lambda(e+x) \in X : \lambda \geqq 0, \|x\| \leqq 1\}, \\ H &= \{f \in X^* : \|f\|=f(e)\}. \end{aligned}$$

It is easy to verify that K_1 and H are always cones, while K_2 is a cone if and only if e is an extreme point of S; the cone K_2 has been used by several authors (e.g. Lindenstrauss [10]). Bohnenblust and Karlin [2] defined e to be a *vertex* of S if H forms a total subset of X^*. We say that e is a *special vertex* of S if H generates X^*.

Lindenstrauss essentially proved (cf. [10, Theorem 4.7]) that X can be partially ordered so that e is an order unit defining the norm in X if and only if

$$S = (2S+e) \cap (2S-e).$$

In fact if S satisfies this relation then the cone K_2 defines such an ordering.

THEOREM 2. (i) *e is an order unit for the partial ordering of X induced by K_1 and e defines an order unit norm equivalent to, and dominating, the original norm in X. Moreover, for this ordering, X is an O.U.B.-space.*

(ii) *If e is a vertex of S then, for the partial ordering of X induced by K_2, e is an order unit defining a norm dominated by the original norm in X; if e is a special vertex of S then the two norms are equivalent, and if H has non-empty interior then X is an O.U.B.-space.*

Conversely if, for some partial ordering of X, e defines an order unit norm dominated by the original norm then e is a vertex of S; if the two norms are equivalent then e is a special vertex of S, and if X is an O.U.B.-space then H has non-empty interior.

PROOF. (i) Let X be partially ordered by K_1. If $3y \in S$ then we have $e+y=\frac{1}{3}(3e+3y) \in K_1$, and hence $S \subseteq 3[-e,e]$. Suppose now that $-e \leqq y \leqq e$. Then

$$y = (3\lambda_1-1)e+\lambda_1 x_1 = (1-3\lambda_2)e-\lambda_2 x_2,$$

with $\lambda_1,\lambda_2 \geqq 0$ and $\|x_1\|,\|x_2\| \leqq 1$. If either $\lambda_1 \leqq \frac{1}{3}$ or $\lambda_2 \leqq \frac{1}{3}$ it follows easily that $\|y\| \leqq 1$; hence we can suppose that $\lambda_1 \geqq \frac{1}{3}$ and $\lambda_2 \geqq \frac{1}{3}$. We now have $\|y\| \leqq 4\lambda_1 - 1$ and $\|y\| \leqq 4\lambda_2 - 1$, and hence $\|y\| \leqq 2(\lambda_1+\lambda_2)-1$. Since

$$(2-3(\lambda_1+\lambda_2))e = \lambda_1 x_1 + \lambda_2 x_2$$

it follows that

$$|2-3(\lambda_1+\lambda_2)| \leqq \lambda_1+\lambda_2 ,$$

and therefore $\lambda_1+\lambda_2 \leqq 1$; this gives $\|y\| \leqq 1$, and so we have shown that $[-e,e] \subseteqq S$.

To prove that X is an O.U.B.-space it is sufficient, by Theorem 1, to show that 0 is strongly exposed in K_1. Choose $f \in X^*$ such that $\|f\| = f(e) = 1$, and suppose that $\{y_n\}$ is a sequence in K_1 such that $f(y_n) \to 0$. Then $y_n = \lambda_n(3e+x_n)$, with $\lambda_n \geqq 0$, $\|x_n\| \leqq 1$, and so

$$f(y_n) = \lambda_n(3+f(x_n)) \geqq 2\lambda_n .$$

Therefore $\lambda_n \to 0$ and, since $\|y_n\| \leqq 4\lambda_n$, it follows that $y_n \to 0$.

(ii) If $f \in H$ and $y = \lambda(e+x) \in K_2$ then

$$f(y) = \lambda(\|f\|+f(x)) \geqq 0 .$$

Conversely if $f \in X^*$ is positive on K_2 then $f(e+x) \geqq 0$ for all $x \in S$, and it follows that $f \in H$. Thus H is the dual cone of K_2.

If e is a vertex of S then the subdual wedge K of H is a proper cone, and K is the closure of K_2 (cf. [7]). If $x \in S$ then we have $e+x \in K_2 \subseteqq K$, and therefore e is an order unit for both cones K_2 and K. Since K is closed it is an archimedean cone, and this implies that K_2 is almost archimedean. It is not difficult to verify that the two order unit norms which e defines in X relative to the cones K and K_2 are equal and are dominated by the original norm.

Now suppose that e is a special vertex of S. Then the dual space $H-H$ of X for its order unit norm coincides with X^*, and since the two norms are comparable on X it follows from the closed graph theorem that they are equivalent. Therefore X^* has an equivalent base norm for the partial ordering defined by H, and hence if H has non-empty interior it follows from Theorem 1 that X is an O.U.B.-space with positive cone K. It is now evident that X is an O.U.B.-space for the positive cone K_2.

Finally suppose that there exists a partial ordering of X for which e is an order unit defining a norm $\|\cdot\|_e$ in X dominated by the original norm. Then the $\|\cdot\|_e$-dual space L of X forms a total subspace of X^*. Since a linear functional f is positive on X if and only if $f(e) = \|f\|_e$, it is clear that H contains the dual cone in L and hence H is a total subset of X^*. If the two norms are equivalent then $L = X^*$, and so H generates X^*. If X is an O.U.B.-space then so is X^*, and therefore the interior of H is non-empty.

If X is any infinite-dimensional AM-space with order unit e, then e is a special vertex of S while the interior of H is empty. We will show below the notions of vertex and special vertex are generally distinct.

For an arbitrary real normed space X we write $Y = X \times R$, where R denotes the real numbers, and we identify X with the subspace $X \times \{0\}$ of Y. If Y is partially ordered by the cone

$$P = \{(x,t) \in Y : \|x\| \leqq t\}$$

then it is easy to see that $(0,1)$ is an order unit for Y which defines the norm $\|(x,t)\|_1 = \|x\| + |t|$. Moreover the set

$$B = \{(x,1) : \|x\| \leqq 1\}$$

is a base for P which defines in Y the norm

$$\|(x,t)\|_\infty = \max\{\|x\|, |t|\} .$$

Since these norms coincide on X it follows that X can be embedded as a normed subspace of co-dimension one of an O.U.B.-space with either a base norm or an order unit norm.

For example let X be the space $C[0,1]$ and let $\|\cdot\|'$ be the supremum norm and $\|\cdot\|''$ the L_1-norm in X. These norms define two non-equivalent order unit norms in Y and it follows easily from Theorem 2 that $(0,1)$ is a vertex but not a special vertex of the unit ball in Y for the first order unit norm.

4. The unique decomposition property for base normed spaces.

A base normed space Y is said to have the *unique decomposition property* if each element y has a unique positive decomposition of the form $y = y_1 - y_2$, with

$$\|y\| = \|y_1\| + \|y_2\| .$$

It was proved in [9] that Y has the unique decomposition property if and only if its base B has the property that whenever $\lambda \geqq 0$ and $y \in Y$ then the set $B \cap (y + \lambda B)$ is either empty, or a single point, or it contains a set of the form $y_1 + \mu B$ for some $y_1 \in Y$ and some $\mu > 0$.

Now if Y is as in the previous section and is equipped with its base norm $\|\cdot\|_\infty$, then the result just quoted easily shows that Y has the unique decomposition property if and only if S has the intersection property in X. The following theorem is therefore a fairly direct consequence.

THEOREM 3. *Y has the unique decomposition property if and only if X is a strictly convex normed space.*

Since Day [3] has given an example of a strictly convex Banach space whose dual space is not smooth, and hence whose second dual space is not strictly convex, we obtain immediately the following result.

COROLLARY. *There exists a base normed Banach space which has the unique decomposition property but whose second dual space does not.*

5. Normal functionals.

Let X denote a base normed Banach space with closed positive cone K and with dual space X^* whose norm is defined by an order unit e. We require the following known result.

LEMMA. *Each norm-bounded monotonic increasing net in X^* converges in the w^*-topology to its least upper bound.*

PROOF. Let $\{f_\alpha\}$ be a net in X^* satisfying the hypotheses. Then this net is conditionally w^*-compact. Moreover $\{f_\alpha\}$ is a w^*-Cauchy net: in fact if $x \in K$ then the net $\{f_\alpha(x)\}$ of real numbers is bounded and monotonic increasing, and therefore converges to its supremum, and since K generates X the result follows. Therefore $\{f_\alpha\}$ converges in the w^*-topology to a limit f such that $f(x) = \sup_\alpha \{f_\alpha(x)\}$ for each $x \in K$. Since X^* has the dual ordering it follows that f is the least upper bound of $\{f_\alpha\}$. (This proof is based on [11, p. 1.8].)

A positive linear functional η on X^* is said to be *normal* if

$$\sup_\alpha \{\eta(f_\alpha)\} = \eta(\sup_\alpha \{f_\alpha\})$$

whenever $\{f_\alpha\}$ is a norm bounded monotonic increasing net in X^*. A linear functional η on X^* is *normal* if $\eta = \eta_1 - \eta_2$, where η_1 and η_2 are both positive and normal. Since every positive linear functional on X^* is continuous, the normal functionals form a linear subspace of X^{**} which, by the above lemma, contains the canonical embedding of X.

The determining factors for the normal functionals on X^* are, of course, the norm, the order unit and the ordering. Since any two order units which define norms in X^* relative to the same ordering necessarily define the same topology, the normal functionals are in a sense determined by the ordering alone. On the other hand since the order unit and norm together determine the ordering of X^*, they together determine the normal functionals. We give below a class of spaces X^* in which the normal functionals are determined by the norm alone, and we also give a class of spaces for which this is not the case.

Let S be a compact hausdorff space and $C(S)$ the Banach space of all real-valued continuous functions on S with the supremum norm. Dixmier [5] proved that $C(S)$ is a Banach dual space if and only if S is hyperstonean, and that in this case the (unique) subdual space consists of the normal functionals on $C(S)$ relative to the natural ordering. We require these facts for the following theorem.

THEOREM 4. *Let S be a hyperstonean space and N the subdual Banach space of $C(S)$.*

(i) *If P is a cone in $C(S)$ with respect to which the norm in $C(S)$ is an order unit norm, then the space of normal functionals on $C(S)$ relative to P coincides with N.*

(ii) *The space $Y=C(S)\times R$ can be ordered in two ways such that the norm $\|(.,.)\|_1$ in Y is an order unit norm, but such that the space of normal functionals relative to the first ordering is $N\times R$ while relative to the second ordering every continuous linear functional is normal.*

PROOF. (i) Let e denote the order unit such that the order interval relative to $P[-e,e]$ is precisely the unit ball U of $C(S)$. Then e is an extreme point of U, and this clearly implies that $|e|=1$. Therefore if

$$S_1 = \{s\in S:\ e(s)=1\} \quad \text{and} \quad S_2 = \{s\in S:\ e(s)=-1\},$$

then $S=S_1\cup S_2$ and S_1,S_2 are compact and disjoint. Since $U+e=[0,2e]$ it follows that $f\in C(S)$ belongs to P if and only if $f(s)\geqq 0$ for all $s\in S_1$ and $f(s)\leqq 0$ for all $s\in S_2$.

By the lemma every functional in N is normal relative to P, and so to prove (i) it will be sufficient to show that each φ which is normal and positive relative to P belongs to N. Define measures φ_1 and φ_2 on S by the relations

$$d\varphi_1 = \chi_1 d\varphi \quad \text{and} \quad d\varphi_2 = \chi_2 d\varphi,$$

where χ_1, χ_2 are the characteristic functions of S_1, S_2 respectively. Then we have $\varphi=\varphi_1+\varphi_2$ and it is clear that φ_1 and $-\varphi_2$ are positive measures on S.

Let $\{f_\alpha\}$ be a norm bounded net in $C(S)$ monotonically increasing in the natural ordering. For each α define $g_\alpha = f_\alpha \chi_1$ and $h_\alpha = f_\alpha \chi_2$. Then evidently $\{g_\alpha - h_\alpha\}$ is norm bounded and monotonic increasing relative to P, and its supremum is $\sup_\alpha\{g_\alpha\} - \sup_\alpha\{h_\alpha\}$. Since φ is positive and normal relative to P we have

$$(1) \qquad \varphi(\sup_\alpha\{g_\alpha\} - \sup_\alpha\{h_\alpha\}) = \sup_\alpha\{\varphi(g_\alpha - h_\alpha)\} = \sup_\alpha\{\varphi_1(f_\alpha)\} + \sup_\alpha\{-\varphi_2(f_\alpha)\}.$$

However, because φ_1 and φ_2 are positive relative to P, we also have

$$\varphi_1(\sup_\alpha\{f_\alpha\}) \geqq \sup_\alpha\{\varphi_1(f_\alpha)\}$$

and

$$(-\varphi_2)(\sup_\alpha\{f_\alpha\}) \geqq \sup_\alpha\{(-\varphi_2)(f_\alpha)\},$$

and these inequalities must in fact be equalities since (1) holds. Therefore φ_1 and $-\varphi_2 \in N$, and this gives $\varphi \in N$.

(ii) We note first that Y is the Banach dual space of $N \times R$ endowed with the norm $\|(.,.)\|_\infty$. Let e_1 be the extreme point $(1,0)$ of the unit ball V of Y, and let P_1 be the cone in Y defined by

$$P_1 = \{\lambda(e_1 + y) : \lambda \geqq 0, y \in V\}.$$

If

$$(f,t) \in (2V - e_1) \cap (2V + e_1),$$

then $\|f \pm 1\| + |t| \leqq 2$ and hence $\|f\| + |t| \leqq 1$. Therefore

$$(2V - e_1) \cap (2V + e_1) = V,$$

and so it follows from the remarks preceding Theorem 2 that the norm in Y is an order unit norm relative to the order unit e_1 and the ordering defined by P_1.

Since all elements of the form $(f,0)$ with $f \geqq 0$ belong to P_1, any positive linear functional in Y must have the form (φ, s) where φ is positive in $C(S)^*$. Now if $\{f_\alpha\}$ is a norm bounded monotonic increasing net in $C(S)$ then $\{(f_\alpha, 0)\}$ is a similar net in Y and therefore, if (φ, s) is a positive normal functional on Y, we have

$$\sup_\alpha\{\varphi(f_\alpha)\} = (\varphi, s)(\sup_\alpha\{(f_\alpha, 0)\}).$$

By the lemma $\{(f_\alpha, 0)\}$ is $\sigma(Y, N \times R)$-convergent to $\sup_\alpha\{(f_\alpha, 0)\}$ and also $\{f_\alpha\}$ is $\sigma(C(S), N)$-convergent to $\sup_\alpha\{f_\alpha\}$. It is clear therefore that

$$\sup_\alpha\{(f_\alpha, 0)\} = (\sup_\alpha\{f_\alpha\}, 0),$$

and so we have

$$\sup_\alpha\{\varphi(f_\alpha)\} = \varphi(\sup_\alpha\{f_\alpha\}).$$

Hence $\varphi \in N$ so that $(\varphi, s) \in N \times R$. Since all functionals in $N \times R$ are normal on Y it follows that $N \times R$ is the space of normal functionals on Y relative to P_1.

Now let

$$P_2 = \{(f,t) \in Y : \|f\| \leqq t\} .$$

Then, as we have noted above, relative to the ordering induced by P_2 the space Y has its norm defined by the order unit $e_2 = (0,1)$. If M denotes the space of functionals on Y which are normal relative to P_2, then we show that M is an order ideal in Y^* with respect to the dual ordering.

If $\eta \in M$ and $\omega \in Y^*$ with $0 \leqq \omega \leqq \eta$, then in order to prove that $\omega \in M$ it will be sufficient to show that $\sup_\alpha \{\omega(y_\alpha)\} = 0$ whenever $\{y_\alpha\}$ is a norm bounded net in Y which, relative to P_2, is monotonic increasing with $\sup_\alpha \{y_\alpha\} = 0$. However in this situation we have $y_\alpha \leqq 0$ for each α, and hence

$$\eta(y_\alpha) \leqq \omega(y_\alpha) \leqq 0 .$$

Since $\sup_\alpha \{\eta(y_\alpha)\} = 0$ it follows that $\sup_\alpha \{\omega(y_\alpha)\} = 0$, and hence M is an order ideal in Y^*.

It is easily verified that the dual cone in Y^* of P_2 is $\{(\varphi, s) : \|\varphi\| \leqq s\}$, and that the element $(0,1) \in M$ is an interior point of this cone. Therefore M is an order ideal in Y^* containing an order unit for Y^*, and so it follows that $M = Y^*$.

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