MATH. SCAND. 23 (1968), 167–168

ON SEMI-EXTREMAL SUBSETS OF CONVEX SETS

T. BAI ANDERSEN and E. THUE POULSEN

In [1, problem 30, p. 324] J. Pryce raised the following problem: Let L be a locally convex topological vector space, K a compact, convex subset of L and X the set of extremal points of K. A non-empty subset C of K is called *semi-extremal* in K if $K \setminus C$ is convex (or equivalently: $\alpha \in [0,1]$, $x,y \in K$, $\alpha x+(1-\alpha)y \in C \Rightarrow x \in C$ or $y \in C$). Can you conclude:

a) $C \cap X \neq \emptyset$?
b) C closed $\Rightarrow C \cap X \neq \emptyset$?
c) $C \cap \bar{X} \neq \emptyset$?
d) C closed $\Rightarrow C \cap \bar{X} \neq \emptyset$?

We prove that b), and hence d), is true, and give an example with X closed, C convex, and $C \cap X = \emptyset$, thus showing in particular that c), and hence a), is false.

The following lemma is well known. For completeness we sketch a proof.

LEMMA. *If H is a convex, compact subset of K with $K \setminus H$ convex, then $H \cap X \neq \emptyset$.*

PROOF. Let x be an extremal point of H. If x is not an extremal point of K, let l be a line such that x is interior to $l \cap K$ (which is a "closed interval"). Then it is easy to see that one of the endpoints of this interval must be in $H \cap X$.

THEOREM. *If C is closed, then $C \cap X \neq \emptyset$.*

PROOF. Let $\mathscr{K} = \{F \subseteq C \mid F \text{ convex}\}$. By an easy application of Zorn's lemma we get a maximal element H in $\mathscr{K}$. Then H is closed, since C is closed. Hence H is convex and compact. Furthermore H is semi-extremal in K. To prove this, let

$$x = \alpha z + (1-\alpha)y \in H ,$$

Received January 8, 1968.

where $z,y \in K$, $z \neq y$, $\alpha \in [0,1]$, and assume that $y \notin H$ and $z \notin H$. Then, by the maximality of H, we have

$$\operatorname{conv}(H \cup \{y\}) \nsubseteq C \quad \text{and} \quad \operatorname{conv}(H \cup \{z\}) \nsubseteq C ,$$

and hence there exist $u,v \in H$ and $\beta,\gamma \in [0,1]$ such that

$$\beta u + (1-\beta) y \notin C \quad \text{and} \quad \gamma v + (1-\gamma) z \notin C .$$

Then the segment

$$S = \operatorname{conv}\{\beta u + (1-\beta)y,\ \gamma v + (1-\gamma)z\} \subseteq K \setminus C$$

because of the semi-extremality of C, and on the other hand, the triangle

$$T = \operatorname{conv}\{u,v,x\} \subseteq H \subseteq C .$$

But it is easy to see that $S \cap T \neq \emptyset$, which is a contradiction. Hence H is semi-extremal in K, and therefore $H \cap X \neq \emptyset$ by the lemma, and a fortiori $C \cap X \neq \emptyset$.

EXAMPLE. Let $L = \mathsf{R}^{\mathsf{N}}$ (with pointwise addition and scalar-multiplication, and the product topology).

Let $K = [0,1]^{\mathsf{N}}$, which is compact and convex in L. Then $X = \{0,1\}^{\mathsf{N}}$, which is closed.

Let

$$C = \{(x_i) \in K \mid \sum x_i < \infty,\ x_i > 0 \text{ for infinitely many } i\} .$$

Then C is semi-extremal in K, and $C \cap X = \emptyset$.

We notice that C itself is convex.

REFERENCE

1. Proceedings of the Colloquium on Convexity 1965, Copenhagen, 1967.

UNIVERSITY OF AARHUS, DENMARK