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RECURRENCE FORMULAE FOR THE COEFFICIENTS OF MODULAR FORMS AND CONGRUENCES FOR THE PARTITION FUNCTION AND FOR THE COEFFICIENTS OF $j(\tau)$, $(j(\tau)-1728)^{\frac{1}{2}}$ AND $(j(\tau))^{\frac{1}{3}}$

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1. Definitions and lemmas.

In this chapter we will give some definitions and also state some well known results that will be used later.

1.1 NOTATIONS AND DEFINITIONS. The function

$$(1.1.1) \qquad \eta(\tau) = \exp(\tfrac{1}{12}\pi i\tau) \prod_{n=1}^{\infty} (1-\exp(2\pi i n\tau)), \qquad \operatorname{Im}\tau > 0 ,$$

is known as Dedekind's η-function. Let

$$(1.1.2) \qquad \eta(\tau)^k = \textstyle\sum_n T_k(n) \exp\big((k,24)\tfrac{1}{12}\pi i n\tau\big) ,$$

$(\cdot,\cdot)$ denoting the greatest common devisor. Here and in the following $\sum_n$ denote $\sum_{n=-\infty}^{+\infty}$. Further, let

$$(1.1.3) \qquad \varphi(x) = \prod_{n=1}^{\infty} (1-x^n) ,$$

$$(1.1.4) \qquad \varphi(x)^k = \textstyle\sum_n p_k(n)x^n, \qquad p(n)=p_{-1}(n) .$$

We make the convention that $p_k(n)=0$ if n is not a non-negativ integer.

Let

$$(1.1.5) \qquad S_{k,\alpha}(\tau) = \eta(q_\alpha\tau)^{-k} \textstyle\sum_n T_k(p^\alpha n) \exp\big((k,24)\tfrac{1}{12}\pi i n\tau\big)$$

where

$$q_\alpha = \begin{cases} p & \text{when } \alpha \text{ is odd ,} \\ 1 & \text{when } \alpha \text{ is even ,} \end{cases}$$

$$(1.1.6) \qquad G_{k,\alpha}(\tau) = S_{k,2\alpha}(\tau) + p^{-1} S_{k,2\alpha-1}(-1/p\tau) .$$

In the following p and q always denote primes unless otherwise stated.

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We use the notations

$$l_p = \begin{cases} 3 & \text{when } p=2\,, \\ 8 & \text{when } p=3\,, \\ 24 & \text{when } p>3\,, \end{cases} \tag{1.1.7}$$

$$g_{p,\nu,\alpha}(\tau) = \eta\big((\tau+l_p\nu)/p^{\alpha}\big)\,, \tag{1.1.8}$$

$[x]$, the greatest integer $\leqq x$,

$$\delta(x) = \begin{cases} 1 & \text{when } x \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$\delta(m/n) = \begin{cases} 1 & \text{when } m\,|\,n\,, \\ 0 & \text{when } m \nmid n\,. \end{cases}$$

Γ shall denote the full modular group, i.e. the group of transformations $(a\tau+b)/(c\tau+d)$ where $ad-bc=1$, and a, b, c, d are integers. Further, let $\Gamma_0(p)$ denote the subgroup of Γ defined by $c\equiv 0 \pmod p$.

1.2 Lemmas. If $f(\tau)$ is a function on $\Gamma_0(p)$, then

$$f(-1/p\tau) + \sum_{\nu=0}^{p-1} f\big((\tau+\nu)/p\big)$$

is a function on Γ. In fact, this is a special case of Theorem 2.2 in [7]. Now, if $24\,|\,kl_p$ then $S_{k,\alpha}(\tau)$ is a function on $\Gamma_0(p)$; this is shown by Newman [8] [10] for $p>2$, and it is easily seen that the same method can be applied for $p=2$. Further, since $(p,l_p)=1$, we have

$$\sum_{\nu=0}^{p-1} S_{k,2\alpha-1}\big((\tau+\nu)/p\big) = \sum_{\nu=0}^{p-1} S_{k,2\alpha-1}\big((\tau+l_p\nu)/p\big)$$

$$= \sum_{\nu=0}^{p-1} \eta(\tau+l_p\nu)^{-k} \sum_{n} T_k(p^{2\alpha-1}n)\exp\big((k,24)\pi i n(\tau+l_p\nu)/12p\big)\,.$$

The condition $24\,|\,kl_p$ implies that $\eta(\tau+l_p\nu)^{-k}=\eta(\tau)^{-k}$, cf. (1.1.1); and changing the order of summation we obtain

$$\sum_{\nu=0}^{p-1} S_{k,2\alpha-1}\big((\tau+\nu)/p\big) = p\,S_{k,2\alpha}(\tau)\,.$$

Hence we have

Lemma 1. *If* $24\,|\,kl_p$, *then* $G_{k,\alpha}(\tau)$ *is a function on the full modular group.*

In the following we shall always assume that $24\,|\,kl_p$.

From the definitions in 1.1 we get

$$\sum_{\nu=0}^{p^\alpha-1} g_{p,\nu,\alpha}(\tau)^k = \sum_{\nu=0}^{p^\alpha-1} \eta((\tau+l_p\nu)/p^\alpha)^k$$

$$= \sum_{\nu=0}^{p^\alpha-1} \sum_n T_k(n) \exp\big(((k,24)\pi i n/12)(\tau+l_p\nu)/p^\alpha\big)$$

$$= \sum_n T_k(n) \exp((k,24)\pi i n\tau/12p^\alpha) \sum_{\nu=0}^{p^\alpha-1} \exp\big(2\pi i((k,24)l_p/24)n\nu/p^\alpha\big) .$$

Since $24 \mid (k,24)l_p$ and $p \nmid (k,24)l_p/24$ we get

$$\sum_{\nu=0}^{p^\alpha-1} \exp\big(2\pi i((k,24)l_p/24)n\nu/p^\alpha\big) = \begin{cases} p^\alpha & \text{when } p^\alpha \mid n, \\ 0 & \text{otherwise} . \end{cases}$$

Hence

$$\sum_{\nu=0}^{p^\alpha-1} g_{p,\nu,\alpha}(\tau)^k = p^\alpha \sum_n T_k(p^\alpha n) \exp\big((k,24)\pi i n\tau/12\big) . \tag{1.2.1}$$

Using (1.1.5) we obtain

LEMMA 2. *We have*

$$S_{k,\alpha}(\tau) = \eta(q_\alpha\tau)^{-k}\, p^{-\alpha} \sum_{\nu=0}^{p^\alpha-1} g_{p,\nu,\alpha}(\tau)^k .$$

For $(a\tau+b)/(c\tau+d) \in \Gamma$, the function $\eta(\tau)$ has the transformation equations ([14] p. 100)

(1.2.2)

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \left(\frac{d}{c}\right)(-i(c\tau+d))^{\frac{1}{2}}\eta(\tau)\exp\big(\pi i(c(a+d)+bd(1-c^2)-3(c-1))/12\big)$$

where $\mathrm{Re}(-i(c\tau+d))^{\frac{1}{2}} > 0$ and (d/c) is the Jacobi symbol, when c is odd, $c > 0$ and $d \neq 0$;

(1.2.3)

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^2 = (c\tau+d)\eta(\tau)^2 \exp\big(\pi i(d(b-c)+ac(1-d^2)+3(d-1))/6\big) ,$$

when d is odd;

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}}\eta(\tau) . \tag{1.2.4}$$

Using (1.2.2)–(1.2.4) we can find transformation equations for $g_{p,\nu,\alpha}(\tau)^k$ as follows: We have

$$g_{p,\nu,\alpha}(-1/\tau) = \eta\left(\frac{-1/\tau+l_p\nu}{p^\alpha}\right) = \eta\left(\frac{A((\tau+l_p\mu)/p^\alpha)+B}{C((\tau+l_p\mu)/p^\alpha)+D}\right),$$

where

$$A = l_p\nu, \quad B = -(1+l_p{}^2\mu\nu)/p^\alpha, \quad C = p^\alpha, \quad D = -l_p\mu .$$

Here $AD-BC=1$, and A, C and D are integers. If $(\nu,p)=1$ we can solve the congruence

(1.2.5) $$l_p{}^2\mu\nu \equiv -1 \pmod{p^\alpha}, \quad 0<\mu<p^\alpha .$$

With this value of μ, B also becomes an integer, and (1.2.2)–(1.2.4) can be used. When $p \geqq 3$ we have

$$\begin{aligned} g_{p,\nu,\alpha}(-1/\tau)^k &= \left(\frac{D}{C}\right)^k \Big(-1\big(C((\tau+l_p\mu)/p^\alpha)+D\big)\Big)^{\frac{1}{2}k}\eta((\tau+l_p\mu)/p^\alpha)^k\cdot \\ &\qquad \cdot\exp\big(\pi ik(C(A+D)+BD(1-C^2)-3C+3)/12\big) \\ &= \left(\frac{l_p\mu}{p^\alpha}\right)^k i^{\frac{1}{2}k(p^\alpha-1)}(-i\tau)^{\frac{1}{2}k}\, g_{p,\mu,\alpha}(\tau)^k . \end{aligned}$$

When $p=2$ we have $8|k$. Hence

$$\begin{aligned} g_{2,\nu,\alpha}(-1/\tau)^k &= \big(C((\tau+l_2\mu)/2^\alpha)+D\big)^{\frac{1}{2}k}\eta((\tau+l_2\mu)/2^\alpha)^k\cdot \\ &\qquad \cdot\exp\big(\pi ik(D(B-C)+AC(1-D^2)+3(D-1))/12\big) \\ &= \tau^{\frac{1}{2}k}g_{2,\mu,\alpha}(\tau)^k \\ &= \left(\frac{l_2\mu}{2^\alpha}\right)^k i^{\frac{1}{2}k(2^\alpha-1)}(-i\tau)^{\frac{1}{2}k}\, g_{2,\mu,\alpha}(\tau)^k , \end{aligned}$$

if we put

$$\left(\frac{a}{2^\alpha}\right)^2 = \begin{cases} 1 & \text{when } a \text{ is odd}, \\ 0 & \text{when } a \text{ is even}. \end{cases}$$

LEMMA 3. *If* $(\nu,p)=1$, *we have*

$$g_{p,\nu,\alpha}(-1/p^r\tau)^k = \left(\frac{l_p\mu}{p^\alpha}\right)^k i^{\frac{1}{2}k(p^\alpha-1)}(-i\tau)^{\frac{1}{2}k}\, p^{\frac{1}{2}kr}\, g_{p,\mu,\alpha}(p^r\tau)^k ,$$

where $l_p{}^2\mu\nu \equiv -1 \pmod{p^\alpha}$, $0<\mu<p^\alpha$.

We shall also need a connection between $T_k(n)$ and $p_k(n)$. Putting $x=\exp(2\pi i\tau)$ in (1.1.2) we get

$$x^{k/24}\varphi(x)^k = \textstyle\sum_n T_k(n)\, x^{(k,24)n/24} ,$$

and hence

LEMMA 4. *We have*

$$T_k(n) = p_k\big(((k,24)n-k)/24\big); \quad p_k(m) = T_k((24m+k)/(k,24)) .$$

Further we shall need the following well-known result,

LEMMA 5. *When* $p \geqq 3$ *we have*

$$\sum_{\substack{1\leqq m\leqq p^k-1\\(m,p)=1}} \left(\frac{m}{p}\right) \exp(2\pi i m/p^k) = \begin{cases} i^{\frac14(p-1)}(-1)^{[\frac14 p]}p^{\frac12} & \text{if } k=1 ,\\ 0 & \text{if } k>1 .\end{cases}$$

2. Recurrence formulae for $T_k(n)$.

The only singularity for $G_{k,\alpha}(\tau)$ is at $\tau = i\infty$. In this chapter we will discuss the behaviour of $G_{k,\alpha}(\tau)$ at $\tau = i\infty$ and use the result to find a class of recurrence formulae for $T_k(n)$.

2.1. FURTHER LEMMAS. Let $(\nu,p^\alpha)=p^t$ and $\nu=\nu_t p^t$ so that $(\nu_t,p)=1$, then

$$g_{p,\nu,\alpha}(\tau) = \eta((\tau+l_p\nu)/p^\alpha) = \eta\big(((\tau/p^t)+l_p\nu_t)/p^{\alpha-t}\big) = g_{p,\nu_t,\alpha-t}(\tau/p^t) .$$

Using lemma 3 we get

$$\begin{aligned} g_{p,\nu,\alpha}(-1/\tau)^k &= g_{p,\nu_t,\alpha-t}(-1/p^t\tau)^k \\ &= \left(\frac{l_p\mu_t}{p^{\alpha-t}}\right)^k i^{\frac12 k(p^{\alpha-t}-1)}(-i\tau)^{\frac12 k} p^{\frac12 kt} g_{p,\mu_t,\alpha-t}(p^t\tau)^k . \end{aligned}$$

Hence we obtain

LEMMA 6. *When* $(\nu,p^\alpha)=p^t$ *and* $\nu_t=\nu/p^t$, *we have*

$$g_{p,\nu,\alpha}(-1/p\tau)^k = \left(\frac{l_p\mu_t}{p^{\alpha-t}}\right)^k i^{\frac12 k(p^{\alpha-t}-1)} p^{\frac12 k(t+1)}(-i\tau)^{\frac12 k} g_{p,\mu_t,\alpha-t}(p^{t+1}\tau)^k ,$$

where $l_p^2\mu_t\nu_t \equiv -1 \pmod{p^{\alpha-t}}$, $0<\mu_t<p^{\alpha-t}$.

Using (1.2.1) we get

$$\begin{aligned} \sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}} g_{p,\mu,\alpha}(\tau)^k &= \sum_{\mu=0}^{p^\alpha-1} g_{p,\mu,\alpha,}(\tau)^k - \sum_{\mu=0}^{p^{\alpha-1}-1} g_{p,\mu,\alpha-1}(\tau/p)^k \\ &= p^\alpha \sum_{n_1} T_k(p^\alpha n_1)\exp((k,24)\pi i n_1\tau/12) - \\ &\quad - p^{\alpha-1}\sum_{n_2} T_k(p^{\alpha-1}n_2)\exp((k,24)\pi i n_2\tau/12p) . \end{aligned}$$

Hence

$$\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}} g_{p,\nu,\alpha}(p^\beta\tau)^k = p^\alpha\sum_{n_1} T_k(p^\alpha n_1)\exp\big((k,24)\pi i p^\beta n_1\tau/12\big)-$$

$$-p^{\alpha-1}\sum_{n_2} T_k(p^{\alpha-1}n_2)\exp\big((k,24)\pi i p^{\beta-1}n_2\tau/12\big)\,.$$

Putting $n=p^\beta n_1$ we get the sum

$$\sum_{\substack{n\\p^\beta|n}}(\ldots)$$

which may be written $\sum_n\delta(n/p^\beta)(\ldots)$. Also putting $n=p^{\beta-1}n_2$ we obtain

LEMMA 7. *We have*

$$\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}} g_{p,\mu,\alpha}(p^\beta\tau)^k$$

$$=\sum_n\{p^\alpha\delta(n/p^\beta)-p^{\alpha-1}\delta(n/p^{\beta-1})\}\,T_k(p^{\alpha-\beta}n)\exp\big((k,24)\pi i n\tau/12\big)\,.$$

Let $p\geqq 3$. Then

$$\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}}\left(\frac{\mu}{p}\right)g_{p,\mu,\alpha}(\tau)^k=\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}}\left(\frac{\mu}{p}\right)\sum_n T_k(n)\exp\Big(\big((k,24)\pi i n/12\big)(\tau+l_p\mu)/p^\alpha\Big)$$

$$=\sum_n T_k(n)\exp\big((k,24)\pi i n\tau/12p^\alpha\big)\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}}\left(\frac{\mu}{p}\right)\exp\big((k,24)\pi i n l_p\mu/12p^\alpha\big)\,.$$

Let $n=n_t p^t$, where $(n,p^\alpha)=p^t$, and let $m=(k,24)l_p n_t\mu/24$. Then by lemma 5,

$$\sum_{\substack{(\mu,p)=1\\1\leqq\mu\leqq p^\alpha-1}}\left(\frac{\mu}{p}\right)\exp\Big(2\pi i\big(l_p(k,24)/24\big)n_t\mu/p^{\alpha-t}\Big)$$

$$=\left(\frac{n_t l_p(k,24)/24}{p}\right)\sum_{\substack{(m,p)=1\\1\leqq m\leqq p^\alpha-1}}\left(\frac{m}{p}\right)\exp\big(2\pi i m/p^{\alpha-t}\big)$$

$$=\begin{cases}\left(\dfrac{l_p(k,24)/24}{p}\right)\left(\dfrac{n_{\alpha-1}}{p}\right)p^{\alpha-1}\,i^{\frac14(p-1)}\,(-1)^{[\frac14 p]}p^{\frac12} & \text{if } t=\alpha-1\,,\\[2ex] 0 & \text{if } t=\alpha \text{ or } t<\alpha-1\,.\end{cases}$$

Hence

$$\sum_{\substack{(\mu,p)=1\\ 1\leqq\mu\leqq p^{\alpha}-1}} \left(\frac{\mu}{p}\right) g_{p,\mu,\alpha}(\tau)^k$$

$$= \sum_{\substack{n\\ (n,p)=1}} \left(\frac{l_p(k,24)/24}{p}\right)\left(\frac{n}{p}\right) i^{\frac{1}{4}(p-1)}(-1)^{[\frac{1}{4}p]}\cdot$$

$$\cdot p^{\alpha-\frac{1}{2}}\, T_k(p^{\alpha-1}n)\exp\bigl((k,24)\pi i n\tau/12p\bigr)\,.$$

Further

$$\sum_{\substack{(\mu,p)=1\\ 1\leqq\mu\leqq p^{\alpha}-1}} \left(\frac{\mu}{p}\right) g_{p,\mu,\alpha}(p^{\beta}\tau)^k = \left(\frac{l_p(k,24)/24}{p}\right) i^{\frac{1}{4}(p-1)}(-1)^{[\frac{1}{4}p]}\, p^{\alpha-\frac{1}{2}}\cdot$$

$$\cdot \sum_{\substack{n\\ (n,p)=1}} \left(\frac{n}{p}\right) T_k(p^{\alpha-1}n)\exp\bigl((k,24)\pi i n p^{\beta-1}\tau/12\bigr)\,.$$

Replacing $np^{\beta-1}$ by n we obtain

LEMMA 8. *When* $p\geqq 3$ *we have*

$$\sum_{\substack{(\mu,p)=1\\ 1\leqq\mu\leqq p^{\alpha}-1}} \left(\frac{l_p\mu}{p}\right) g_{p,\mu,\alpha}(p^{\beta}\tau)^k$$

$$= \left(\frac{l_p(k,24)/24}{p}\right) i^{\frac{1}{4}(p-1)}(-1)^{[\frac{1}{4}p]}\, p^{\alpha-\frac{1}{2}} \sum\nolimits_n \left(\frac{\delta(n/p^{\beta-1})\, n/p^{\beta-1}}{p}\right) T_k(p^{\alpha-\beta}n)\cdot$$

$$\cdot\exp\bigl((k,24)\pi i n\tau/12\bigr)\,.$$

2.2 DISCUSSION OF $G_{k,\alpha}(\tau)$ AT $\tau=i\infty$. From lemma 4 we easily find

$$(2.2.1)\qquad \begin{cases} T_k(p^{2\alpha}n) = p_k\bigl(p^{2\alpha}m+k(p^{2\alpha}-1)/24\bigr) & \text{if } n=(24m+k)/(k,24)\\ T_k(p^{2\alpha}n) = 0 & \text{otherwise}\,. \end{cases}$$

Let $x=\exp(2\pi i\tau)$. Then we get

THEOREM 1. *When* $24\,|\,kl_p$ *we have*

$$S_{k,2\alpha}(\tau) = \varphi(x)^{-k}\sum\nolimits_m p_k\bigl(p^{2\alpha}m+k(p^{2\alpha}-1)/24\bigr)\, x^m\,.$$

We shall also need an expression for $S_{k,2\alpha-1}(-1/p\tau)$. Lemmas 2 and 6 yield

$$p^{-1} S_{k,2\alpha-1}(-1/p\tau) = \eta(-1/\tau)^k\, p^{-2\alpha} \sum_{\nu=0}^{p^{2\alpha}-1} g_{p,\nu,2\alpha-1}(-1/p\tau)^k$$

$$= (-i\tau)^{-\frac{1}{2}k}\, \eta(\tau)^{-k}\, p^{-2\alpha} \left\{ \eta(-1/p^{2\alpha}\tau)^k + \sum_{t=0}^{2\alpha-2} \sum_{\substack{(\nu,p^{2\alpha-1})=p^t \\ 1 \leqq \nu \leqq p^{2\alpha-1}-1}} g_{p,\nu,2\alpha-1}(-1/p\tau)^k \right\}$$

$$= (-i\tau)^{-\frac{1}{2}k}\, \eta(\tau)^{-k}\, p^{-2\alpha} \Bigg\{ p^{k\alpha}(-i\tau)^{\frac{1}{2}k}\, \eta(p^{2\alpha}\tau)^k +$$

$$+ \sum_{t=0}^{2\alpha-2} \sum_{\mu_t} \left(\frac{l_p \mu_t}{p^{2\alpha-1-t}}\right)^k i^{\frac{1}{2}k(p^{2\alpha-1-t}-1)} p^{\frac{1}{2}(t+1)k} (-i\tau)^{\frac{1}{2}k}\, g_{p,\mu_t,2\alpha-1-t}(p^{t+1}\tau)^k \Bigg\}.$$

The summation conditions on $\nu_t = \nu/p^t$ are $1 \leqq \nu_t \leqq p^{2\alpha-1-t} - 1$ and $(\nu_t, p) = 1$. Since $l_p{}^2 \mu_t \nu_t \equiv -1 \pmod{p^{2\alpha-1-t}}$ and $0 < \mu_t < p^{2\alpha-1-t}$ the conditions on μ_t may also be written $1 \leqq \mu_t \leqq p^{2\alpha-1-t} - 1$ and $(\mu_t, p) = 1$. Letting $\mu_t = \mu$ and $s = 2\alpha - 1 - t$ we obtain

$$p^{-1} S_{k,2\alpha-1}(-1/p\tau) = p^{k\alpha-2\alpha}\, \eta(\tau)^{-k}\, \eta(p^{2\alpha}\tau)^k +$$

$$+ \eta(\tau)^{-k} \sum_{s=1}^{2\alpha-1} p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}\, i^{\frac{1}{2}k(p^s-1)} \sum_{\substack{(\mu,p)=1 \\ 1 \leqq \mu \leqq p^s-1}} \left(\frac{l_p \mu}{p^s}\right)^k g_{p,\mu,s}(p^{2\alpha-s}\tau)^k .$$

In the following we treat the cases k even and k odd separately.

k even. Then $(l_p\mu/p^s)^k = 1$ and hence, by lemma 7,

$$(2.2.2) \quad p^{-1} S_{k,2\alpha-1}(-1/p\tau) = p^{(k-2)\alpha}\, \eta(\tau)^{-k} \eta(p^{2\alpha}\tau)^k +$$

$$+ \eta(\tau)^{-k} \sum_{s=1}^{2\alpha-1} p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}\, i^{\frac{1}{2}k(p^s-1)} \cdot$$

$$\cdot \sum\nolimits_n \{p^s \delta(n/p^{2\alpha-s}) - p^{s-1}\delta(n/p^{2\alpha-s-1})\}\, T_k(p^{2s-2\alpha}n) \exp\big((k,24)\pi i n\tau/12\big).$$

Using (2.2.1) we obtain

THEOREM 2. *When $24 \mid kl_p$ and k is even, we have*

$$p^{-1} S_{k,2\alpha-1}(-1/p\tau) = p^{(k-2)\alpha}\, x^{k(p^{2\alpha}-1)/24}\, \varphi(x)^{-k}\, \varphi(x^{p^{2\alpha}})^k +$$

$$+ \varphi(x)^{-k} \sum_{s=1}^{2\alpha-1} p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}\, i^{\frac{1}{2}k(p^s-1)} \cdot$$

$$\cdot \sum\nolimits_m \{p^s \delta\big((24m+k)/(k,24)p^{2\alpha-s}\big) - p^{s-1}\delta\big((24m+k)/(k,24)p^{2\alpha-s-1}\big)\} \cdot$$

$$\cdot\, p_k\big(p^{2s-2\alpha}m + k(p^{2s-2\alpha}-1)/24\big)\, x^m .$$

k odd. Then $(l_p\mu/p^s)^k = (l_p\mu/p)^s$. Hence, putting $s = 2u+1$ and $s = 2u$, we obtain

$$p^{-1}S_{k,2\alpha-1}(-1/p\tau) = p^{(k-2)\alpha}\eta(\tau)^{-k}\eta(p^{2\alpha}\tau)^k +$$
$$+\eta(\tau)^{-k}\sum_{u=0}^{\alpha-1} p^{(\frac{1}{2}(2\alpha-2u-1)k)-2\alpha}\, i^{\frac{1}{2}k(p^{2u+1}-1)}\cdot$$
$$\cdot \sum_{\substack{(\mu,p)=1\\ 1\leq\mu\leq p^{2u+1}-1}} \left(\frac{l_p\mu}{p}\right) g_{p,\mu,2u+1}(p^{2\alpha-2u-1}\tau)^k +$$
$$+\eta(\tau)^{-k}\sum_{u=1}^{\alpha-1} p^{(\alpha-u)k-2\alpha}\, i^{\frac{1}{2}k(p^{2u}-1)} \sum_{\substack{(\mu,p)=1\\ 1\leq\mu\leq p^{2u}-1}} g_{p,\mu,2u}(p^{2\alpha-2u}\tau)^k .$$

When k is odd we have $p\geqq 3$, and hence $p^2\equiv 1 \pmod 8$ so that $i^{\frac{1}{2}(p^{2u+1}-1)} = i^{\frac{1}{2}(p-1)}$ and $i^{\frac{1}{2}(p^{2u}-1)}=1$. This, together with lemmas 7 and 8, yields

$$(2.2.3)\quad p^{-1}S_{k,2\alpha-1}(-1/p\tau) = p^{(k-2)\alpha}\,\eta(\tau)^{-k}\,\eta(p^{2\alpha}\tau)^k +$$
$$+\eta(\tau)^{-k}\sum_{u=0}^{\alpha-1} p^{(\frac{1}{2}(2\alpha-2u-1)k)-2\alpha}\, i^{\frac{1}{2}k(p-1)} \left(\frac{l_p{}^2(k,24)/24}{p}\right) i^{\frac{1}{2}(p-1)}(-1)^{[\frac{1}{4}p]}p^{2u+\frac{1}{2}}\cdot$$
$$\cdot\Sigma_n \left(\frac{\delta(n/p^{2\alpha-2u-2})\; n/p^{2\alpha-2u-2}}{p}\right) T_k(p^{4u-2\alpha+2}n)\exp\big((k,24)\pi i n\tau/12\big) +$$
$$+\eta(\tau)^{-k}\sum_{u=1}^{\alpha-1} p^{(\alpha-u)k-2\alpha}\cdot\Sigma_n\{p^{2u}\delta(n/p^{2\alpha-2u}) - p^{2u-1}\delta(n/p^{2\alpha-2u-1})\}\cdot$$
$$\cdot T_k(p^{4u-2\alpha}n)\exp\big((k,24)\pi i n\tau/12\big).$$

Using (2.2.1) we obtain

THEOREM 3. *When* $24\,|\,kl_p$ *and* k *is odd, we have*

$$p^{-1}S_{k,2\alpha-1}(-1/p\tau) = p^{(k-2)\alpha}\, x^{k(p^{2\alpha}-1)/24}\,\varphi(x)^{-k}\,\varphi(x^{p^{2\alpha}})^k +$$
$$+\ \varphi(x)^{-k}\sum_{u=0}^{\alpha-1} p^{(\frac{1}{2}(2\alpha-2u-1)k)-2\alpha+2u+\frac{1}{2}}\, i^{\frac{1}{2}(p-1)(k+1)}(-1)^{[\frac{1}{4}p]}\left(\frac{l_p{}^2(k,24)/24}{p}\right)\cdot$$
$$\cdot\Sigma_m\left(\frac{\delta\big((24m+k)/(k,24)p^{2\alpha-2u-2}\big)(24m+k)/(k,24)p^{2\alpha-2u-2}}{p}\right)\cdot$$
$$\cdot p_k\big(p^{4u-2\alpha+2}m + k(p^{4u-2\alpha+2}-1)/24\big)x^m +$$
$$+\varphi(x)^{-k}\sum_{u=1}^{\alpha-1} p^{(\alpha-u)k-2\alpha}\cdot\Sigma_m\{p^{2u}\delta\big((24m+k)/(k,24)p^{2\alpha-2u}\big) -$$
$$-p^{2u-1}\delta\big((24m+k)/(k,24)p^{2\alpha-2u-1}\big)\}\cdot$$
$$\cdot p_k\big(p^{4u-2\alpha}m + k(p^{4u-2\alpha}-1)/24\big)\,x^m .$$

The power series for $S_{k,2\alpha}(\tau)$ in theorem 1 shows that the coefficients are zero when $p^{2\alpha}m+k(p^{2\alpha}-1)/24<0$. Hence we get a contribution only when $m \geqq -k(1-1/p^2)/24$ so that $S_{k,2\alpha}(\tau)$ has a pole of order $<k/24$ or is regular. From theorems 2 and 3 we conclude that $S_{k,2\alpha-1}(-1/p\tau)$ also has a pole of order $<k/24$ or is regular. Using (1.1.6) and observing that $G_{k,\alpha}(\tau)$ is regular in the inner of the upper half plane, we obtain

THEOREM 4. *At $\tau=i\infty$ the function $G_{k,\alpha}(\tau)$ is regular or have a pole of order $\leqq[(k-1)/24]$. Further $G_{k,\alpha}(\tau)$ is regular in the interior of the upper half plane.*

2.3 ALGEBRAIC EQUATIONS. We put

(2.3.1) $$a = a_k = [(k-1)/24]\,.$$

Let

$$G_{k,\alpha}(\tau) = \sum_{r=-a}^{\infty} b_{\alpha,r}x^r, \qquad A_\alpha = \{b_{\alpha,-a},\ldots,b_{\alpha,-1}\}\,,$$

so that A_α is a vector of dimension a associated with $G_{k,\alpha}(\tau)$. Choosing $a+1$ such vectors A_{α_j}, $j=1,2,\ldots,a+1$, they must be linearly dependant. Hence there exists an equation

$$\sum_{i=1}^{a+1} \Delta_{\alpha_j}A_{\alpha_j} = 0$$

where not all Δ_{α_j} are zero. Since $b_{\alpha,r}$ is rational, we may assume

(2.3.2) $$(\Delta_{\alpha_1},\ldots,\Delta_{\alpha_{a+1}}) = 1, \quad \Delta_{\alpha_j}\text{ integral}\,.$$

The function

$$\sum_{j=1}^{a+1} \Delta_{\alpha_j}G_{k,\alpha_j}(\tau)$$

is regular at $\tau=i\infty$. Hence it is a modular function on Γ which is regular in the fundamental domain and thus reduces to a constant,

(2.3.3) $$\sum_{j=1}^{a+1} \Delta_{\alpha_j}G_{k,\alpha_j}(\tau) = \sum_{j=1}^{a+1} \Delta_{\alpha_j}b_{\alpha_j,0}\,.$$

Letting $\alpha_j=j$ and changing variable we obtain

THEOREM 5. *There exist constants Δ_α, not all zero, such that*

$$\sum_{\alpha=1}^{a+1} \Delta_\alpha G_{k,\alpha}(\tau) = \Delta\,,$$

$(\Delta_1,\Delta_2,\ldots,\Delta_{a+1})=1$, Δ_α *integral.*

2.4 Recurrence formulae for $T_k(n)$. We have

$$\begin{aligned}\eta(p^{2\alpha}\tau)^k &= \textstyle\sum_n T_k(n)\exp\left((k,24)\pi i n p^{2\alpha}\tau/12\right)\\ &= \textstyle\sum_n T_k(n/p^{2\alpha})\exp\left((k,24)\pi i n\tau/12\right).\end{aligned}$$

From (2.2.2) we obtain, when k is even,

$$\begin{aligned}\eta(\tau)^k G_{k,\alpha}(\tau) &= \textstyle\sum_n T_k(p^{2\alpha}n)\exp\left((k,24)\pi i n\tau/12\right)+\\ &\quad + p^{(k-2)\alpha}\textstyle\sum_n T_k(n/p^{2\alpha})\exp\left((k,24)\pi i n\tau/12\right)+\\ &\quad + \sum_{s=1}^{2\alpha-1} p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}\, i^{\frac{1}{2}k(p^s-1)}\textstyle\sum_n \{p^s\delta(n/p^{2\alpha-s})-p^{s-1}\delta(n/p^{2\alpha-s-1})\}\cdot\\ &\quad \cdot T_k(p^{2s-2\alpha}n)\exp\left((k,24)\pi i n\tau/12\right).\end{aligned}$$

This, together with theorem 5, yields

Theorem 6. *When $24\,|\,kl_p$ and k is even, there exist integers Δ_α, not all zero, such that $(\Delta_1,\Delta_2,\ldots,\Delta_{a+1})=1$, where $a=[(k-1)/24]$, and*

$$\begin{aligned}&\sum_{\alpha=1}^{a+1}\Delta_\alpha\Big\{T_k(p^{2\alpha}n)+p^{(k-2)\alpha}T_k(n/p^{2\alpha})+\\ &+\sum_{s=1}^{2\alpha-1}p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}\,(-1)^{\frac{1}{4}k(p^s-1)}\{p^s\delta(n/p^{2\alpha-s})-p^{s-1}\delta(n/p^{2\alpha-s-1})\}\,T_k(p^{2s-2\alpha}n)\Big\}\\ &=\Delta T_k(n).\end{aligned}$$

Replacing n by $p^{2a+1}n$, and noticing that $2\alpha-s-1<2\alpha-s\leqq 2\alpha-1\leqq 2a+1$, we obtain

Corollary 1. *We have*

$$\begin{aligned}&\sum_{\alpha=1}^{a+1}\Delta_\alpha\Big\{T_k(p^{2a+2\alpha+1}n)+p^{(k-2)\alpha}T_k(p^{2a-2\alpha+1}n)+\\ &+\sum_{s=1}^{2\alpha-1}p^{(\frac{1}{2}(2\alpha-s)k)-2\alpha}(-1)^{\frac{1}{4}k(p^s-1)}(p^s-p^{s-1})T_k(p^{2a+2s-2\alpha+1}n)\Big\} = \Delta T_k(p^{2a+1}n).\end{aligned}$$

Observing that $2\alpha-s\geqq 1$ and that $2\alpha-s-1>0$ unless $s=2\alpha-1$, we obtain

Corollary 2. *When $(n,p)=1$ we have*

$$\sum_{\alpha=1}^{a+1}\Delta_\alpha\{T_k(p^{2\alpha}n)-p^{\frac{1}{2}k-2}(-1)^{\frac{1}{4}k(p-1)}T_k(p^{2\alpha-2}n)\} = \Delta T_k(n).$$

When k is odd, we use (2.2.3) and obtain

Theorem 7. *When* $24 \mid kl_p$ *and* k *is odd, there exist integers* Δ_α, *not all zero, such that* $(\Delta_1, \Delta_2, \ldots, \Delta_{a+1}) = 1$, *where* $a = [(k-1)/24]$, *and*

$$\sum_{\alpha=1}^{a+1} \Delta_\alpha \Big\{ T_k(p^{2\alpha} n) + p^{(k-2)\alpha} T_k(n/p^{2\alpha}) + \\ + \sum_{u=0}^{\alpha-1} p^{(\frac{1}{2}(2\alpha-2u-1)k)-2\alpha+2u+\frac{1}{2}} (-1)^{(\frac{1}{4}(p-1)(k+1))+[\frac{1}{4}p]} \left(\frac{l_p{}^2 (k,24)/24}{p} \right) \cdot \\ \cdot \left(\frac{\delta(n/p^{2\alpha-2u-2}) n/p^{2\alpha-2u-2}}{p} \right) T_k(p^{4u-2\alpha+2} n) + \\ + \sum_{u=1}^{\alpha-1} p^{(\alpha-u)k-2\alpha} \{ p^{2u} \delta(n/p^{2\alpha-2u}) - p^{2u-1} \delta(n/p^{2\alpha-2u-1}) \} \, T_k(p^{4u-2\alpha} n) \Big\} \\ = \Delta T_k(n) .$$

Corollary 1. *We have*

$$\sum_{\alpha=1}^{a+1} \Delta_\alpha \Big\{ T_k(p^{2a+2\alpha+1} n) + p^{(k-2)\alpha} T_k(p^{2a-2\alpha+1} n) + \\ + \sum_{u=1}^{\alpha-1} p^{(\alpha-u)k-2\alpha} (p^{2u} - p^{2u-1}) T_k(p^{2a+4u-2\alpha+1} n) \Big\} = \Delta T_k(p^{2a+1} n) .$$

Corollary 2. *When* $(n,p) = 1$, *we have*

$$\sum_{\alpha=1}^{a+1} \Delta_\alpha \left\{ T_k(p^{2\alpha} n) + p^{\frac{1}{2}(k+1)-2} (-1)^{(\frac{1}{4}(k+1)(p-1))+[\frac{1}{4}p]} \left(\frac{l_p{}^2 (k,24)/24}{p} \right) \left(\frac{n}{p} \right) T_k(p^{2\alpha-2} n) \right\} \\ = \Delta T_k(n) .$$

From the corollaries 1 of theorems 6 and 7 we obtain

Theorem 8. *When* $24 \mid kl_p$ *there exist integers* b_β, *not all zero, such that*

$$\sum_{\beta=-(a+1)}^{a+1} b_\beta T_k(p^{2a+2\beta+1} n) = 0, \quad a = [(k-1)/24] ,$$

where b_β *is dependent of* k *and* p, *but independent of* n.

3. Some results on $p(n)$ and $p_k(n)$.

3.1 Congruences for $p(n)$ modulo 7. All congruences in 3.1 are modulo 7 unless otherwise stated.

Kolberg [4] has shown

$$p(7n+s) \equiv p_{23}(7n+s-1) + 2p_{47}(7n+s-2) \tag{3.1.1}$$

when $s=1,\ 3,\ 4$. Let $q>3$ and $q\neq 7$. Let

(3.1.2) $$c_n^m = q^{2n}m-(q^{2n}-1)/24 .$$

Then

(3.1.3) $$c_{n+n_1}^m = q^{2n_1}c_n^m - (q^{2n_1}-1)/24 .$$

When m runs through the residues 1, 3, 4 modulo 7, then c_n^m runs through the same set of residues. Hence

(3.1.4) $$p(c_n^m) \equiv p_{23}(c_n^m-1)+2p_{47}(c_n^m-2), \qquad \text{when } m\equiv 1,3,4 .$$

From theorem 7 we obtain

$$\Delta_1\left\{T_{47}(q^2n)+q^{45}T_{47}(n/q^2)+q^{22}(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{n}{q}\right)T_{47}(n)\right\}+$$

$$+\Delta_2\left\{T_{47}(q^4n)+q^{90}T_{47}(n/q^4)+q^{67}(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{\delta(n/q^2)n/q^2}{q}\right)T_{47}(n/q^2)+\right.$$

$$\left.+q^{22}(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{n}{q}\right)T_{47}(q^2n)+q^{43}\big(q^2\delta(n/q^2)-q\delta(n/q)\big)T_{47}(n)\right\}$$

$$= \Delta T_{47}(n) .$$

We put $n=24m-1$ and notice that

$$T_{47}(q^{2\alpha}(24m-1)) = p_{47}(q^{2\alpha}m-(q^{2\alpha}-1)/24-2) = p_{47}(c_\alpha^m-2) ,$$

and that $q^6\equiv 1$. Hence

(3.1.5)

$$\Delta_1\left\{p_{47}(c_1^m-2)+q^3p_{47}(c_{-1}^m-2)+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p_{47}(c_0^m-2)\right\}+$$

$$+\Delta_2\left\{p_{47}(c_2^m-2)+p_{47}(c_{-2}^m-2)+q(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\cdot\right.$$

$$\cdot\left(\frac{\delta((24m-1)/q^2)(24m-1)/q^2}{q}\right)p_{47}(c_{-1}^m-2)+$$

$$+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p_{47}(c_1^m-2)+$$

$$\left.+q\left(q^2\delta((24m-1)/q^2)-q\delta((24m-1)/q)\right)p_{47}(c_0^m-2)\right\}$$

$$\equiv \Delta p_{47}(c_0^m-2) .$$

Equating coefficients of $\eta(\tau)^{23}(\Delta_1G_{23,1}(\tau)+\Delta_2G_{23,2}(\tau))=\Delta'\eta(\tau)^{23}$, we obtain

(3.1.6)

$$\Delta_1\left\{p_{23}(c_1^m-1)+q^3p_{23}(c_{-1}^m-1)+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p_{23}(c_0^m-1)\right\}+$$

$$+\Delta_2\left\{p_{23}(c_2^m-1)+p_{23}(c_{-2}^m-1)+q(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\cdot\right.$$

$$\cdot\left(\frac{\delta((24m-1)/q^2)(24m-1)/q^2}{q}\right)p_{23}(c_{-1}^m-1)+$$

$$+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p_{23}(c_1^m-1)+$$

$$\left.+q\big(q^2\delta((24m-1)/q^2)-q\delta((24m-1)/q)\big)\,p_{23}(c_0^m-1)\right\}$$

$$\equiv \Delta' p_{23}(c_0^m-1)\,.$$

(3.1.4)–(3.1.6) yield

THEOREM 9. *When q is a prime, $q>3$, $q\neq 7$, $c_n^m=q^{2n}m-(q^{2n}-1)/24$, and $m\equiv 1,3,4 \pmod 7$, we have*

$$\Delta_1\left\{p(c_1^m)+q^3p(c_{-1}^m)+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p(c_0^m)\right\}+$$

$$+\Delta_2\left\{p(c_2^m)+p(c_{-2}^m)+q(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\cdot\right.$$

$$\cdot\left(\frac{\delta((24m-1)/q^2)(24m-1)/q^2}{q}\right)p(c_{-1}^m)+$$

$$+q^4(-1)^{[\frac14 q]}\left(\frac{24}{q}\right)\left(\frac{24m-1}{q}\right)p(c_1^m)+$$

$$\left.+\big(q^3\delta((24m-1)/q^2)-q^2\delta((24m-1)/q)\big)\,p(c_0^m)\right\}$$

$$\equiv \Delta' p_{23}(c_0^m-1)+2\Delta\, p_{47}(c_0^m-2)$$

$$\equiv \Delta\, p(c_0^m)+(\Delta'-\Delta)\,p_{23}(c_0^m-1) \pmod 7\,.$$

When $q=5$ the coefficients Δ_1, Δ_2, Δ and Δ' have been determined, using theorem 1 and 3, and the methode of the proof of theorem 5. The result is $\Delta_2/\Delta_1\equiv 4$, $\Delta/\Delta_1\equiv 6$ and $\Delta'/\Delta_1\equiv 6$ which yields

THEOREM 10. *When $m\equiv 1,3,4 \pmod 7$ we have*

$$p(5^4m-26)+\left\{5\binom{m+1}{5}+2\right\}p(5^2m-1)+$$
$$+\left\{6\delta((m+1)/25)+3\delta((m+1)/5)+3\binom{m+1}{5}+2\right\}p(m)+$$
$$+\left\{2\binom{\delta((m+1)/25)(24m-1)/25}{5}+5\right\}p((m+1)/5^2)+p((m+26)/5^4)$$
$$\equiv 0 \pmod 7 .$$

Repeated use of theorem 10 yields

LEMMA 9.

$$p(c_2^m)\equiv\left\{2\binom{24m-1}{5}+5\right\}p(c_1^m)+$$
$$+\left\{\delta((24m-1)/25)+4\delta((24m-1)/5)+4\binom{24m-1}{5}+5\right\}p(c_0^m)+$$
$$+\left\{5\binom{\delta((24m-1)/25)(24m-1)/25}{5}+2\right\}p(c_{-1}^m)+6p(c_{-2}^m),$$

$$p(c_3^m)\equiv 5p(c_2^m)+3p(c_1^m)+\left\{5\binom{24m-1}{5}+2\right\}p(c_0^m)+6p(c_{-1}^m),$$

$$p(c_n^m)\equiv 5p(c_{n-1}^m)+3p(c_{n-2}^m)+2p(c_{n-3}^m)+6p(c_{n-4}^m)\quad\text{when } n\geqq 4 .$$

[15] is used to find $p(m)$ and $p(c_1^m)$ when $m<40$. Using lemma 9, we find table I which give $p(c_n^m)$ (mod 7). The case $m=24\equiv 3$ is left out since $c_n^{24}=c_{n+1}^1$.

Table I.

n \ m	1	3	4	8	10	11	15	17	18	22	25	29	31	32	36	38	39
−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	3	5	1	0	0	1	3	0	1	5	1	3	5	1	3	0
1	0	5	3	4	0	0	4	0	0	0	6	2	0	0	0	5	0
2	1	6	4	2	0	0	2	3	0	1	3	5	3	5	1	6	0
3	2	3	4	1	0	0	1	6	0	2	5	5	6	3	2	3	0
4	5	5	5	4	0	0	4	1	0	5	6	1	1	4	5	5	0
5	5	6	0	2	0	0	2	1	0	5	3	0	1	4	5	6	0
6	1	3	5	1	0	0	1	3	0	1	5	1	3	5	1	3	0
7	0	5	3	4	0	0	4	0	0	0	6	2	0	0	0	5	0
8	1	6	4	2	0	0	2	3	0	1	3	5	3	5	1	6	0
9	2	3	4	1	0	0	1	6	0	2	5	5	6	3	2	3	0

From this we conclude

THEOREM 11. *When* $c_n^m = 5^{2n}m-(5^{2n}-1)/24$ *we have, modulo* 7,

m	1	4	17	22	29	31	32	36
$p(c_{6n}^m)$	1	5	3	1	1	3	5	1
$p(c_{6n+1}^m)$	0	3	0	0	2	0	0	0
$p(c_{6n+2}^m)$	1	4	3	1	5	3	5	1
$p(c_{6n+3}^m)$	2	4	6	2	5	6	3	2
$p(c_{6n+4}^m)$	5	5	1	5	1	1	4	5
$p(c_{6n+5}^m)$	5	0	1	5	0	1	4	5

m	3	8	15	25	38
$p(c_{3n}^m)$	3	1	1	5	3
$p(c_{3n+1}^m)$	5	4	4	6	5
$p(c_{3n+2}^m)$	6	2	2	3	6

$$p(c_n^m) \equiv 0 \quad \textit{when } m=10,\ 11,\ 18 \textit{ and } 39\ .$$

COROLLARY. The sequence $\{p(n)\}$ *fills each residue class modulo* 7 *infinitely often.*

3.2 A LEMMA. In the following we shall repeatedly require the following well-known lemma from the theory of linear recurrence:

LEMMA 10. *Let* $f(n)$ *be an arithmetic function such that*

$$f(n) \equiv \alpha_1 f(n-1)+\ldots+\alpha_r f(n-r) \pmod{N}\ ,$$

where $(\alpha_r, N)=1$ *and* $n-r \geqq r_0$. *Then there exists a constant* μ_0 *so that*

$$f(\mu\mu_0+r_1) \equiv f(r_1) \pmod{N}$$

for all $\mu \geqq 0$ *and* $r_1 \geqq r_0$.

COROLLARY. *If there exists an integer* r_1 *such that* $f(r_1) \equiv s \pmod{N}$, *then* $\{f(n)\}$ *fills the residue class* s *modulo* N *infinitely often.*

Let

$$\tau_t(n) = T_{24t}(n), \quad \tau(n) = \tau_1(n)\ . \tag{3.2.1}$$

To show how lemma 10 applies, we will prove a theorem on $\tau(n)$. Mordell [13] has shown

$$\tau(pn) = \tau(p)\,\tau(n) - p^{11}\,\tau(n/p)\ .$$

Let N be an integer and p a prime, $p \equiv 1 \pmod{N}$. Let $a_n = p^n a$ where $a > 0$. Then $a_n \equiv a \pmod{N}$ and

$$\tau(a_n) \equiv \tau(p)\tau(a_{n-1}) - \tau(a_{n-2}) \pmod{N}$$

when $n \geqq 1$. Putting $f(n) = \tau(a_n)$, lemma 10 applies and we obtain

THEOREM 12. *If there exists an integer* n_0 *such that* $\tau(Nn_0 + A) \equiv s \pmod{N}$ *and* $Nn_0 + A > 0$, *then* $\{\tau(Nn + A)\}$ *fills the residue class* s *modulo* N *infinitely often.*

3.3 ON THE RESIDUE CLASSES OF $p(n)$ MODULO 7. Putting $f(n) = p(c_n^m)$ and using lemmas 9 and 10, we conclude that there exists a μ_0 so that $p(c_{\mu\mu_0}^m) \equiv p(c_0^m) \pmod{7}$ and further, that $p(c_{3\mu\mu_0}^m) \equiv p(c_0^m) \pmod{7}$. However, $c_{3\mu\mu_0}^m \equiv m \pmod{7}$ hence, if $p(c_0^m) = p(m) \equiv s \pmod{7}$ and $m \equiv a = 1, 3, 4 \pmod{7}$, then $\{p(7n + a)\}$ fills the residue class s modulo 7 infinitely often. Table II lists $p(m) \pmod{7}$. The values of m in the first column are $\equiv 1 \pmod{7}$, in the second column $\equiv 3 \pmod{7}$ and in the third column $\equiv 4 \pmod{7}$.

Table II.

$p(m)$	m	m	m
0	225	10	11
1	1	171	144
2	43	129	214
3	85	3	88
4	113	199	200
5	134	143	4
6	148	73	172

From the foregoing discussion and table II we obtain

THEOREM 13. *Each of the sequences* $\{p(7n+1)\}$, $\{p(7n+3)\}$, *and* $\{p(7n+4)\}$ *fills each residue class modulo 7 infinitely often.*

3.4 ON THE RESIDUE CLASSES OF $p_k(n)$. Let $p > 3$ and $q \neq p$. Let $\Delta_{a'+1}$ be the first coefficient in theorem 6 (when k is even) or 9 (when k is odd) such that $\Delta_{a'+1} \not\equiv 0 \pmod{q}$. Hence $0 \leqq a' \leqq a$. We obtain

$$\sum_{\alpha=1}^{a'+1} \Delta_\alpha \Big\{ T_k(p^{2\alpha} n) + p^{(k-2)\alpha} T_k(n/p^{2\alpha}) + \\ + \sum_{s=1}^{2\alpha-1} p^{\frac{1}{2}(2\alpha-s)k-2\alpha} (-1)^{\frac{1}{4}k(p^s-1)} \{ p^s \delta(n/p^{2\alpha-s}) - \\ - p^{s-1} \delta(n/p^{2\alpha-s-1}) \} T_k(p^{2s-2\alpha} n) \Big\} \\ \equiv \Delta T_k(n) \pmod{q}$$

when k is even and a similar result when k is odd. Replacing n by $p^{2a'+2}n$ and rearranging we obtain

$$(3.4.1) \quad T_k(p^{4a'+4}n) \equiv \alpha_1 T_k(p^{4a'+2}n) + \ldots + \alpha_{2a'+2} T_k(n) \pmod{q},$$

where $\alpha_{2a'+2} = -p^{(k-2)(a'+1)}$ and $\alpha_j, j = 1, 2, \ldots, 2a'+2$, are independent of n. Replacing n by $(24a_n + k)/(k,24)$, where $a_n = p^{2n}a_0 + k(p^{2n}-1)/24$, we obtain

$$p_k(a_{2a'+2+n}) \equiv \alpha_1 p_k(a_{2a'+1+n}) + \ldots + \alpha_{2a'+2} p_k(a_n) \pmod{q}.$$

By lemma 10 we conclude

$$(3.4.2) \qquad p_k(a_{\mu\mu_0}) \equiv p_k(a_0) \pmod{q}.$$

Let q_i, $i = 1, 2, \ldots, u$, be different primes, $q_i \neq p$. To each q_i we may associate an integer μ_i such that

$$p_k(a_{\mu\mu_i}) \equiv p_k(a_0) \pmod{q_i}.$$

Let $Q = q_1 q_2 \ldots q_u$ and $\mu^* = (q_1 - 1) \ldots (q_u - 1)\mu_1 \ldots \mu_u$. Then

$$p_k(a_{\mu\mu^*}) \equiv p_k(a_0) \pmod{Q}, \quad a_{\mu\mu^*} \equiv a_0 \pmod{Q}.$$

As before we conclude

THEOREM 14. *If Q is a square free number and there exists an integer n_0 such that $p_k(Qn_0 + A) \equiv s \pmod{Q}$, then $\{p_k(Qn + A)\}$ fills the residue class s modulo Q infinitely often.*

3.5 ON THE RESIDUE CLASSES OF $p(n)$ MODULO 17, 19, 29 AND 31. Kolberg [5] has shown

$$\begin{aligned}
p(17n+5) &\equiv p_{95}(17n+1) \pmod{17},\\
p(19n+4) &\equiv p_{71}(19n+1) \pmod{19},\\
p(29n+23) &\equiv 7p_{167}(29n+16) \pmod{29},\\
p(31n+22) &\equiv 22p_{119}(31n+17) \pmod{31}.
\end{aligned}$$

Suppose $p(17n_0 + 5) \equiv s \pmod{17}$. Then we have $p_{95}(17n_0 + 1) \equiv s \pmod{17}$, and hence $\{p_{95}(17n+1)\}$ fills the residue class s modulo 17 infinitely often and the same is true for $\{p(17n+5)\}$. The other cases are similar. Table III gives representatives for each residue class modulo 17 of $\{p(17n+5)\}$, modulo 19 of $\{p(19n+4)\}$, modulo 29 of $\{p(29n+23)\}$ and modulo 31 of $\{p(31n+22)\}$.

Table III.

	17	19	29	31
0	413	669	893	332
1	73	23	284	146
2	1467	365	1357	1200
3	226	745	1647	84
4	634	1372	342	704
5	464	4	545	394
6	328	1980	2227	1076
7	5	289	690	580
8	566	1106	23	890
9	430	80	197	1107
10	56	99	1038	22
11	821	61	1763	115
12	158	42	313	1324
13	107	194	2401	301
14	549	270	226	2998
15	209	137	1125	797
16	22	650	168	208
17		156	951	425
18		574	255	177
19			429	2626
20			1531	735
21			603	518
22			980	1820
23			516	270
24			139	983
25			81	1696
26			110	549
27			400	363
28			52	239
29				53
30				487

From the foregoing disucession and table III we obtain

THEOREM 15. *If* $q=17,\ 19,\ 29,\ 31$ *and* $24s\equiv 1 \pmod{q}$, $0<s<q$, *then* $\{p(qn+s)\}$ *fills each residue class modulo q infinitely often.*

3.6 ON THE RESIDUE CLASSES OF $\frac{1}{11}p(11n+6)$ MODULO 11. It is well known ([3]) that

$$\tfrac{1}{11}p(11n+6) \equiv 6p_{119}(11n+1) \pmod{11}.$$

Table IV gives $\frac{1}{11}p(11n+6)$ modulo 11.

Table IV.

$11n+6$	$\frac{1}{11}p(11n+6)$
116	0
6	1
248	2
160	3
567	4
17	5
182	6
61	7
28	8
50	9
83	10

From table IV and the foregoing discussion we conclude, as in 3.5, that

THEOREM 16. $\{\frac{1}{11}p(11n+6)\}$ *fills each residue class modulo* 11 *infinitely often.*

3.7 A GENERAL THEOREM. Kolberg [5] has shown

LEMMA 11. *When* q *is a prime* >3, $24s\equiv 1 \pmod q$, $0<s<q$, $t=(q-1)/(q-1,12)$, $v=[(q+11)/24]$, *there exist integers* a_k, *not all* $\equiv 0 \pmod q$, *such that*

$$\sum_{k=0}^{v} a_k p_{24kt-1}(qn+s-kt) \equiv 0 \pmod q .$$

If there exists a set of integers a_k such that $a_0 \not\equiv 0 \pmod q$ we define q as p-regular. When q is p-regular we have

$$p(qn+s) \equiv \sum_{k=1}^{v} c_k p_{24kt-1}(qn+s-kt) \pmod q . \tag{3.7.1}$$

Let $b_n^k = p^{2n}\nu-(p^{2n}+24kt-1)/24$ where $p \neq q$. Then we have

$$b_{m+n}^k = p^{2m}b_n^k+(24kt-1)(p^{2m}-1)/24 ,$$

$$b_n^k = b_n^0-kt .$$

From (3.4.2) we obtain

$$p_{24kt-1}(b_{\mu\mu_k}^k) \equiv p_{24kt-1}(b_0^k) \pmod q . \tag{3.7.2}$$

Let $\nu\equiv s \pmod q$. Then $b_n^k \equiv s-kt \pmod q$. Hence

$$\begin{aligned} p(b^0_{\mu\mu_1\ldots\mu_v}) &\equiv \sum_{k=1}^{v} c_k p_{24kt-1}(b^k_{\mu\mu_1\ldots\mu_v}) \\ &\equiv \sum_{k=1}^{v} c_k p_{24kt-1}(b^k_0) \\ &\equiv p(b^0_0) \pmod{q} . \end{aligned}$$

Letting $\lambda = \mu_1\mu_2 \ldots \mu_v$ we obtain

(3.7.3) $$p(b^0_{\mu\lambda}) \equiv p(b^0_0) \pmod{q} .$$

The proof applies only when $q > 11$, but (3.7.3) is correct also when $q = 5$, 7 and 11, since then each side $\equiv 0$ by a well known result due to Ramanujan. In 3.3 we also proved (3.7.3) when $q = 7$ and $\nu \equiv 1, 3, 4 \pmod 7$. When $q = 5$ (3.7.3) is correct when $\nu \equiv 4 \pmod 5$, but also when $\nu \equiv 1, 2 \pmod 5$, ([2], [11]). Therefore, we define

(3.7.4) $$\begin{cases} 24 s_q \equiv 1 \pmod{q}, \; 0 < s_q < q, \text{ when } q > 7 , \\ s_7 = 1, 3, 4, 5 , \\ s_5 = 1, 2, 4 . \end{cases}$$

Let q_i, $i = 1, 2, \ldots r$ be different p-regular primes and let λ_i be an associated number given by (3.7.3). Let

$$Q = q_1 q_2 \ldots q_r, \quad \Lambda = \lambda_1 \lambda_2 \ldots \lambda_r (q_1 - 1)(q_2 - 1) \ldots (q_r - 1) .$$

Let $S \equiv s_{qi} \pmod{q_i}$. Then

$$b^0_{\mu\Lambda} \equiv s_{qi} \equiv S \pmod{q_i}, \quad i = 1, 2, \ldots r .$$

Hence

$$b^0_{\mu\Lambda} \equiv S \pmod{Q} .$$

From (3.7.3) we get

$$p(b^0_{\mu\Lambda}) \equiv p(b^0_0) \pmod{Q} .$$

Hence we obtain

THEOREM 17. *If Q is a product of different p-regular primes, $S \equiv s_q \pmod{q}$ when $q \mid Q$, $0 < S < Q$, and there exists an integer n_0 such that $p(Qn_0 + S) \equiv u \pmod{Q}$, then $\{p(Qn + S)\}$ fills the residue class u modulo Q infinitely often.*

3.8 FURTHER CONGRUENCES FOR $p(n)$. Let $p = 11$, 13, 17, 19, 29, or 31. Define

$$b_p = \begin{cases} \frac{1}{11} \text{ when } p = 11, \\ 1 \text{ otherwise.} \end{cases}$$

Then we have ([3], [5])

$$b_p p(pn+s) \equiv c p_{24t-1}(pn+s-t) \pmod{p} \tag{3.8.1}$$

with the following values of c, s, and t, respectively:

p	11	13	17	19	29	31
c	6	6	1	1	7	22
s	6	6	5	4	23	22
t	5	1	4	3	7	5.

Let q be a prime >3 and $k=24t-1$. Then $a=t-1$, and from theorem 7 we obtain

$$\sum_{\alpha=1}^{t} \Delta_\alpha \Big\{ T_k(q^{2\alpha}n) + q^{(k-2)\alpha} T_k(n/q^{2\alpha}) + \\ + \sum_{u=0}^{\alpha-1} q^{\frac{1}{2}(2\alpha-2u-1)k-2\alpha+2u+\frac{1}{2}} (-1)^{[\frac{1}{2}q]} \left(\frac{24}{q}\right) \cdot \\ \cdot \left(\frac{\delta(n/q^{2\alpha-2u-2})n/q^{2\alpha-2u-2}}{q}\right) T_k(q^{4u-2\alpha+2}n) + \\ + \sum_{u=1}^{\alpha-1} q^{(\alpha-u)k-2\alpha} \{q^{2u}\delta(n/q^{2\alpha-2u}) - \\ - q^{2u-1}\delta(n/q^{2\alpha-2u-1})\Big\} T_k(q^{4u-2\alpha}n) = \Delta T_k(n). \tag{3.8.2}$$

Let $n=24(pm+s)-1$, and $r_\beta = q^{2\beta}(pm+s)-(q^{2\beta}-1)/24$. Then $T_k(q^{2\beta}n) = p_k(r_\beta - t)$, and $r_\beta \equiv s \pmod{p}$. Hence

$$c T_k(q^{2\beta}n) = c p_k(r_\beta - t) \equiv b_p p(r_\beta) \pmod{p}.$$

Using this together with (3.8.2) we obtain

THEOREM 18. *If* $p=11, 13, 17, 19, 29, 31$; $t=(p-1)/(p-1,12)$, $24s\equiv 1 \pmod{p}$, $0<s<p$, $b_{11}=\frac{1}{11}$, $b_p=1$ for $p>11$, *then there exist integers* $\Delta_1, \ldots \Delta_t$ *not all* $\equiv 0 \pmod{p}$ *and an integer* Δ *such that*

$$b_p \sum_{\alpha=1}^{t} \Delta_\alpha \Big\{ p(r_\alpha) + q^{(24t-3)\alpha} p(r_{-\alpha}) + \\ + \sum_{u=0}^{\alpha-1} q^{(24t-3)(\alpha-u)-(12t-1)} (-1)^{[\frac{1}{2}q]} \left(\frac{24}{q}\right) \cdot \\ \cdot \left(\frac{\delta(n/q^{2\alpha-2u-2})\, n/q^{2\alpha-2u-2}}{q}\right) p(r_{2u-\alpha+1}) + \\ + \sum_{u=1}^{\alpha-1} q^{(24t-1)(\alpha-u)-2\alpha} \{q^{2u}\delta(n/q^{2\alpha-2u}) - \\ - q^{2u-1}\delta(n/q^{2\alpha-2u-1})\} p(r_{2u-\alpha})\Big\} \equiv b_p \Delta p(r_0) \pmod{p},$$

where q *is a prime* >3, $n=24(pm+s)-1$, $r_\beta=q^{2\beta}(pm+s)-(q^{2\beta}-1)/24$.

COROLLARY 1. *We have*

$$b_p\sum_{\alpha=1}^{t}\Delta_\alpha\{p(p^{2\alpha}(pm+s)-(p^{2\alpha}-1)/24)\} \equiv b_p\Delta p(pm+s) \pmod p .$$

When $q\neq p$ we have $q^{12t}\equiv 1 \pmod p$. Hence we obtain

COROLLARY 2. *When* $(24(pm+s)-1,\, q)=1$ *we have*

$$b_p\sum_{\alpha=1}^{t}\Delta_\alpha\Big\{p(q^{2\alpha}(pm+s)-(q^{2\alpha}-1)/24)+ \\ +(-1)^{[\frac{1}{4}q]}q^{p-3}\left(\frac{pm+s+(q^2-1)/24}{q}\right)p(q^{2\alpha-2}(pm+s)-(q^{2\alpha-2}-1)/24)\Big\} \\ \equiv b_p\Delta p(pm+s) \pmod p .$$

4. Further applications.

4.1 ON THE RESIDUE CLASSES OF $c(n)$. The modular invariant $j(\tau)$ is defined by

$$(4.1.1)\qquad j(\tau) = x^{-1}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)x^n\right)^3\varphi(x)^{-24} = \sum_{n=-1}^{\infty}c(n)\,x^n ,$$

where $x=\exp(2\pi i\tau)$ and $\sigma_k(n)=\sum_{d|n}d^k$. Kolberg [6] has shown

LEMMA 12. *When* $q\geqq 13$ *is a prime*, $r=[\frac{1}{12}q]$, $t=(q-1)/(q-1,12)$, *there exist integers* a_k, *not all* $\equiv 0 \pmod q$, *such that*

$$a_0c(qn) \equiv \sum_{k=1}^{r}a_k\tau_{kt}(qn) \pmod q, \qquad n>0 .$$

If there exists such a set of integers a_k with $a_0\not\equiv 0 \pmod q$, we define q as c-regular. When q is c-regular we have

$$c(qn) \equiv \sum_{k=1}^{r}b_k\tau_{kt}(qn) \pmod q, \qquad n>0 .$$

Replacing n by $c_n^m=p^{2n}m$, where $m>0$, and letting $k=24t$ and $p\neq q$, we obtain from (3.4.1)

$$\tau_t(c^m_{2a'+2+n}) \equiv \alpha_1\tau_t(c^m_{2a'+1+n})+\ldots+\alpha_{2a'+2}\tau_t(c^m_n) \pmod q .$$

This, together with lemma 10, yields

$$\tau_t(c^m_{\mu\mu_0}) \equiv \tau_t(c^m_0) \pmod q .$$

Let μ_k be associated with $\tau_{kt}(n)$ and let $\lambda = \mu_1\mu_2 \ldots \mu_r$. Then

$$c(c_{\mu\lambda}^{qm}) \equiv \sum_{k=1}^{r} b_k \tau_{kt}(c_{\mu\mu_1 \ldots \mu_r}^{qm}) \equiv \sum_{k=1}^{r} b_k \tau_{kt}(c_0^{qm}) \equiv c(c_0^{qm}) \pmod{q} .$$

Let q_i, $i = 1, 2, \ldots, s$, be different c-regular primes and choose $p \neq q_i$, $i = 1, 2, \ldots s$. Let λ_i be associated with q_i and let

$$Q = q_1 q_2 \ldots, q_s, \quad \Lambda = \lambda_1 \lambda_2 \ldots \lambda_s (q_1 - 1)(q_2 - 1) \ldots (q_s - 1) .$$

Then we have

$$c(c_{\mu\Lambda}^{Qm}) \equiv c(c_0^{Qm}) \pmod{q_i} .$$

Hence

$$c(c_{\mu\Lambda}^{Qm}) \equiv c(c_0^{Qm}) \pmod{Q} .$$

Further

$$c_{\mu\Lambda}^{Qm} \equiv c_0^{Qm} \pmod{Q} ,$$

and hence we obtain

THEOREM 19. *If Q is a product of different c-regular primes and there exists a $m_0 > 0$ such that $c(Qm_0) \equiv u \pmod{Q}$, then $\{c(Qm)\}$ fills the residue class u modulo Q infinitely often.*

4.2 CONGRUENCES FOR $c(n)$. Let $p = 13$, 17, 19 or 23. Then we have ([6])

$$c(pn) \equiv d\,\tau_t(pn) \pmod{p}, \quad n > 0 , \tag{4.2.1}$$

with the following values of d and t:

p	13	17	19	23
d	8	7	4	13
t	1	4	3	11 .

From theorem 6 we get

$$\sum_{\alpha=1}^{t} \Delta_\alpha \Big\{ \tau_t(q^{2\alpha} n) + q^{(24t-2)\alpha} \tau_t(q^{-2\alpha} n) + \\ + \sum_{s=1}^{2\alpha-1} q^{12t(2\alpha-s)-2\alpha} \{q^2 \delta(n/q^{2\alpha-s}) - q^{s-1} \delta(n/q^{2\alpha-s-1})\Big\} \tau_t(q^{2s-2\alpha} n) = \Delta \tau_t(n) .$$

Hence we obtain

THEOREM 20. *When $p = 13$, 17, 19, 23 and $t = (p-1)/(p-1, 12)$ there exists integers Δ_α, Δ, not all $\equiv 0 \pmod{p}$, such that*

$$\sum_{\alpha=1}^{t} \varDelta_\alpha \Big\{ c(q^{2\alpha}pn) + q^{(24t-2)\alpha} c(q^{-2\alpha}pn) + \\ + \sum_{s=1}^{2\alpha-1} q^{12t(2\alpha-s)-2\alpha} \{q^s \delta(pn/q^{2\alpha-s}) - q^{s-1}\delta(pn/q^{2\alpha-s-1})\} c(q^{2s-2\alpha}pn) \Big\} \\ \equiv \varDelta c(pn) \mod p), \quad n > 0 .$$

COROLLARY 1. *We have, for* $n > 0$

$$\sum_{\alpha=1}^{t} \varDelta_\alpha c(p^{2\alpha+1}n) \equiv \varDelta c(pn) \pmod{p} .$$

COROLLARY 2. *When* $(n,q) = 1$ *we have*

$$\sum_{\alpha=1}^{t} \varDelta_\alpha \{c(q^{2\alpha}pn) - q^{p-3} c(q^{2\alpha-2}pn)\} \equiv \varDelta c(pn) \pmod{p} .$$

4.3 RESULTS ON $(j(\tau))^{\frac{1}{3}}$ AND $(j(\tau)-1728)^{\frac{1}{2}}$. Let

$$\sum_{-1}^{\infty} a(n) x^{\frac{1}{3}n} = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) x^n\right) x^{-\frac{1}{3}} \varphi(x)^{-8} = (j(\tau))^{\frac{1}{3}} ,$$

$$\sum_{-1}^{\infty} b(n) x^{\frac{1}{2}n} = \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) x^n\right) x^{-\frac{1}{2}} \varphi(x)^{-12} = (j(\tau) - 1728)^{\frac{1}{2}} .$$

Erevik [1] has shown

(4.3.1) $$a(11n) \equiv 4T_{40}(11n) \pmod{11} ,$$

(4.3.2) $$b(11n) \equiv 10T_{60}(11n) \pmod{11} .$$

From theorem 6 we conclude

THEOREM 21. *When p is a prime* $\neq 3$ *we have*

$$\sum_{\alpha=1}^{2} \varDelta_\alpha \Big\{ a(p^{2\alpha}\cdot 11n) + p^{8\alpha} a(p^{-2\alpha}\cdot 11n) + \\ + \sum_{s=1}^{2\alpha-1} p^{10-2\alpha} \{p^s \delta(11n/p^{2\alpha-s}) - p^{s-1}\delta(11n/p^{2\alpha-s-1})\} a(p^{2s-2\alpha}\cdot 11n) \Big\} \\ \equiv \varDelta a(11n) \pmod{11} .$$

THEOREM 22. *When p is a prime* > 2 *we have*

$$\sum_{\alpha=1}^{3} \varDelta_\alpha \Big\{ b(p^{2\alpha}\cdot 11n) + p^{8\alpha} b(p^{-2\alpha}\cdot 11n) + \\ + \sum_{s=1}^{2\alpha-1} p^{10-2\alpha} \{p^s \delta(11n/p^{2\alpha-s}) - p^{s-1}\delta(11n/p^{2\alpha-s-1})\} b(p^{2s-2\alpha}\cdot 11n) \Big\} \\ \equiv \varDelta b(11n) \pmod{11} .$$

From (4.3.1) and (4.3.2) we obtain

$$a(33n+11) \equiv 4p_{40}(11n+2) \pmod{11},$$
$$b(22n+11) \equiv 10p_{60}(11n+3) \pmod{11}.$$

Table V gives one representative from each residue class modulo 11 of $\{p_{40}(11n+2)\}$ and $\{p_{60}(11n+3)\}$.

Table V.

	$11n+2$
0	35
1	24
2	156
3	2
4	13
5	101
6	90
7	189
8	46
9	57
10	266

	$11n+3$
0	234
1	311
2	113
3	14
4	80
5	58
6	25
7	102
8	201
9	69
10	3

By theorem 14 we conclude that, $\{p_{40}(11n+2)\}$ and $\{p_{60}(11n+3)\}$ both fill each residue class modulo 11 infinitely often, and hence

THEOREM 23. *Each of the sequences* $\{a(33n+11)\}$ *and* $\{b(22n+11)\}$ *fills each residue class modulo* 11 *infinitely often.*

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