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ON THE RADICAL THEORY OF A DISTRIBUTIVELY GENERATED NEAR-RING

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Introduction.

The following concepts have been defined in connection with the radical of a distributively generated (d.g.) near-ring R satisfying the descending chain condition (d.c.c.) for left R-modules [7], [8]:

(i) The radical J itself, which contains all nilpotent left R-modules and for which the factor d.g. near-ring R/J is semi-simple.

(ii) The quasi-radical Q, which is a nilpotent left ideal containing all the nilpotent left ideals of R.

(iii) The ideal-radical P, which is a nilpotent (two sided) ideal containing all the nilpotent ideals of R.

We have the inclusions $J \supseteqq Q \supseteqq P$. If $J = Q$, then all three are equal and this occurs, in particular, when R is a ring.

In [8] R. R. Laxton gave an example of a finite d.g. near-ring with identity in which all three are distinct. There remains the possibility, discussed in [8], that Q be equal to P without at the same time being the radical J. We will give an example of a finite d.g. near-ring in which this occurs. Such a near-ring R has non-zero nilpotent left R-modules but no non-zero nilpotent left ideals. This shows that the radical need not be the least two-sided ideal containing all the nilpotent left ideals of R.

These examples lead us to introduce a further ideal S of R such that the factor near-ring R/S has no non-zero nilpotent left ideals (though it will, in general, contain non-zero nilpotent left (R/S)-modules). The near-rings R for which the ideal $S = (0)$ are of special interest since they have, after the semi-simple ones, the simplest structure. We will discuss the structure of these near-rings in a later paper.

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Throughout the paper we will assume that the near-ring R has a multiplicative identity.

1. s-primitivity.

Let R be a (right) d.g. near-ring (cf. [7]). We refer the reader to [7] for the concepts (R, U)-group (where U is some distributive semi-group generating R), R-group, faithful R-group, cyclic R-group and R-subgroup of an R-group.

A *minimal* R-group is a non-zero R-group which contains no proper, non-zero R-subgroups. An *irreducible* R-group is a non-zero R-group which contains no proper, non-zero normal R-subgroups (see [8]). Of special interest are the cyclic irreducible R-groups. Clearly a minimal R-group is also cyclic irreducible.

A primitive (*primitively prime*) d.g. near-ring R is a d.g. near-ring which has a faithful representation on a minimal (respectively, cyclic irreducible) R-group (cf. [8]).

The concepts of R-homomorphisms, near-ring homomorphisms, left (right) R-modules and left (right, two sided) ideals of a d.g. near-ring R are given in [3], [4].

An ideal A of a d.g. near-ring R is called *primitive* (*primitively prime*) if the factor near-ring R/A is primitive (respectively, primitively prime).

An ideal C of R is called *prime* if whenever A, B are ideals of R and $AB \subseteq C$, then either A or B is contained in C. A d.g. near-ring is called prime if its zero ideal is prime. In the class of d.g. near-rings R which satisfy the d.c.c. for left R-modules, primitive is equivalent to simple and primitively prime is equivalent to prime [7], [8].

The radical J (ideal-radical P) of R is defined to be the intersection of all primitive (resp., primitively prime) ideals of R. The quasi-radical Q is defined to be the intersection of all the maximal left ideals of R.

DEFINITION 1. A cyclic irreducible R-group Ω is called *s-irreducible* if every non-zero cyclic R-subgroup of Ω is a direct sum of cyclic irreducible R-subgroups. A d.g. near-ring R will be called *s-primitive* if it has a faithful representation on an s-irreducible R-group and an ideal A of R will be called s-primitive if the factor near-ring R/A is s-primitive.

Clearly a minimal R-group is s-irreducible and so a primitive near-ring (ideal) is s-primitive and an s-primitive near-ring (resp., ideal) is prime.

In order to prove theorem 1 we shall need the following lemma (cf. [8]):

LEMMA. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. If A is an intersection of maximal left ideals, then the R-group*

$$R^+ - A = \sum_{i=1}^{k} \oplus \mathfrak{l}_i$$

where each $\mathfrak{l}_i$ is a cyclic irreducible R-group.

We point out that if A is an (two sided) ideal and $e = e_1 + e_2 + \ldots + e_k$ where e_i is an R/A generator of $\mathfrak{l}_i$ for $i = 1, \ldots, k$ and e the multiplicative identity of R/A, then $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$ (see [5]).

THEOREM 1. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. A prime ideal C of R is s-primitive if, and only if, it is an intersection of maximal left ideals.*

PROOF. Let C be a prime ideal which is an intersection of maximal left ideals; without loss of generality we may assume $C = (0)$. Then by the above lemma we have

$$R = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \ldots \oplus \mathfrak{l}_r \tag{1}$$

where each left ideal $\mathfrak{l}_i$ is a cyclic irreducible R-group with generator e_i. Since R is prime and satisfies the d.c.c. it is primitively prime [8] and so there is a faithful, cyclic irreducible R-group Ω. For any $\omega \in \Omega$ we have

$$R\omega = \mathfrak{l}_1 \omega + \mathfrak{l}_2 \omega + \ldots + \mathfrak{l}_r \omega . \tag{2}$$

Since each $\mathfrak{l}_i$ is normal in R^+, each $\mathfrak{l}_i \omega$ is normal in $R\omega$. The mappings $x \to x\omega$ of R onto $R\omega$ is an R-homomorphism and induces a map of $\mathfrak{l}_i$ into $\mathfrak{l}_i \omega$. As each $\mathfrak{l}_i$ is irreducible we have $\mathfrak{l}_i \cong \mathfrak{l}_i \omega$ in which case $\mathfrak{l}_i \omega$ is cyclic irreducible, or $\mathfrak{l}_i \omega = 0$. In either case, we have either

$$\mathfrak{l}_i \omega \subseteq \sum_{j \neq i} \mathfrak{l}_j \omega \quad \text{or} \quad \mathfrak{l}_i \omega \cap \sum_{j \neq i} \mathfrak{l}_i \omega = 0 .$$

Hence, if $\omega \neq 0$ it follows that $R\omega$ is a direct sum by dropping (if necessary) some of the $\mathfrak{l}_i \omega$ in (2). Therefore $R\omega$ is a direct sum of cyclic irreducible R-groups for every $\omega \in \Omega$, $\omega \neq 0$. Hence, Ω is s-irreducible.

Now assume that C is s-primitive. Again we may take C to be the zero ideal. Let Ω be a faithful s-irreducible R-group. If ω is a non-zero element of Ω, then $R\omega$ is a direct sum of cyclic, irreducible R-groups and we may write

$$R\omega = R\omega_1 \oplus R\omega_2 \oplus \ldots \oplus R\omega_r \tag{3}$$

for some $\omega_i \in \Omega$, where each $R\omega_1$ is cyclic irreducible. We will prove that the left ideal

$$\mathfrak{l}(\omega) = \{x \in R \text{ such that } x\omega = 0\}$$

is equal to an intersection of maximal left ideals of R. Since ω is an arbitrary non-zero element of Ω and R acts faithfully on Ω this will prove that the zero ideal is an intersection of maximal left ideals and the proof will be complete.

From (3) we may write $\omega = e_i\omega_i + \ldots + e_r\omega_r$ for some $e_i \in R$. Consequently,

$$\begin{aligned} R\omega &= \{y(e_i\omega_i + \ldots + e_r\omega_r) \text{ for all } y \in R\} \\ &= \{ye_1\omega_1 + \ldots + ye_r\omega_r \text{ for all } y \in R\}. \end{aligned}$$

(The left distributive law is valid in this case since the sum (3) is direct (cf. [2], [5]).) But

$$Re_i\omega_i \subseteqq R\omega_i \quad \text{and so} \quad Re_i\omega_i = R\omega_i \quad \text{for } i = 1,\ldots,r.$$

Hence,

$$R\omega = Re_i\omega_i + \ldots + Re_r\omega_r,$$

and if

$$0 = x\omega = xe_1\omega_1 + \ldots + xe_r\omega_r$$

we have $xe_i\omega_i = 0$ for all $i = 1,\ldots,r$. Thus

$$\mathfrak{l}(\omega) = \bigcap_{i=1}^{r} \mathfrak{l}(e_i\omega_i)$$

where each $\mathfrak{l}(e_i\omega_i)$ is a maximal left ideal of R since $e_i\omega_i$ is an R-generator of the irreducible R-group $R\omega_i$.

THEOREM 2. *Let R be a d.g. near-ring satisfying the d.c.c. for left R-modules. If the quasi-radical Q of R is the zero ideal, then every ideal of R is an intersection of s-primitive ideals.*

PROOF. Since Q, the intersection of all maximal left ideals of R, is the zero ideal we have

$$R^+ = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \ldots \oplus \mathfrak{l}_r,$$

by the lemma. We can write $e = e_1 + e_2 + \ldots + e_r$ where $e_i \in \mathfrak{l}_i$, $\mathrm{R}e_i = \mathfrak{l}_i$ for $i = 1,\ldots,r$, and e is the multiplicative identity of R. The e_i are such that $e_ie_j = 0$ if $i \neq j$ and $e_i^2 = e_i$, by the remark following the lemma.

Let A be any ideal of R. For any $a \in A$, $a = x_1e_1 + \ldots + x_re_r$ and so $x_ie_i = x_ie_i^2 = ae_i \in A$ because A is a right ideal. Consequently,

$$A = (\mathfrak{l}_1 \cap A) \oplus (\mathfrak{l}_2 \cap A) \oplus \ldots \oplus (\mathfrak{l}_r \cap A).$$

Since the $\mathfrak{l}_i$ are irreducible, $\mathfrak{l}_i \cap A = (0)$ or $\mathfrak{l}_i \cap A = \mathfrak{l}_i$ and so, reindexing if necessary, we may write

$$A = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \ldots \oplus \mathfrak{l}_d$$

for some d, $0 \leqq d \leqq n$. Thus

$$R = \mathfrak{l}_{d+1} \oplus \ldots \oplus \mathfrak{l}_r \oplus A$$

and so

(4) $$R/A = \mathfrak{l}'_{d+1} \oplus \ldots \oplus \mathfrak{l}_r'$$

where the $\mathfrak{l}_i'$ are cyclic irreducible R-groups. It is easily shown that

$$A = \bigcap_{i=d+1}^{r} (0:\mathfrak{l}_i')$$

where each $(0:\mathfrak{l}_i')$ is prime [8]. For any prime ideal B of R we have a decomposition as in (4) above and consequently it is s-primitive. This proves the theorem.

2. The s-radical.

DEFINITION 2. The *s-radical* S of a d.g. near-ring R is the intersection of all the s-primitive ideals of R.

It is clear that we have the inclusions $J \supseteqq S \supseteqq Q \supseteqq P$. Using theorem 2 we obtain

THEOREM 3. *The left ideal* Q *is an ideal if, and only if,* $S = Q = P$.

The s-radical of a d.g. near-ring R is precisely the intersection of those ideals A of R such that the factor near-ring R/A has no non-zero nilpotent left ideals. Again this is a ready consequence of theorem 2.

We point out that if the s-radical of a d.g. near-ring R, which satisfies the d.c.c. for left R-modules, is the zero ideal, then the left ideal structure of R is

$$R = \mathfrak{l}_1 \oplus \ldots \oplus \mathfrak{l}_r \oplus \mathfrak{l}_{r+1} \oplus \ldots \oplus \mathfrak{l}_n$$

where each left ideal $\mathfrak{l}_i$ is an s-irreducible R-group. Furthermore, if $\mathfrak{l}_1, \ldots, \mathfrak{l}_r$ are *all* the minimal left R-modules among these left ideals, then $J = \mathfrak{l}_{r+1} \oplus \ldots \oplus \mathfrak{l}_n$.

3. Some examples.

We shall give an example of a finite d.g. near-ring with identity which is s-primitive but not simple; this will then show that there exist finite d.g. near-rings R with $J \neq S = Q = P$ and thereby answers in the affirmative the question posed in [8].

Consider the alternating group A_6 on the six symbols $\{1,2,3,4,5,6\}$ and the alternating group A_5 on the symbols $\{1,2,3,4,5\}$ which we regard as a subgroup of A_6. Let U be the semi-group of all inner automorphisms of the symmetric group S_6 which induce automorphisms on A_5. Thus U consists of the maps

$$\Phi_x : a \rightarrow x+a-x \quad \text{for} \quad a \in S_6$$

where x is any sum of cycles of S_6 involving only the symbols $\{1,2,3,4,5\}$. (We are using the additive notation for S_6.) Let R be the d.g. near-ring generated by U (cf. [3]). Then clearly A_5 is a minimal R-group. It is easily shown that A_5 is the only proper, non-zero R-subgroup of A_6 and A_6 is a cyclic irreducible R-group (by construction it is faithful). Hence, A_6 is an s-irreducible R-group which is not minimal.

In [8] a large class of finite d.g. near-rings was constructed in which $J \neq Q \supseteqq P$ and it was shown that among them were near-rings with $J \neq Q \neq P$. We mention that in this class there are also near-rings with $J = S \neq Q \neq P$. (For example with the notation of [8], section 4, take $\Omega = A_5$ and use the fact that every proper subgroup of A_5 is soluble.)

4. Concluding remarks.

The theory of s-primitivity can readily be extended to general (not necessarily distributively generated) near-rings. This was done in [5]. In this wider class of near-rings it is a relatively easy matter to construct s-primitive near-rings which are not simple.

It is an open question whether or not the s-radical S is the least ideal containing the quasi-radical Q of a d.g. near-ring.

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