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VERTICES OF CHOQUET SIMPLEXES

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The purpose of this note is to prove that every extreme point of a Choquet simplex is a polyhedral vertex, and to apply this result to clarify the relationship between Choquet simplexes and Bastiani polyhedra [2].

In the sequel we shall assume that K is a convex compact subset of a locally convex Hausdorff space E over the reals, and we shall use the symbol $A(K)$ to denote the Banach subspace of $C_{\mathsf{R}}(K)$ consisting of all continuous affine functions.

The *support cone* of K at a point $x \in E$ is the set

$$\mathrm{Cone}(x,K) \;=\; x + \textstyle\bigcup_{\lambda \geqq 0} \lambda(K-x)\,. \tag{1}$$

A point $x \in K$ is said to be a *polyhedral vertex* of K if $\mathrm{Cone}(x,K)$ is closed and proper ("saillant").

Clearly every polyhedral vertex is an extreme point. In fact it is a vertex in the sense of Bohnenblust–Karlin [3], by virtue of:

Proposition 1. *If x is a polyhedral vertex of a convex set K, then the collection $\mathscr{H}_x$ of supporting closed hyperplanes at x is separating; in symbols:*

$$\textstyle\bigcap_{H \in \mathscr{H}_x} H \;=\; \{x\}\,. \tag{2}$$

Proof. Without lack of generality we assume $x=0$, and we consider a point $y \neq 0$.

Assume first $y \notin \mathrm{Cone}(0,K)$. Since $\mathrm{Cone}(0,K)$ is closed, there is an $f \in E^*$ and an $\alpha \in \mathsf{R}$ such that

$$f(y) \;<\; \alpha \;\leqq\; f(z) \quad \text{for all } z \in \mathrm{Cone}(0,K)\,. \tag{3}$$

Clearly (3) prevails with 0 in the place of α, and so $H=f^{-1}(0)$ is a closed supporting hyperplane at 0 which excludes y.

Assume next that $y \in \mathrm{Cone}(0,K)$. Since $\mathrm{Cone}(0,K)$ is proper, $-y \notin \mathrm{Cone}(0,K)$. By the first part of the proof, there is a supporting

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hyperplane H at 0 which excludes $-y$ and hence also y. The proof is complete.

Note that the definition of a polyhedral vertex is non-intrinsic since it depends on the particular embedding of K in the surrounding locally convex space E. In the theory of Choquet simplexes it is often assumed that K is (canonically) embedded in the Banach dual space $A(K)^*$ endowed with the w^*-topology [5]. Then the simplex K is located on the closed hyperplane

$$H_1 = \{\varphi \in A(K)^* \mid \varphi(1)=1\} ,$$

$\mathrm{Cone}(0,K)$ is a lattice cone, and every continuous affine function on K can be (uniquely) extended to a linear functional. Moreover, the closed unit ball of $A(K)^*$ is equal to $\mathrm{conv}(K \cup -K)$, and if $x \in K$ and $[x]$ is the linear span of x, then

$$\mathrm{Cone}(x,K) = H_1 \cap (\mathrm{Cone}(0,K)+[x]) . \tag{4}$$

We shall also need the following simple consequence of the decomposition theorem for Choquet-simplexes [1]:

Proposition 2. *If K is a Choquet simplex canonically embedded in $E=A(K)^*$, and if F is a closed face with complementary face F', then*

$$E = [F] \oplus [F'] . \tag{5}$$

Proof. We have $E=\mathrm{Cone}(0,K)-\mathrm{Cone}(0,K)$ and $K=\mathrm{conv}(F\cup F')$ [1]. It follows that $E=[F]+[F']$. It remains to be proved that $[F]\cap[F']=\{0\}$.

We assume that $z \in [F]\cap[F']$, say

$$z = (\alpha_1 u_1 - \alpha_2 u_2) = (\beta_1 v_1 - \beta_2 v_2) , \tag{6}$$

where $\alpha_i,\beta_i \geqq 0$, $u_i \in F$, $v_i \in F'$ for $i=1,2$. Now $z \in \gamma H_1$ for some $\gamma \in \mathsf{R}$. Then

$$\gamma = \alpha_1 - \alpha_2 = \beta_1 - \beta_2 ,$$

and so we may define a number $\delta \geqq 0$ by

$$\delta = \alpha_1 + \beta_2 = \alpha_2 + \beta_1 .$$

If $\delta=0$, then $\alpha_i=\beta_i=0$ for $i=1,2$. Hence $z=0$, and there is nothing more to prove. If $\delta>0$, then we may rewrite (6) as follows:

$$\alpha_1\delta^{-1}u_1 + \beta_2\delta^{-1}v_2 = \alpha_2\delta^{-1}u_2 + \beta_1\delta^{-1}v_1 . \tag{7}$$

By the unique decomposition theorem in Choquet simplexes [1], we obtain $\alpha_1 u_1 = \alpha_2 u_2$ and $\beta_2 v_2 = \beta_1 v_1$. Hence $z=0$, and the proof is complete.

Note that the space E in Proposition 2 is not a topological direct sum of $[F]$ and $[F']$ in general.

THEOREM 1. *If K is a Choquet simplex canonically embedded in $A(K)^*$, then every extreme point of K is a polyhedral vertex.*

PROOF. We claim that the following formula is valid for any extreme point x of K:

$$(8) \qquad (\mathrm{Cone}(0,K)+[x]) \cap \mathrm{conv}(K \cup -K) \;=\; \mathrm{conv}(K \cup \{-x\}) .$$

To prove this claim we only have to verify that an arbitrary point z of the left hand side of (8) also belongs to the right hand side. Since $z \in \mathrm{conv}(K \cup -K)$, we shall have an expression

$$z \;=\; \alpha u_1 - (1-\alpha) u_2 ,$$

where $u_1, u_2 \in K$ and $0 \leqq \alpha \leqq 1$.

Let F be the (possibly non-closed) complementary face of $\{x\}$ in K. By the decomposition theorem for Choquet simplexes, we have an expression

$$u_i \;=\; \lambda_i x + (1-\lambda_i) y_i ,$$

where $y_i \in F$ and $0 \leqq \lambda_i \leqq 1$ for $i=1,2$. Hence

$$(9) \qquad z \;=\; (\alpha\lambda_1 - (1-\alpha)\lambda_2)x + \alpha(1-\lambda_1)y_1 - (1-\alpha)(1-\lambda_2)y_2 .$$

Since $z \in \mathrm{Cone}(0,K)+[x]$, we also have an expression

$$z \;=\; \beta x + \gamma v ,$$

where $v \in K$, $\beta \in \mathsf{R}$ and $\gamma \in \mathsf{R}^+$. Again the decomposition theorem yields

$$v \;=\; \mu x + (1-\mu) w ,$$

where $w \in F$ and $0 \leqq \mu \leqq 1$. Hence

$$(10) \qquad z \;=\; (\beta + \gamma\mu)x \;+\; \gamma(1-\mu)w .$$

Now $z \in \delta H_1$ for some $\delta \in \mathsf{R}$. Since all the points at the right hand sides of (9) and (10) are located on the hyperplane H_1, the sum of the coefficients must be equal to δ in both of these equations. Thus we obtain an equation in $\alpha,\beta,\gamma,\lambda_1,\lambda_2,\mu$ which we solve for the (possibly) non-positive parameter β, obtaining

$$\beta \;=\; 2\alpha - \gamma - 1 .$$

We may rewrite (10) as follows:

$$z = (2\alpha-\gamma-1+\gamma\mu)x + \gamma(1-\mu)w . \tag{11}$$

Assume first that $\alpha=0$. By Proposition 2, the space $A(K)^*$ is the direct sum of $[x]$ and $[F]$. Hence we may compare the expressions (9) and (11). Elimination of the parameter λ_2 from the two resulting equations, yields

$$\gamma - \gamma\mu = 0 .$$

Substituting into (11), we obtain that $z=-x$ if $\alpha=0$. Hence $z \in \operatorname{conv}(K \cup \{-x\})$ in this case.

Assume next that $\alpha>0$. Then we may write (11) in the form

$$z = \alpha t - (1-\alpha)x , \tag{12}$$

where

$$t = \alpha^{-1}(\alpha-\gamma+\gamma\mu)x + \alpha^{-1}\gamma(1-\mu)w . \tag{13}$$

We claim that the right hand side of (13) is a convex combination. It is easily verified that the sum of the coefficients is 1, and the last coefficient is trivially positive. To prove that the first coefficient is positive, we define

$$\xi = \alpha - \gamma + \gamma\mu ,$$

and substitute in (12), obtaining

$$z = \big(\xi-(1-\alpha)\big)x + \gamma(1-\mu)w .$$

Again we may compare this expression with formula (9), obtaining

$$\xi - (1-\alpha) = \alpha\lambda_1 - (1-\alpha)\lambda_2 .$$

Solving for ξ, we get

$$\xi = \alpha\lambda_1 + (1-\alpha)(1-\lambda_2) \geqq 0 .$$

This completes the proof that the right hand side of (13) is a convex combination. Hence $t \in K$, and $z \in \operatorname{conv}(K \cup \{-x\})$ by virtue of (12). Formula (8) is proved.

The rest of the proof is a simple application of the Krein–Šmulyan Theorem [4, p. 429]. By virtue of (8), $\operatorname{Cone}(0,K)+[x]$ has a compact intersection with the closed unit ball $\operatorname{conv}(K \cup -K)$. Since we are in a Banach dual space endowed with the w^*-topology, we can conclude that $\operatorname{Cone}(0,K)+[x]$ itself is closed. Now it follows from (4) that $\operatorname{Cone}(x,K)$ is closed. This completes the proof since $\operatorname{Cone}(x,K)$ is proper, x being an extreme point of K.

In [2] A. Bastiani has defined the concept of an infinite dimensional polyhedron in a locally convex space. Bastiani's definition is based on the (non trivial) geometrical fact that a closed convex set K in R^n is a polyhedron iff $\mathrm{Cone}(x,K)$ is closed for every $x \in K$ [2, p. 271]. This characterization can easily be translated to the statement that K is a polyhedron in R^n iff every extreme point x of K is a polyhedral vertex. (Pass to R^n/M, where M is the affine span of $\mathrm{face}(x)$.) This statement could also reasonably be transferred to the infinite dimensional case as definition of a polyhedron. In fact, this might be a more natural definition since it would comprise all compact simplexes K (canonically embedded in $A(K)^*$), whereas the original definition turns out to be very restrictive, admitting no compact simplexes other than the finite dimensional ones.

THEOREM 2. *If K is a (compact) simplex such that* $\mathrm{Cone}(x,K)$ *is closed for every point x in K, then K is finite dimensional.*

PROOF. We first observe that $\mathrm{face}(x)$ is closed for every x in K since

$$\mathrm{face}(x) = K \cap (2x - \mathrm{Cone}(x,K)) .$$

Next we claim that for any countable subset A of the extreme boundary $\partial_e K$ there exists a point $x \in K$ such that

$$A = \mathrm{face}(x) \cap \partial_e K . \tag{14}$$

In fact, let $A = \{x_i \mid x_i \in \partial_e K,\ i = 1,2,\ldots\}$, and let x be the barycenter of some probability measure μ of the form

$$\mu = \sum_{i=1}^{\infty} \alpha_i \varepsilon_{x_i}, \qquad \alpha_i > 0 \text{ for } i = 1,2,\ldots .$$

Clearly $A \subset \mathrm{face}(x) \cap \partial_e K$, and if there was an element

$$y \in (\mathrm{face}(x) \cap \partial_e K) \setminus A ,$$

then for some $\lambda \in]0,1]$ and some $z \in K$

$$x = \lambda y + (1-\lambda) z ,$$

and so x would be the barycenter of a boundary measure

$$\nu = \lambda \varepsilon_y + (1-\lambda)\mu_z ,$$

which is different from μ since it charges the point $y \notin A$. This contradicts the hypothesis that K be a simplex, and formula (14) is established.

Now it follows that for every countable subset A of $\partial_e K$ there is a closed face F of K such that

$$A = F \cap \partial_e K .$$

Otherwise stated, every countable subset of $\partial_e K$ is closed in the *structure topology* of Effros [6]. This rules out the possibility that $\partial_e K$ be countably infinite, for then $\partial_e K$ would be discrete in the compact structure topology.

Assume finally that $\partial_e K$ is uncountable. Then we can construct a sequence $\{A_n\}$ of countable subsets of $\partial_e K$ with the finite intersection property and empty intersection. For example we may start by choosing an arbitrary countable set of points $A_1 = \{x_1, x_2, x_3, \ldots\}$, then form $A_2 = \{x_1', x_2, x_3, \ldots\}$ where the first point x_1 is replaced by a new point x_1' different from all $x_1, x_2, \ldots$, next form $A_3 = \{x_1'', x_2'', x_3, \ldots\}$ by replacing the first two points by two new points x_1'', x_2'' different from each other and from all points previously chosen, and so on. Again this contradicts the compactness, since the sets A_n are closed in the structure topology.

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