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# ON TWO CONSEQUENCES OF A THEOREM OF HOFFMAN AND WERMER

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## 1.

Let $A$ be a closed separating subalgebra of $C(X)$, with $X$ compact and $1 \in A$. Some time ago Hoffman and Wermer [6] showed that the set of real parts $\operatorname{Re} A$ cannot be closed in $C^{\mathsf{R}}(X)$ unless $A = C(X)$; using this result, Sidney and Stout [8] have recently obtained the following extension:

*For any closed set $F \subset X$, the restriction of* $\operatorname{Re} A$ *to* $F$, $\operatorname{Re} A \mid F$, *is not closed in* $C^{\mathsf{R}}(F)$ *unless* $A \mid F = C(F)$.

Thus the set of real parts is not uniformly closed on any closed subset of $X$ which is not a set of interpolation for $A$.

Along with the Hoffman–Wermer result Sidney and Stout used several known facts, in particular a criterion for interpolation given in [5]. The first purpose of the present note is to point out a simpler proof of the Sidney–Stout theorem, based on the cited result of [5] and a real linear analogue, as well as the Hoffman–Wermer result itself. (Since this was written a much more general result has been obtained by A. Bernard in C. R. Acad. Sci. 267, 634–635.) Secondly we give another consequence of the Hoffman–Wermer theorem which asserts that $A + \bar{I}$ is closed for a closed ideal $I$ in $A$ only for conjugate closed ideals.

## 2. Proof of the Sidney-Stout theorem.

Suppose $\operatorname{Re} A \mid F$ $\big(= (\operatorname{Re} A) \mid F = \operatorname{Re}(A \mid F)\big)$ is closed in $C^{\mathsf{R}}(F)$. The closure $(A \mid F)^-$ of $A \mid F$ in $C(F)$ then clearly has

$$\operatorname{Re}(A \mid F)^- \subset (\operatorname{Re} A \mid F)^- = \operatorname{Re} A \mid F \subset \operatorname{Re}(A \mid F)^- ,$$

so $\operatorname{Re}(A \mid F)^-$ is closed in $C^{\mathsf{R}}(F)$. By the Hoffman–Wermer result [3, 6] $(A \mid F)^- = C(F)$, so

$$\operatorname{Re} A \mid F = \operatorname{Re}(A \mid F)^- = C^{\mathsf{R}}(F) ,$$

as in the Sidney–Stout proof [8].

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Thus the real linear map $a \to \operatorname{Re} a|F$ is onto $C^{\mathsf{R}}(F)$, which has topological import for the adjoint map which takes $M^{\mathsf{R}}(F)$, the space of real measures on $F$, into the real dual of $A$. As a real vector space $C(X)$ is the direct sum of two copies of $C^{\mathsf{R}}(X)$, and so has a real dual $C(X)^{*\mathsf{R}}$ isomorphic to $M^{\mathsf{R}}(X) \oplus M^{\mathsf{R}}(X)$; hence each element of the dual is a pair $(\lambda, \nu)$ of measures, which takes $u + iv \in C(X)$, for $u, v$ real valued, into

$$\lambda(u) - \nu(v) = \operatorname{Re}[(\lambda + i\nu)(u + iv)] .$$

The functional norm is evidently equivalent to (but $\leqq$) $\|\lambda\| + \|\nu\|$.

Clearly the real dual $A^{*\mathsf{R}}$ of $A$ is the quotient of $C(X)^{*\mathsf{R}}$ modulo the subspace orthogonal to $A$:

$$N = \{(\lambda, \nu) : \operatorname{Re}(\lambda + i\nu)(a) = 0,\ a \in A\} .$$

Now the fact that $a \to \operatorname{Re} a|F$ is onto $C^{\mathsf{R}}(F)$ implies the $(1-1)$ adjoint has closed range [4, p. 488], hence is topological by the open mapping theorem. But the adjoint takes the element $\mu$ in $M^{\mathsf{R}}(F)$ into $(\mu, 0) + N$; thus $\theta\|\mu\| \leqq \|(\mu, 0) + N\|$ for some $\theta$, $0 < \theta < 1$, and

$$\theta\|\mu\| \leqq \|\mu + \lambda\| + \|\nu\| \quad \text{if} \quad \operatorname{Re}(\lambda + i\nu)(a) = 0,\ a \in A .$$

Setting $\mu = -\lambda_F$, we obtain in particular

$$\theta\|\lambda_F\| \leqq \|\lambda_{F'}\| + \|\nu\| \quad \text{for} \quad \lambda + i\nu \in A^{\perp}, \quad \lambda, \nu \text{ real} . \tag{2.1}$$

Now if $A|F \neq C(F)$ we conclude from [5, 3.2] that for each $n \geqq 1$ we have a measure $\lambda + i\nu$ in $A^{\perp}$ for which

$$\|(\lambda + i\nu)_F\| > n\|(\lambda + i\nu)_{F'}\| . \tag{2.2}$$

From the open mapping theorem and the fact that $\operatorname{Re} A|F = C^{\mathsf{R}}(F)$ we have a constant $k$ for which, for each $u \in C^{\mathsf{R}}(F)$ there is an $a$ in $A$ with

$$\operatorname{Re} a = u \quad \text{on } F, \qquad \|a\| \leqq k\|u\|; \tag{2.3}$$

in particular this is true for $u$ in $C^{\mathsf{R}}(X)$.

For $\lambda + i\nu$ as in (2.2) we have $\lambda + i\nu = \varrho|\lambda + i\nu|$ where $\varrho$ is a unimodular Baire function, and for $\eta > 0$ by Lusin's and Tietze's theorems we can find a continuous function $p$ on the support of $|\lambda + i\nu|$ coinciding with $\varrho$ except on a set of $|\lambda + i\nu|$-measure $< \eta$, and with values in $\Gamma \setminus I$, where $\Gamma$ is the unit circle in $\mathsf{C}$ and $I$ is a small arc. Extending $p$ continuously to all of $X$, with values in $\Gamma \setminus I$, we have a well-defined element $u = i \log p$ of $C^{\mathsf{R}}(X)$, and then an $a$ in $A$ satisfying (2.3) for that $u$.

In the support of $|\lambda + i\nu|$ less our set of measure $< \eta$,

$$e^{ia}\varrho = e^{-\operatorname{Im} a} e^{-\log p} \varrho = e^{-\operatorname{Im} a} > 0 ,$$

so that $\lambda^* + i\nu^* = e^{ia}(\lambda + i\nu)$ is an element of $A^\perp$ real valued (hence $=\lambda^*$) off that set, and

$$\|(\lambda^* + i\nu^*)_F - \lambda^*_F\| = \|\nu^*_F\| \leqq \eta\|e^{ia}\| \leqq \eta e^{k2\pi} = \eta c \tag{2.4}$$

since

$$\|e^{ia}\| \leqq e^{\|a\|} \leqq e^{k\|\log p\|} \leqq e^{k2\pi}$$

by (2.3). Since $c^{-1} \leqq |e^{ia}| \leqq c$ we have

$$c\eta + \|\lambda^*_F\| \geqq \|(\lambda^* + i\nu^*)_F\| \geqq c^{-1}\|(\lambda + i\nu)_F\| > 0$$

by (2.2), and thus taking $\eta$ sufficiently small we obtain a $\lambda^* + i\nu^*$ satisfying

$$c\eta < \tfrac{1}{2}\theta\|\lambda^*_F\| < \|\lambda^*_F\|$$

as well, so by (2.2)

$$\|\lambda^*_F\| \geqq \tfrac{1}{2}c^{-1}\|(\lambda + i\nu)_F\| > \tfrac{1}{2}nc^{-1}\|(\lambda + i\nu)_{F'}\| \geqq \tfrac{1}{2}nc^{-2}\|(\lambda^* + i\nu^*)_{F'}\| . \tag{2.5}$$

Now by (2.1), applied to $\lambda^* + i\nu^* \in A^\perp$, and the fact that

$$\|\nu^*_F\| \leqq c\eta < \tfrac{1}{2}\theta\|\lambda^*_F\|$$

(cf. (2.4)) we have

$$\|\lambda^*_{F'}\| + \|\nu^*_{F'}\| + \tfrac{1}{2}\theta\|\lambda^*_F\| \geqq \|\lambda^*_{F'}\| + \|\nu^*\| \geqq \theta\|\lambda^*_F\|$$

so that $\|\lambda^*_{F'}\| + \|\nu^*_{F'}\| \geqq \frac{1}{2}\theta\|\lambda^*_F\|$. From (2.5) we now have

$$\begin{aligned} 2\|(\lambda^* + i\nu^*)_{F'}\| &\geqq \|\lambda^*_{F'}\| + \|\nu^*_{F'}\| \\ &\geqq \tfrac{1}{2}\theta\|\lambda^*_F\| > \tfrac{1}{4}n\theta c^{-2}\|(\lambda^* + i\nu^*)_{F'}\| , \end{aligned}$$

so that $8c^2 > n\theta$ for all $n$, our contradiction, which completes the proof.

## 3.

A trivial consequence of the Hoffman–Wermer theorem is that $A + \bar{A}$ is closed only if $A = C(X)$ (where the bar denotes conjugation). Another consequence is a variant which is not trivial:

**Theorem 3.1.** *Suppose $X$ is metric and $I \subset A$ is a closed ideal. Then $A + \bar{I}$ is closed (if and) only if $\bar{I} = I \subset A$.*

Thus $A + \bar{I}$ closed implies $A$ contains all continuous functions vanishing on the hull $hI$ of $I$ (by Stone–Weierstrass), so that $hI$ contains the essential set for $A$ [1]; in particular if $A$ is an essential algebra on $X$ then $A + \bar{I}$ is closed only for the trivial ideal.

Our proof of 3.1 uses some variants of arguments given in [5]. First

LEMMA 3.2. *Suppose* $\nu \in \mathscr{E}$, *the set of extreme points of* $\mathrm{ball}(A+\bar{I})^{\perp}$, *the unit ball of the measures orthogonal to* $A+\bar{I}$. *Then the carrier* $K$ *of* $\nu$ *is a set of antisymmetry for* $A$.

Suppose there is a $g$ in $A$ non-constant and real on $K$; we can of course assume $0<g<1$ on $K$. Then $g\nu \perp A$, and since $\overline{gI} \subset \bar{I}$ while $g=\bar{g}$ on $K$ we have $g\nu = \bar{g}\nu \perp \bar{I}$. So $g\nu \perp (A+\bar{I})$, and exactly as in [5, 2.1] we conclude $\nu$ cannot be extreme.

Let $\mathscr{K}$ denote the collection of maximal sets of antisymmetry for $A$, as in [5]. For $\nu \in \mathscr{E}$ and $K \in \mathscr{K}$ we have $\nu_K = \nu$ or $=0$ by 3.2, so $\nu_K \perp (A+\bar{I})$ for each $\nu \in \mathscr{E}$. By the Bishop–de Leeuw theorem [2, 7] we have each $\mu$ in $(A+\bar{I})^{\perp}$ given by an integral

$$\mu = \int_{\mathscr{E}} \nu \, \lambda(d\nu)$$

so that, since $X$ is metric and thus $K \in \mathscr{K}$ Baire, exactly as in [5, 3.3] we have

$$\mu_K = \int_{\mathscr{E}} \nu_K \, \lambda(d\nu) \, ,$$

hence orthogonal to $A+\bar{I}$; again as in [5, 3.3] this implies $(A+\bar{I})\,|\,K$ is closed in $C(K)$.

Now suppose for the moment $A$ were antisymmetric. The real valued elements of $A+\bar{I}$ form the closed subspace $(A+\bar{I}) \cap C^{\mathsf{R}}(X)$ of $C^{\mathsf{R}}(X)$, and $f \in (A+\bar{I}) \cap C^{\mathsf{R}}(X)$ implies $f = a+\bar{b} = \mathrm{Re}\,a + \mathrm{Re}\,b$, $\mathrm{Im}\,a = \mathrm{Im}\,b$, where $a \in A$, $b \in I$. Since $a-b$ is real valued it is constant and $f = b+\bar{b}+r$, $r \in \mathsf{R}$. Thus $(A+\bar{I}) \cap C^{\mathsf{R}}(X) \subset \mathrm{Re}(I+\mathsf{C})$. But $\mathrm{Re}\,I \subset I+\bar{I} \subset A+\bar{I}$, so

$$\mathrm{Re}(I+\mathsf{C}) = (A+\bar{I}) \cap C^{\mathsf{R}}(X) \, .$$

Since $\mathrm{Re}(I+\mathsf{C})$ is thus uniformly closed, as is $I+\mathsf{C}$, the algebra $\mathsf{C}+I$ satisfies the hypotheses of the Hoffman–Wermer theorem (except for separation of $X$), so that result implies $\mathsf{C}+I$ is self-adjoint. Since $A$ is antisymmetric $\mathsf{C}+I = \mathsf{C}$, and either $I=0$, or $I=\mathsf{C}$ and thus $A=\mathsf{C}$.

Returning to the general case, the fact that $(A+\bar{I})\,|\,K$ is closed for each $K \in \mathscr{K}$, while $A\,|\,K$ and $I\,|\,K$ are also [5, 1.1 and 2.5], allows us to apply our conclusion for $A$ antisymmetric to $A\,|\,K$ and the ideal $I\,|\,K$: we have $I\,|\,K=0$, or $A\,|\,K=\mathsf{C}$ whence $K$ is a singleton. In any case $\bar{I}\,|\,K = I\,|\,K$ for each $K \in \mathscr{K}$, so $\bar{I} \subset I$ by [5, 2.5], and $\mathrm{Re}\,I \subset I$,

$$I \subset \mathrm{Re}\,I + i\,\mathrm{Re}\,I \subset I$$

follow, so $I = \bar{I}$.

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