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ON THE MOMENTS OF FUNCTIONS SATISFYING A LIPSCHITZ CONDITION

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0. Introduction.

An infinite sequence $\mu = (\mu_k)_0^\infty$ of complex numbers is called the moment sequence of the integrable function f on $0 < x < 1$, if

$$\mu_k = \int_0^1 x^k f(x)\, dx, \qquad k = 0, 1, \ldots .$$

If $\mu = (\mu_k)_0^\infty$ is any given sequence, introduce the functional

$$\lambda_{r,k}(\mu) = \binom{r}{k} \sum_{j=0}^{r-k} (-1)^j \binom{r-k}{j} \mu_{k+j}, \tag{0.1}$$

where $0 \leqq k \leqq r$

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!},$$

(r is a non-negative integer). It is well known that μ is a moment sequence of a bounded function on $0 < x < 1$, if and only if

$$\sup_{r,k} (r+1)|\lambda_{r,k}(\mu) < \infty ,$$

(see Widder [4, pp. 111–112] or Shohat–Tamarkin [3, pp. 99–101]). Moreover

$$\sup_{r,k} (r+1)|\lambda_{r,k}(\mu)| = \|f\| ,$$

if μ is the moment sequence of f and

$$\|f\| = \sup_{0<x<1} |f(x)| .$$

The object of our paper is to generalize this result in the following direction. If f is a bounded function on $0 < x < 1$, denote its modulus of continuity by

$$\omega(t,f) = \sup_{|h| \leqq t} \sup_{\substack{0<x<1 \\ 0<x+h<1}} |f(x+h) - f(x)| .$$

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Our aim is to characterize the moments of functions f satisfying a Lipschitz condition

$$\omega(t,f) = O(t^\theta) \quad \text{as } t \to +0, \qquad 0<\theta\leqq 1 .$$

We can also treat moments of functions belonging to certain generalized Lipschitz' classes. In doing this we give an answer to a question raised by Cotlar (personal communication), who also suggested that the technique of interpolation spaces might be useful in connection with moment problems. (It might be possible that also other cases than the one contained here can be treated with the aid of interpolation spaces).

In our proof we need an alternative definition of the Lipschitz spaces, as interpolation spaces. This equivalent description is essentially due to Lions [1], (see also Peetre [2]). For completeness we include a proof of the equivalence between the two definitions. This proof as well as the exact definition of the interpolation spaces in question, can be found in section 1. Our main result (theorem 2.1) is stated and proved in section 2. The method of the proof is quite elementary and the paper is essentially self-contained.

1. Preliminaries on interpolation spaces.

We shall let $\mathscr{C}$ denote the space of all bounded, continuous functions on $0<x<1$. It is a Banach space with norm

$$\|f\| = \sup_{0<x<1}|f(x)| .$$

By $\mathscr{C}^1$ we shall mean the space of all functions f such that the first derivative f' belongs to $\mathscr{C}$. Then $\mathscr{C}^1$ is a semi-normed space, with semi-norm

$$\|f\|_1 = \sup_{0<x<1}|f'(x)| .$$

We shall also write

$$\|f\|^* = \inf_{a\in N}\|f-a\| ,$$

where N is the space of all constant functions.

On the space $\mathscr{C}+\mathscr{C}^1$ we consider the family of semi-norms

$$K(t,f) = \inf_{f_0+f_1=f}(\|f_0\|+t\|f_1\|_1), \quad 0<t<\infty .$$

It is easy to see that

$$\|f\|^* = K(t,f), \qquad t\geqq 1 . \tag{1.1}$$

In fact, we have

$$f(x)-f(\tfrac{1}{2}) = \int_{\frac{1}{2}}^{x} f'(y)dy, \quad f\in\mathscr{C}^1 ,$$

and hence

$$\|f\|^* \leqq \|f-f(\tfrac{1}{2})\| \leqq \|f'\| = \|f\|_1 .$$

It follows that

$$\begin{aligned} t\|f\|^* &\leqq t\|f_0\| + t\|f_1\|^* \leqq \|f_0\| + t\|f_1\|_1, & 0<t\leqq 1 , \\ \|f\|^* &\leqq \|f_0\| + \|f_1\|^* \leqq \|f_0\| + t\|f_1\|_1, & t\geqq 1 , \end{aligned}$$

if $f=f_0+f_1$. Thus

$$\min(1,t)\,\|f\|^* \leqq K(t,f) . \tag{1.2}$$

On the other hand we see, by taking $f_0=f-a$, $f_1=a$, that

$$K(t,f) \leqq \|f-a\|, \quad a \in N$$

and thus

$$K(t,f) \leqq \|f\|^* .$$

Now (1.1) follows

We shall now consider the function norm $\Phi_{\theta,q}$, $0\leqq\theta\leqq 1$, $1\leqq q\leqq\infty$, defined on positive measurable function $\varphi(t)$, $0<t<\infty$, by

$$\Phi_{\theta,q}(\varphi) = \left(\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^q \frac{dt}{t}\right)^{1/q}, \quad 1\leqq q<\infty ,$$

$$\Phi_{\theta,\infty}(\varphi) = \sup_{0<t<\infty} \frac{\varphi(t)}{t^\theta} .$$

By means of the function norm $\Phi_{\theta,q}$ we define the interpolation space $\mathscr{C}^{\theta,q}$ of all functions f for which $K(t,f)$ is defined and the semi-norm

$$f \to \Phi_{\theta,q}(K(t,f)) ,$$

is finite (see Peetre [2]). From (1.1) we deduce that

$$\Phi_{\theta,q}(K(t,f)) = \left(\int_0^1 \left(\frac{K(t,f)}{t^\theta}\right)^q \frac{dt}{t} + \frac{1}{\theta q}\|f\|^*\right)^{1/q}, \quad 1\leqq q<\infty ,$$

$$\Phi_{\theta,\infty}(K(t,f)) = \max\left(\sup_{0<t<1} \frac{K(t,f)}{t^\theta};\ \|f\|^*\right).$$

In particular we see that $\mathscr{C}^{0,q}=N=$ the space of constant functions, $q<\infty$. It is clear that

$$\Phi_{0,\infty}(K(t,f)) = \|f\|^* ,$$

so that $\mathscr{C}^{0,\infty}$ is essentially identical with $\mathscr{C}$. From the inequality (1.2) we get that $\mathscr{C}^{1,q}=N$ if $1\leqq q<\infty$. Consequently we shall only consider

$\mathscr{C}^{\theta,q}$ for $0<\theta<1$, $1\leqq q<\infty$ or $0<\theta\leqq 1$, $q=\infty$. For simplicity we consider on $\mathscr{C}^{\theta,q}$ the equivalent semi-norms

$$\|f\|_{\theta,q} = \left(\int_0^1 \left(\frac{K(t,f)}{t^\theta}\right)^q \frac{dt}{t}\right)^{1/q}, \quad 0<\theta<1, \quad 1\leqq q<\infty,$$

$$\|f\|_{\theta,\infty} = \sup_{0<t<1} \frac{K(t,f)}{t^\theta}, \quad 0<\theta\leqq 1, \quad q=\infty.$$

It is quite obvious from these expressions, that

$$\mathscr{C}^{\theta',q} \subset \mathscr{C}^{\theta'',q}, \quad \theta''\leqq\theta',$$

$$\mathscr{C}^{\theta,q'} \supset \mathscr{C}^{\theta,q''}, \quad q''\leqq q'.$$

We shall need an equivalent characterization of $\mathscr{C}^{\theta,q}$, in terms of the modulus of continuity

$$\omega(t,f) = \sup_{|h|\leqq t} \sup_{\substack{0<x<1\\ 0<x+h<1}} |f(x+h)-f(x)|.$$

By means of the modulus of continuity we form the generalized Lipschitz' space $\mathrm{Lip}(\theta,q)$, $0<\theta<1$, $1\leqq q<\infty$ or $0<\theta\leqq 1$, $q=\infty$, of all functions f such that the semi-norm

$$f\to\left(\int_0^1 \left(\frac{\omega(t,f)}{t^\theta}\right)^q \frac{dt}{t}\right)^{1/q}, \quad 0<\theta<1, \quad 1\leqq q<\infty,$$

$$f\to \sup_{0<t<1} \frac{\omega(t,f)}{t^\theta}, \quad 0<\theta\leqq 1, \quad q=\infty,$$

is finite. We shall prove

THEOREM 1.1. (Essentially Lions–Peetre). *For any* $f\in\mathscr{C}+\mathscr{C}^1$ *and* $0<t<1$ *we have*

$$\tfrac{1}{2}K(t,f) \leqq \omega(t,f) \leqq 2K(t,f). \tag{1.3}$$

In particular $\mathscr{C}^{\theta,q}=\mathrm{Lip}(\theta,q)$ *with equivalent semi-norms.*

For the conveniance of the reader we reproduce the proof, which is a slight modification of the proof in Lions [1] and Peetre [2], valid for the case of an unbounded interval.

PROOF OF THEOREM 1.1. The right hand inequality of (1.3) is quite trivial for if $f=f_0+f_1$, $f_0\in\mathscr{C}$, $f_1\in\mathscr{C}^1$ we have

$$\omega(t,f) \leqq \omega(t,f_0) + \omega(t,f_1)\,,$$

and since obviously

$$\omega(t,f_0) \leqq 2\|f_0\|, \quad \omega(t,f_1) \leqq t\|f_1'\| = t\|f_1\|_1$$

we get

$$\omega(t,f) \leqq 2(\|f_0\|+t\|f_1\|_1)\,.$$

Thus the right hand inequality follows.

To prove the left inequality we introduce the function

$$f^*(x) = \begin{cases} f(x), & 0<x<1\,, \\ f(2-x), & 1<x<2\,. \end{cases}$$

Here f is a given function in $\mathscr{C}$. We write

$$f_1(t;x) = \frac{1}{t}\int_0^t f^*(x+h)\,dh\,,$$

$$f_0(t;x) = f(x) - f_1(t;x)\,,$$

for $0<x<1$, $0<t<1$. Then $f_0+f_1=f$.

Obviously we have

$$t\left|\frac{d}{dx}f_1(t;x)\right| = |f^*(x+t)-f(x)| = \Delta(x,t), \qquad 0<x<1\,.$$

But if $x+t<1$ it is clear that $\Delta(x,t) \leqq \omega(t,f)$. If $x+t>1$ we have $\Delta(x,t) = |f(2-x-t)-f(x)|$ and since in this case $|(2-x-t)-x| \leqq t$ we have again $\Delta(x,t) \leqq \omega(t,f)$. Thus

$$t\|f_1\|_1 \leqq \omega(t,f)\,.$$

On the other hand

$$f_0(t;x) = t^{-1}\int_0^t (f(x)-f^*(x+h))\,dh\,,$$

so that in the same way as above

$$\|f_0\| \leqq t^{-1}\int_0^t \omega(h;f)\,dh \leqq \omega(t,f)\,.$$

Since it is obvious that $f_0 \in \mathscr{C}$ and $f_1 \in \mathscr{C}^1$ we have proved

$$K(t,f) \leqq \|f_0\| + t\|f_1\|_1 \leqq 2\omega(t,f)\,,$$

and our proof is complete.

2. Moments of functions in Lip(θ,q).

Suppose $\mu=(\mu_k)_0^\infty$ is the moment sequence of the bounded function $f(x)$, $0<x<1$. Then we introduce the functional $\lambda_{r,k}(\mu)$ by means of formula (0.1), i.e.

$$\lambda_{r,k}(\mu) = \binom{r}{k}\sum_{j=0}^{r-k}(-1)^j\binom{r-k}{j}\mu_{k+j} = \binom{r}{k}\int_0^1 x^k(1-x)^{r-k}f(x)\,dx$$

for $0\leqq k\leqq r$. Then

$$\sup_{r,k}(r+1)\,|\lambda_{r,k}(\mu)| = \sup_{0<x<1}|f(x)|\,, \tag{2.1}$$

as was mentioned in the introduction.

Let us now define a new sequence $\mu'=(\mu_k')_0^\infty$ by

$$\mu_0' = 0, \quad \mu_k' = -k\mu_{k-1}, \quad k=1,2,\ldots .$$

Then we introduce the „modulus of continuity''

$$\Omega(t,\mu) = \sup_{\substack{0<j<k<r\\ k-j<tr}}\left|\sum_{i=j+1}^{k}\lambda_{r,i}(\mu')\right|, \quad 0<t<1\,.$$

Our object is to prove

THEOREM 2.1. *For any $f\in\mathscr{C}$ and $0<t<1$ we have the inequalities*

$$\Omega(t,\mu) \leqq K(t,f)\,, \tag{2.2}$$

and

$$\omega(t,f) \leqq \Omega(t,\mu)\,. \tag{2.3}$$

For the proof we need

LEMMA 2.1. *For $0<i<r$ we have*

$$\sum_{i=j+1}^{k}\lambda_{r,i}(\mu') = r(\lambda_{r-1,k}(\mu)-\lambda_{r-1,j}(\mu))\,. \tag{2.4}$$

If μ is the moment sequence of a function f in $\mathscr{C}^1$ we have

$$\lambda_{r,i}(\mu') = \binom{r}{i}\int_0^1 t^i(1-t)^{r-i}f'(t)\,dt, \quad 0<i<r\,. \tag{2.5}$$

PROOF OF LEMMA 2.1. By the definition of $\lambda_{r,i}(\mu')$ we have

(2.6)
$$\lambda_{r,i}(\mu') = -\binom{r}{i}\sum_{l=0}^{r-i}(-1)^l\binom{r-i}{l}(i+l)\mu_{i+l-1}$$
$$= -i\binom{r}{i}\sum_{l=0}^{r-i}(-1)^l\binom{r-i}{l}\mu_{i+l-1}+$$
$$+(r-i)\binom{r}{i}\sum_{l=1}^{r-i}(-1)^{l-1}\binom{r-i-1}{l-1}\mu_{i+l-1}\,.$$

Thus
$$\lambda_{r,i}(\mu') = -r\binom{r-1}{i-1}\sum_{l=0}^{r-i}(-1)^l\binom{r-i}{l}\mu_{i-1+l}+$$
$$+r\binom{r-1}{i}\sum_{l=0}^{r-1-i}(-1)^l\binom{r-1-i}{l}\mu_{i+l}$$
$$= -r\lambda_{r-1,i-1}(\mu)+r\lambda_{r-1,i}(\mu)\,.$$

Now we get (2.4) by summation.

To prove (2.5) we note that (2.6) gives
$$\lambda_{r,i}(\mu') = -\binom{r}{i}\int_0^1\left(ix^{i-1}(1-x)^{r-i}-(r-i)x^i(1-x)^{r-i-1}\right)f(x)\,dx$$
$$= -\binom{r}{i}\int_0^1\left(\frac{d}{dx}x^i(1-x)^{r-i}\right)f(x)\,dx\,.$$

Integrating by parts we get (2.5), since f is bounded, and f' is continuous on $0<x<1$.

Proof of Theorem 2.1. We begin by proving (2.2). Let $f=f_0+f_1$, $f_0\in\mathscr{C}$, $f_1\in\mathscr{C}^1$ be any decomposition of f. Let μ_0 and μ_1 be the corresponding moment sequences, i.e.
$$\mu_{0k} = \int_0^1 x^k f_0(x)\,dx, \qquad \mu_{1k} = \int_0^1 x^k f_1(x)\,dx\,.$$

Then $\mu=\mu_0+\mu_1$ is the moment sequence of f. Since $\mu'=\mu_0'+\mu_1'$ and since the functional $\mu\to\lambda_{r,k}(\mu)$ is linear we have
$$\left|\sum_{i=j+1}^{k}\lambda_{r,i}(\mu')\right| \leqq \left|\sum_{i=j+1}^{k}\lambda_{r,i}(\mu_0')\right| + \left|\sum_{i=j+1}^{k}\lambda_{r,i}(\mu_1')\right|,$$

and consequently

$$\Omega(t,\mu) \leqq \Omega(t,\mu_0) + \Omega(t,\mu_1) \, . \tag{2.7}$$

Now formula (2.4) shows that

$$\sum_{i=j+1}^{k} \lambda_{r,i}(\mu_0') = r\left(\lambda_{r-1,k}(\mu_0) - \lambda_{r-1,j}(\mu_0)\right) .$$

From (2.1) we get however that $r|\lambda_{r-1,l}(\mu_0)| \leqq \|f_0\|$, and thus

$$\left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu_0') \right| \leqq 2\|f_0\| \, .$$

It follows that

$$\Omega(t,\mu_0) \leqq 2\|f_0\| \, . \tag{2.8}$$

From (2.5) we get

$$\lambda_{r,i}(\mu_1') = \binom{r}{i} \int_0^1 t^i (1-t)^{r-i} f_1'(t)\, dt, \qquad 0<i<r ,$$

and thus, again by (2.1),

$$(r+1)|\lambda_{r,i}(\mu_1')| \leqq \|f_1'\| = \|f_1\|_1, \qquad 0<i<r \, .$$

Consequently

$$\left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu_1') \right| \leqq \frac{k-j}{r} \|f_1\|_1 \leqq t\|f_1\|_1 ,$$

if $k-j<tr$. Thus

$$\Omega(t,\mu_1) \leqq t\|f_1\|_1 \, . \tag{2.9}$$

Combining (2.8) and (2.9) with (2.7) we get $\Omega(t,\mu) \leqq 2(\|f_0\| + t\|f_1\|_1)$, which gives (2.2).

Next we prove (2.3). We shall prove that if x and y are rational numbers, $0<x<y<1$, then

$$|f(x)-f(y)| = \lim_{\substack{r=sq\to\infty \\ j/r=x,\, k/r=y}} \left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu') \right|, \qquad x = p_1/q,\ y = p_2/q \, . \tag{2.10}$$

Having done this we can easily conclude the proof of (2.3). In fact, we note that since $f \in \mathscr{C}$ we have

$$\omega(t,f) = \sup_{\substack{0<y-x<t \\ 0<x<y<1}} |f(x)-f(y)| ,$$

where x and y are rational numbers; $x=p_1/q, y=p_2/q$. From (2.10) we then get

$$\omega(t,f) \leqq \sup_{0<y-x<t} \lim_{\substack{r=sq\to\infty \\ j/r=x,\, k/r=y}} \left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu') \right|$$

$$\leqq \lim_{s\to\infty} \sup_{\substack{0<j<k<r=sq \\ 0<k-j<tr}} \left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu') \right|$$

$$\leqq \sup_{\substack{0<j<k<r \\ k-j<tr}} \left| \sum_{i=j+1}^{k} \lambda_{r,i}(\mu') \right| = \Omega(t,\mu) .$$

It remains to prove (2.10). By formula (2.4) we see that it is enough to show that

$$r\lambda_{r-1,k}(\mu) \to f(x), \quad x = k/r, \quad r \to \infty . \tag{2.11}$$

However

$$r\lambda_{r-1,k}(\mu) = \int_0^1 \alpha_r(x,y) f(y)\, dy ,$$

where

$$\alpha_r(x,y) = r \binom{r-1}{rx} y^{rx} (1-y)^{r(1-x)-1} .$$

Since $\alpha_r(x,y) \geqq 0$ and

$$\int_0^1 \alpha_r(x,y)\, dy = 1 ,$$

we have only to prove

$$\lim_{r\to\infty} \int_{|x-y|\geqq\delta x} \alpha_r(x,y)\, dy = 0, \quad \delta > 0 . \tag{2.12}$$

To prove (2.12) we use Stirlings formula which gives

$$C^{-1}A_r(x) \leqq r \binom{r-1}{rx} \leqq CA_r(x),$$

where

$$A_r(x) = \frac{r^{\frac{1}{2}}}{x^{rx+\frac{1}{2}}(1-x)^{r(1-x)-\frac{1}{2}}} .$$

The constant C is positive and independent of x and r.

Let us first consider the integral

$$\int_0^{x(1-\delta)} \alpha_r(x,y)\,dy \leqq C A_r(x) \int_0^{x(1-\delta)} y^{rx}(1-y)^{r(1-x)-1}\,dy\,.$$

Substituting $y=x(1-u)$ the integral on the right hand side becomes

$$CA_r(x)x^{rx+1}\int_\delta^1 (1-u)^{rx}(1-x-xu)^{r(1-x)-1}\,du$$

$$= Cr^{\frac{1}{2}}x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\int_\delta^1 (1-u)^{rx}\left(1+\frac{x}{1-x}u\right)^{r(1-x)-1} du$$

$$\leqq C' r^{\frac{1}{2}}\int_\delta^1 (1-u)^{rx}\left(1+\frac{x}{1-x}u\right)^{r(1-x)} du\,.$$

Since

$$rx\log(1-u) + r(1-x)\log\left(1+\frac{x}{1-x}u\right)$$

$$\leqq rx(-u-\tfrac{1}{2}u^2) + r(1-x)\left(\frac{x}{1-x}u\right) = -rx\tfrac{1}{2}u^2,$$

we finally get the estimate

$$\int_0^{x(1-\delta)} \alpha_r(x-y)\,dy \leqq C' r^{\frac{1}{2}}\int_\delta^1 e^{-\frac{1}{2}rxu^2}\,du\,.$$

It is clear that the right hand side tends to zero as $r\to\infty$, for any $\delta>0$.

In the same way we get

$$\int_{x(1+\delta)}^1 \alpha_r(x,y)\,dy \leqq Cr^{\frac{1}{2}}x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\int_{x\delta/(1-x)}^1 \left(1+\frac{1-x}{x}u\right)^{rx}(1-u)^{r(1-x)}\,du$$

$$\leqq C' r^{\frac{1}{2}}\int_{x\delta/(1-x)}^1 e^{-\frac{1}{2}ru^2(1-x)}\,du\,.$$

Since the right hand side tends to zero as $r\to\infty$ we conclude

$$\int_{|y-x|\geqq x\delta} \alpha_r(x,y)\,dy \to 0 \quad \text{as } r\to\infty\,.$$

Our theorem is proved.

Combining theorem 2.1 with theorem 1.1 we get

COROLLARY 2.1. *For any* $f \in \mathscr{C}$ *and* $0 < t < 1$ *we have*

$$\tag{2.13} \tfrac{1}{2}K(t,f) \leqq \Omega(t,\mu) \leqq K(t,f)\,,$$

$$\tag{2.14} \omega(t,f) \leqq \Omega(t,\mu) \leqq 2\,\omega(t,f)\,.$$

In particular μ *is the moment sequence of a function* $f \in \mathrm{Lip}\,(\theta,q)$ *if and only if*

$$\tag{2.15} \int_0^1 \left(\frac{\Omega(t,\mu)}{t^\theta}\right)^q \frac{dt}{t} < \infty, \qquad 1 \leqq q < \infty\,,$$

$$\tag{2.16} \sup_{0<t<1} \frac{\Omega(t,\mu)}{t^\theta} < \infty, \qquad q = \infty\,.$$

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