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CONGRUENCE PROPERTIES AND DENSITY PROBLEMS FOR THE FOURIER COEFFICIENTS OF MODULAR FORMS

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1. Introduction.

Let

$$x = e^{2\pi i\tau}, \quad \operatorname{Im}\tau > 0 ,$$

$$\varphi(x) = \prod_{n=1}^{\infty} (1-x^n) ,$$

$$\sum_{n=0}^{\infty} p_k(n)\, x^n = \varphi(x)^k ,$$

where k is an integer. Then $p_{-1}(n) = p(n)$ is the number of unrestricted partitions of n. Further, let $c(n)$ be the Fourier coefficient of Klein's modular invariant $j(\tau)$ given by

$$j(\tau) = x^{-1}\varphi(x)^{-24}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)x^n\right)^3,$$

where $\sigma_3(n) = \sum_{d|n} d^3$. Atkin and O'Brien [2] have proposed the question:

(A) *Given a,m, is $p(n) \equiv a \pmod{m}$ soluble for values of n with positive density?*

They also note that the best hope of establishing (A) is that one may exhibit *explicit* congruences of the form

$$p(bn+c) \equiv a \pmod{m} .$$

The same questions, of course, arise for $c(n)$, and indeed for the Fourier coefficients of other modular forms and functions.

In this paper we make some contribution to the solution of these problems for $p_k(n)$ with $k>0$, $p(n)$ and $c(n)$, when $a=0$.

2. The theorems.

Dedekind's modular form $\eta(\tau)$ is given by

$$\eta(\tau) = e^{\pi i\tau/12}\,\varphi(x) .$$

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Now, put

$$\eta(\tau)^k = \sum_{n=k/(k,24)}^{\infty} T_k(n)\, e^{(k,24)\pi i n\tau/12},$$

where k is an integer, and (a,b) is the greatest common divisor of the integers a,b. The $T_k(n)$'s have the following congruence property:

THEOREM 1. *Let k be a positive integer, and let Q be a square-free number. Then, to each prime p such that $p \nmid Q$, $p^2 \equiv 1 \pmod{24/(k,24)}$, there exists an even integer M such that*

$$T_k(p^{mM-1}n) \equiv T_k(n/p) \pmod{Q}$$

for all n and all $m \geqq 0$.

The $T_k(n)$'s are closely connected to the $p_k(n)$'s, viz.

$$\begin{aligned} T_k(n) &= p_k\big(((k,24)n-k)/24\big)\,, \\ p_k(m) &= T_k\big((24m+k)/(k,24)\big) \end{aligned} \tag{1}$$

(see lemma 4 of Kløve [3]). As an immediate consequence of Theorem 1 we therefore have

COROLLARY 1.

$$p_k(p^{mM-1}n + k(p^{mM}-1)/24) \equiv 0 \pmod{Q}$$

for all n prime to p and all $m \geqq 1$.

Now, if $f(n)$ is any arithmetical function with integral values, put

$$d(f\,|\,m) = \liminf_{x\to\infty} x^{-1} \sum_{\substack{n\leq x \\ f(n)\equiv 0 \pmod{m}}} 1\,.$$

Then corollary 1 implies

COROLLARY 2.

$$d(p_k\,|\,Q) > 0$$

for all $k \geqq 1$ and all square-free Q.

For $T_{-1}(n)$ we use the special notation

$$T_{-1}(n) = P(n)\,.$$

We shall prove the following congruence property of $P(n)$ (for the definition of the class of p-regular primes, see section 4):

THEOREM 2. *Let Q be a product of different p-regular primes. Then to each prime p such that $p \nmid Q$, $p \geqq 5$, there exists an even integer L such that*

$$P(p^{mL-1}Qn) \equiv P(Qn/p) \pmod{Q}$$

for all n and all $m \geqq 0$.

By (1), $P(n)$ is connected to $p(n)$ through

$$P(n) = p((n+1)/24), \quad p(m) = P(24m-1).$$

Thus Theorem 2 implies

COROLLARY 1. *If $24S \equiv 1 \pmod{Q}$, $0 < S < Q$, then*

$$p(p^{mL-1}(Qn+pS) - (p^{mL}-1)/24) \equiv 0 \pmod{Q}$$

for all n prime to p and all $m \geqq 1$.

This, in turn, gives

COROLLARY 2. *If Q is a product of different p-regular primes, then*

$$d(p|Q) > 0. \tag{2}$$

In particular the primes

5, 7, 11, 13, 17, 19, 29, 31, 37, 41, 43, 53, 59

are p-regular. Now the well known Ramanujan congruences for $p(n)$ $(\text{mod}\, 5\cdot 7\cdot 11)$ imply (2) for $Q = 5\cdot 7\cdot 11$. Further, the results of Atkin and O'Brien [2] imply (2) for $Q = 13$, and the results announced in Atkin [1] imply (2) for $Q = 17\cdot 19\cdot 23\cdot 29\cdot 31$ (note that Atkin [1] has a result for the modulus 23, which we cannot get by our methods). However, the result for $Q = 37\cdot 41\cdot 43\cdot 53\cdot 59$ seems to be new.

Similarly, we shall prove the following congruence property for $c(n)$ (for the definition of the class of c-regular primes, see section 4):

THEOREM 3. *Let Q be a product of different c-regular primes. Then, to each prime p such that $p \nmid Q$, there exists an integer N such that*

$$c(p^{mN-1}Qn) \equiv c(Qn/p) \pmod{Q}$$

for all n and all $m \geqq 0$.

An immediate consequence of Theorem 3 is

COROLLARY 1.

$$c(p^{mN-1}Qn) \equiv 0 \pmod{Q}$$

for all n prime to p and all $m \geqq 1$.

Therefore we have

COROLLARY 2. *If Q is a product of different c-regular primes, then*

$$d(c\,|\,Q) > 0\,. \tag{3}$$

In particular the primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31

are c-regular. Now the results of Lehmer [6] imply (3) for $Q=2\cdot 3\cdot 5$, and the results of Lehner [7] imply (3) for $Q=7\cdot 11$. Further, the results of Newman [8] imply (3) for $Q=13$, and lately Kolberg has proved a result which implies (3) for $Q=17\cdot 19\cdot 23$. However, the result for $Q=29\cdot 31$ seems to be new.

3. Proof of Theorem 1.

Let k be an even positive integer. Then, if p is a prime such that $p^2\equiv 1 \pmod{24/(k,24)}$, there exist integers Δ_α such that $(\Delta_1,\Delta_2,\ldots,\Delta_{a+1})=1$, where $a=[(k-1)/24]$, and

$$\begin{cases} \sum_{\alpha=1}^{a+1}\Delta_\alpha\Big\{T_k(p^{2\alpha}n)+p^{(k-2)\alpha}T_k(n/p^{2\alpha})+ \\ \qquad +\sum_{s=1}^{2\alpha-1}p^{\frac{1}{2}(2\alpha-s)k-2\alpha}(-1)^{\frac{1}{4}k(p^s-1)}\cdot \\ \qquad\qquad \cdot\{p^s\delta(n/p^{2\alpha-s})-p^{s-1}\delta(n/p^{2\alpha-s-1})\}T_k(p^{2s-2\alpha}n)\Big\} \\ \qquad =\Delta_0T_k(n) \end{cases} \tag{4}$$

(Theorem 6 of Kløve [3]). Here

$$\delta(x)=\begin{cases}1 & \text{if } x \text{ is an integer}\,,\\ 0 & \text{otherwise,}\end{cases}$$

and $[x]$ is the largest integer in x. A quite similar result exists, when k is an odd positive integer (Theorem 7 of Kløve [3]).

Let now k be a positive integer and q a given prime. Then there exists an integer $b=b(p)$ such that $\Delta_b\not\equiv 0 \pmod q$, while $\Delta_\alpha\equiv 0 \pmod q$ for $\alpha>b$. Solving (4) (or the similar equation, if k is odd), we get

$$T_k(p^{2b}n)\equiv a_1(n)T_k(p^{2b-2}n)+\ldots+a_{2b}(n)T_k(p^{-2b}n) \pmod q\,,$$

where in particular $a_{2b}(n)=-p^{(k-2)b}$. Replacing n by np^{2b-1} we obtain

$$T_k(p^{4b-1}n)\equiv a_1T_k(p^{4b-3}n)+\ldots+a_{2b}T_k(p^{-1}n) \pmod q\,,$$

where now all $a_i=a_i(np^{2b-1})$ are independent of n. This shows that for all n the function $f(r)=T_k(p^{2r-1}n)$ is a solution of the linear recurrence relation

$$f(r) \equiv a_1 f(r-1) + \ldots + a_{2b} f(r-2b) \pmod q \quad \text{for } r \geqq 2b .$$

Using now a well known result on linear recurrence, we conclude that if $p \neq q$ (so that $(a_{2b}, q) = 1$) there exists an even integer μ (independent of n) such that

$$T_k(p^{m\mu-1}n) \equiv T_k(n/p) \pmod q \tag{5}$$

for all n and all $m \geqq 0$.

Let now $q_1, q_2, \ldots, q_r$ be different primes, $Q = q_1 q_2 \ldots q_r$ and p a prime such that $p \nmid Q$, $p^2 \equiv 1 \pmod{24/(k,24)}$. To each q_i we associate an even integer μ_i given by (5). Then, with $M = \{\mu_1, \mu_2, \ldots, \mu_r\}$ (the least common multiple of $\mu_1, \mu_2, \ldots, \mu_r$), Theorem 1 follows.

4. Proofs of Theorems 2 and 3.

The following two lemmas are due to Kolberg [4], [5]:

LEMMA 1. *Let q be a prime $\geqq 5$, and put $t = (q-1)/(q-1, 12)$, $v = [(q+11)/24]$. Then there exist constants a_k, not all $\equiv 0 \pmod q$, such that*

$$a_0 P(qn) \equiv \sum_{k=1}^{v} a_k T_{24kt-1}(qn) \pmod q ,$$

where the sum is empty when $v = 0$.

LEMMA 2. *Let q be a prime, and put $t = (q-1)/(q-1, 12)$, $r = [q/12]$. Then there exist constants α_k, not all $\equiv 0 \pmod q$, such that*

$$\alpha_0 c(qn) \equiv \sum_{k=1}^{r} \alpha_k T_{24kt}(qn) \pmod q ,$$

where the sum is empty when $r = 0$.

If the set of integers a_k in lemma 1 can be chosen such that $a_0 \not\equiv 0 \pmod q$, we define q as p-regular. Obviously, if q is p-regular, we get a congruence of the form

$$P(qn) \equiv \sum_{k=1}^{v} b_k T_{24kt-1}(qn) \pmod q . \tag{6}$$

Similarly, if the set of integers α_k in lemma 2 can be chosen such that $\alpha_0 \not\equiv 0 \pmod q$, we define q as c-regular; and if q is c-regular, we get a congruence of the form

$$c(qn) \equiv \sum_{k=1}^{r} \beta_k T_{24kt}(qn) \pmod q . \tag{7}$$

Before completing the proofs of Theorems 2 and 3 we shall give several instances of (6) and (7) (written out in the p_k-notation). We have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These are the cases of (6) with $v=0$ and are recognized as the well known Ramanujan congruences. Further

$$\begin{aligned} p(13n+6) &\equiv 6p_{23}(13n+5) \pmod{13}, \\ p(17n+5) &\equiv p_{95}(17n+1) \pmod{17}, \\ p(19n+4) &\equiv p_{71}(19n+1) \pmod{19}, \\ p(29n+23) &\equiv 7p_{167}(29n+16) \pmod{29}, \\ p(31n+22) &\equiv 22p_{119}(31n+17) \pmod{31}. \end{aligned}$$

These are the cases of (6) with $v=1$ and are given by Kolberg [4]. Further

$$\begin{aligned} p(37n+17) &\equiv p_{71}(37n+14)+19p_{143}(37n+11) \pmod{37}, \\ p(41n+12) &\equiv 35p_{239}(41n+2)+3p_{479}(41n-8) \pmod{41}, \\ p(43n+9) &\equiv 23p_{167}(43n+2)+5p_{335}(43n-5) \pmod{43}, \\ p(53n+42) &\equiv 8p_{311}(53n+29)+14p_{623}(53n+16) \pmod{53}, \\ p(59n+32) &\equiv 27p_{695}(59n+3)+58p_{1391}(59n-26) \pmod{59}. \end{aligned}$$

These are the cases of (6) with $v=2$ and seem to be new.

Similarly, we have

$$\begin{aligned} c(2n) &\equiv 0 \pmod 2, \\ c(3n) &\equiv 0 \pmod 3, \\ c(5n) &\equiv 0 \pmod 5, \\ c(7n) &\equiv 0 \pmod 7, \\ c(11n) &\equiv 0 \pmod{11}, \end{aligned}$$

$$\begin{aligned} c(13n) &\equiv 8p_{24}(13n-1) \pmod{13}, \\ c(17n) &\equiv 7p_{96}(17n-4) \pmod{17}, \\ c(19n) &\equiv 4p_{72}(19n-3) \pmod{19}, \\ c(23n) &\equiv 13p_{264}(23n-11) \pmod{23}, \end{aligned}$$

$$\begin{aligned} c(29n) &\equiv 4p_{168}(29n-7)+23p_{336}(29n-14) \pmod{29}, \\ c(31n) &\equiv p_{120}(31n-5)+25p_{240}(31n-10) \pmod{31}. \end{aligned}$$

The above results for $q=2$ and 3 are implied by the congruence $(n+1)$ $c(n)\equiv 0 \pmod{24}$ given by Lehmer [6]. The other cases of (7) with $r=0$ are implied by the congruences of Lehner [7], and the cases of (7) with $r=1$ are given by Kolberg [5] (the congruence $c(13n)\equiv -\tau(n) \pmod{13}$ of Newman [8], where $\tau(n)$ is Ramanujan's function, implies the above result for $q=13$, as noted by Kolberg [5]). The cases of (7) with $r=2$ seem to be new.

The necessary computation for establishing (6) and (7) in the cases with $v=2$ and $r=2$ was performed by the second author on the IBM 360/50 computer at the University of Bergen.

We now turn to the proof of Theorem 2. Let $q \geqq 13$ be a p-regular prime, and let p be a prime $\neq q$ such that $p^2 \equiv 1 \pmod{24/(24kt-1,24)}$ $(k=1,\ldots,v)$, that is, $p \geqq 5$. To each of the functions $T_{24kt-1}(n)$ $(k=1,\ldots,v)$ we associate an even integer μ_k given by (5). Put $\Lambda=\{\mu_1,\mu_2,\ldots,\mu_v\}$; then we have

$$P(p^{m\Lambda-1}qn) \equiv \sum_{k=1}^{v} b_k T_{24kt-1}(p^{m\Lambda-1}qn)$$

$$\equiv \sum_{k=1}^{v} b_k T_{24kt-1}(qn/p) \pmod{q},$$

that is,

$$P(p^{m\Lambda-1}qn) \equiv P(qn/p) \pmod{q}. \tag{8}$$

If $q \leqq 11$ is a p-regular prime, then $P(qn) \equiv 0 \pmod{q}$, so that (8) is obvious for any integral Λ.

Let now $q_1,q_2,\ldots,q_r$ be different p-regular primes, $Q=q_1q_2\cdots q_r$ and p a prime such that $p \nmid Q$, $p \geqq 5$. Put $L=\{\Lambda_1,\Lambda_2,\ldots,\Lambda_r\}$, where Λ_i is an even integer associated to q_i through (8), and Theorem 2 follows.

Starting from lemma 2, Theorem 3 is proved in a similar way.

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