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A CLASS OF RINGS HAVING ALL SINGULAR SIMPLE MODULES INJECTIVE

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A ring R with unit will be called a *T-ring* if every nonzero R-module has nonzero socle (all modules are unitary left R-modules). In this paper we investigate T-rings having the following property:

(*) Every singular simple module is injective.

This is of course equivalent to the condition that every simple module is either projective or injective. Theorem 1.1 gives necessary and sufficient conditions for a T-ring to have property (*) and from this we deduce some properties of essential left ideals of R (Theorem 1.2). Applying these results we show that a right perfect ring R has property (*) if and only if R is hereditary and $N^2=0$, where N is the radical of R, while a commutative T-ring has (*) if and only if it is regular. Moreover a T-ring with (*) is regular if and only if $N=0$. In Section 2, we examine R_n the ring of $n\times n$ matrices over R, $n \geqq 1$, and show that if R is a T-ring, then so is R_n while if R is a T-ring, then R satisfies (*) if and only if R_n satisfies (*).

1. Singular simple injectives.

For an R-module M, let $Z(M)$ denote the singular submodule of M. Thus

$$Z(M) \;=\; \{m \in M \mid (0:m) \text{ is essential in } R\}\,,$$

where [7]

$$(0:m) \;=\; \{x \in R \mid xm=0\}\,.$$

As is known, $Z({}_RR)$ is an ideal of R containing no nonzero idempotent elements. An R-module M is *singular* if $Z(M)=M$. For any R-module M, $\operatorname{rad}(M)$ denotes the intersection of the maximal submodules of M and

$$N \;=\; \operatorname{rad}({}_RR) \;=\; \text{(Jacobson) radical of } R\,.$$

For an R-module M, $R\text{-}\operatorname{soc}(M)$ denotes the R-socle of M, which is the sum of the simple submodules of M.

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THEOREM 1.1. *For a T-ring R the following are equivalent:*

a) *All singular simple R-modules are injective.*
b) $Z({}_RR)=0$, and $\operatorname{rad}(R/I)=0$ for any essential left ideal I of R.

PROOF. Assume a). If $S \subseteq Z({}_RR)$ is a simple R-module, then $Z(S)=S$, hence S is injective and thus a direct summand of R. Since $Z({}_RR)$ has no nonzero idempotents we conclude that $S=0$ and so $Z({}_RR)=0$. If I is an essential left ideal of R then R/I is a singular module and thus so is every submodule of R/I. Hence every simple submodule of R/I is injective, and so every simple submodule of R/I is excluded by some maximal submodule of R/I. It follows that $\operatorname{rad}(R/I)=0$.

Conversely, suppose b) holds and let S be a singular simple module. In order to show that S is injective we must show that for every left ideal I of R, every $f \in \operatorname{Hom}_R(I,S)$ extends to some $g \in \operatorname{Hom}_R(R,S)$, and for this we may assume I is essential. If $K=\operatorname{Ker} f$, then K is an essential left ideal of R, for suppose $K \cap J=0$ for $J \neq 0$ a left ideal of R. Then

$$J' = J \cap I \neq 0 \quad \text{and} \quad J' \cap K = 0 .$$

Hence $J' \approx f(J') \subseteq S$ is singular contrary to $Z({}_RR)=0$. Now f induces $f^*: I/K \to S$ which is an isomorphism if $f \neq 0$, and so $I/K \neq 0$ is a simple submodule of R/K. Since $\operatorname{rad}(R/K)=0$ there is a maximal submodule M/K with $I/K \nsubseteq M/K$. Hence $I/K \cap M/K=0$ and so

$$R/K = I/K \oplus M/K .$$

Let $h: R \to R/K$ be the natural map and $p: R/K \to I/K$ the projection map. Then $g: R \to S$ given by $g=f^*ph$ is the desired map.

We use this result to obtain some properties of essential left ideals.

THEOREM 1.2. *Let R be a T-ring having all singular simple R-modules injective. For any essential left ideal I of R, $I^2=I$ and I is an intersection of maximal left ideals. Moreover, $N^2=0$.*

PROOF. If I is an essential left ideal of R by b) above, $\operatorname{rad}(R/I)=0$. If A and B are R-modules and $f \in \operatorname{Hom}_R(A,B)$ is an epimorphism, then

$$f(\operatorname{rad} A) \subseteq \operatorname{rad} B .$$

Thus in our case $N \subseteq I$ for every essential left ideal I of R. Since the intersection of all essential left ideals of R is the socle of R, we have $N \subseteq R\text{-soc}(R)$ and so $N^2=0$. Note also that by b) every essential left ideal is an intersection of maximal left ideals. Finally, suppose $I^2 \neq I$ for I an essential left ideal of R. Now $Z({}_RR)=0$ so the essential left ideals of R form an idempotent filter [6]. Thus I^2 is essential and is an intersection of maximal left ideals. If $x \in I$, $x \notin I^2$, there is a maximal left ideal $M \supseteqq I^2$ with $x \notin M$. Hence $R = Rx + M$, so $l = rx + m$ implies $x = xrx + xm \in M$, a contradiction. Thus $I^2 = I$.

Since right perfect rings are T-rings [2], we now have

THEOREM 1.3. *For a right perfect ring R the following are equivalent:*

(a) *All singular simple R-modules are injective.*
(b) *R is left-hereditary and $N^2 = 0$.*

PROOF. Assume (a); then by Theorem 1.2, $N^2 = 0$. We will show that any singular module is injective. Let M be a singular module, I an essential left ideal of R and $f \in \mathrm{Hom}_R(I, M)$. Then as in Theorem 1.1, $K = \mathrm{Ker} f$ is essential and so $N \subseteqq K$. Now R/N is a completely reducible R-module since R is right perfect, and so $R/K \cong (R/N)/(K/N)$ is a completely reducible R-module. Thus I/K is a direct summand of R/K and as in Theorem 1.1, this yields an extension $g \in \mathrm{Hom}_R(R, M)$ of f. Thus M is injective as claimed. Now for any R-module A, if $E(A)$ denotes the minimal injective containing A, then $E(A)/A$ is singular and hence injective. Thus $\mathrm{l.gl.dim}\, R \leqq 1$ and so R is left-hereditary by [4].

Now suppose (b) holds. Since R is hereditary, as in [7], for any $x \in R$, the sequence

$$0 \to (0:x) \to R \to Rx \to 0$$

splits and so $Z({}_R R) = 0$. Since $N^2 = 0$ and R/N is semisimple with d.c.c., we conclude that $N \subseteqq \mathrm{socle}\, R$. But any essential left ideal I of R contains the socle of R and so $N \subseteqq I$. It follows that R/I is a completely reducible R-module for any essential left ideal I of R and so $\mathrm{rad}(R/I) = 0$. By Theorem 1.1, all singular simple R-modules are injective.

An interesting consequence of the proof of Theorem 1.3 is the following

COROLLARY 1.4. *A semiprimary ring R with $N^2 = 0$ is hereditary if and only if $Z({}_R R) = 0$.*

We note that Theorem 1.3 has been obtained by A. Zaks [13] for semiprimary rings, however the methods used are distinct from ours; the methods used in our proofs of Theorems 1.1 and 1.3 are similar to those in [5].

Looking next at commutative T-rings we first have

PROPOSITION 1.5. *For any commutative ring R, the following are equivalent:*

a) *R is regular.*
b) *$I^2=I$ for each essential ideal I of R.*
c) *$I^2=I$ for each ideal I of R.*

PROOF. The equivalence of a) and c) is well known and certainly a) implies b). Thus we need only show that b) implies c). If I is a non-essential ideal of R choose A maximal with respect to $I \cap A=0$. Then $I+A$ is an essential ideal of R and so

$$I+A = (I+A)^2 = I^2+A^2 .$$

Hence if $x \in I$,

$$x = \sum x_i y_i + \sum a_i b_i ,$$

where $x_i, y_i \in I$ and $a_i, b_i \in A$. Then

$$x-\sum x_i y_i \in I \cap A = 0$$

and so $x \in I^2$. Thus $I^2=I$.

THEOREM 1.6. *If R is a commutative T-ring, all singular simple R-modules are injective if and only if R is regular.*

PROOF. If R is regular then all simple R-modules are injective by a theorem of Kaplansky (see [12]). On the other hand if R has all singular simple modules injective then $I^2=I$ for each essential ideal I of R by Theorem 1.2, and so R is regular by the previous proposition.

Combining the previous theorem with Theorem 1.3 we have

COROLLARY 1.7. *A commutative perfect ring R has all singular simple R-modules injective if and only if R is a finite direct sum of fields.*

We remark that, as a consequence of Theorem 1.3 not all T-rings having all singular simples injective will be regular. Our next result shows that this is the case if and only if $N=0$. We will make use of the following facts concerning idempotent left ideals of a ring R. Suppose I and J are left ideals of R with $I \subseteq J$. If $I^2=I$, then $I=JI$. Thus if $I_1, \ldots, I_n$ are idempotent left ideals of R and $J=I_1+\ldots+I_n$, then $JI_k=I_k$ for $k=1, \ldots, n$ and so

$$J^2 = JI_1+\ldots+JI_n = J .$$

THEOREM 1.8. *Let R be a T-ring having all singular simple R-modules injective. Then R is regular if and only if $N=0$.*

PROOF. If R is regular, then certainly $N=0$. Thus assume $N=0$ and let $S=R\text{-soc}(R)$. If I is a left ideal of R contained in S, then I is a completely reducible R-module since S is completely reducible, and so $I=\sum_{\beta\in B}A_\beta$, where each A_β is a simple R-module. Now $N=0$ hence $A_\beta=Re_\beta$, where e_β is a nonzero idempotent for each $\beta\in B$. Thus $A_\beta{}^2=A_\beta$ for each $\beta\in B$. If $x\in I$, then

$$x \in A_{\beta_1}+\ldots+A_{\beta_n}$$

for some finite subset $\{\beta_1,\ldots,\beta_n\}$ of B and so

$$x \in (A_{\beta_1}+\ldots+A_{\beta_n})^2 \subseteqq I^2 .$$

Thus every left ideal of R contained in S is idempotent. If I is a left ideal of S, then $I'=I+RI$ is a left ideal of R contained in S and, since S is an ideal of R, $I'=(I')^2\subseteqq I$, hence I is a left ideal of R. Hence S is a ring coinciding with its S-socle and since $N=0$, the radical of the ring S is zero. Thus S is a ring direct sum of its homogeneous components $\{S_\delta \mid \delta\in D\}$, where each S_δ is a two-sided ideal of S (and hence of R) and each S_δ is a simple idempotent ring [8, p. 65]. For each $\delta\in D$, S_δ contains a simple R-module $J\neq 0$. If $0\neq K\subseteqq J$ is a left ideal of S_δ, then $S_\delta K\subseteqq K$ is a nonzero R-submodule of J and so $K=J$. Thus J is a simple left ideal of S_δ. By [11, Theorem 7.13], a simple ring having nonzero socle is a regular ring and so S being a ring direct sum of regular rings is a regular ring. If V denotes the maximal regular ideal of R [3], then $S\subseteqq V$ since S is a regular ideal of R. If $V\neq R$, then since S is essential in R, V is essential in R, hence the ring R/V has zero radical and nonzero R/V-socle. It follows as above that the R/V-socle of R/V is a regular ideal contrary to R/V having no nonzero regular ideals. Hence $R=V$ and so R is regular.

Note that we have shown the following

COROLLARY 1.9. *Let R be a T-ring having all singular simple R-modules injective. If $S=R$-$\mathrm{soc}({}_RR)$, then R/S is a regular ring. Hence R/I is a regular ring for every essential ideal I of R.*

2. Matrix rings.

For a ring R, define the left ideal $T^\alpha(R)$ of R for all ordinals α as follows:

1) $T^0(R)=0$ and $T^1(R)=R$-$\mathrm{soc}\,R$.
2) If $\alpha=\beta+1$ is not a limit ordinal define $T^\alpha(R)$ by $T^\alpha(R)/T^\beta(R)=R$-$\mathrm{soc}(R/T^\beta(R))$.
3) If α is a limit ordinal, let $T^\alpha(R)=\bigcup_{\beta<\alpha}T^\beta(R)$.

Note that $T^\alpha(R)$ is a two-sided ideal of R for each ordinal α.

PROPOSITION 2.1. *The ring R is a T-ring if and only if $T^{\alpha}(R)=R$ for some ordinal α.*

PROOF. Clearly if R is a T-ring the condition holds. Conversely, assume that $T^{\alpha}(R)=R$ for the ordinal α. To show that R is a T-ring it suffices to show that any non-zero cyclic R-module R/I has a simple submodule. Let β be the least ordinal such that $T^{\beta}(R)\nsubseteq I$. Then $\beta=\delta+1$ is not a limit ordinal and $T^{\delta}(R)\subseteq I$. Thus the map

$$T^{\beta}(R)/T^{\delta}(R) \to R/I$$

is nonzero, and since $T^{\beta}(R)/T^{\delta}(R)$ is a direct sum of simple R-modules, R/I has a simple submodule. This completes the proof.

Note that in the previous proof if R/I is simple we get that R/I is isomorphic to a submodule of $T^{\beta}(R)/T^{\delta}(R)$ and hence

COROLLARY 2.2. *If R is a T-ring, then any simple R-module is isomorphic to a submodule of $T^{\alpha+1}(R)/T^{\alpha}(R)$ for some ordinal α.*

In what follows we use R_n to denote the ring of $n\times n$ matrices over the ring R and if L is a left ideal of R, L_n denotes the the left ideal of R_n consisting of all matrices with entries in L. In addition, e_{ij} denotes the matrix unit with a 1 in the ith row and jth column.

THEOREM 2.3. a) *If R is a T-ring, then R_n is a T-ring.*
b) *If R is a T-ring, then R satisfies (*) if and only if R_n satisfies (*).*

PROOF. We first make the following observations:
1) If N is an essential submodule of the R-module M and if N is generated by simple modules, then $N=R\text{-soc}(M)$.
2) Let I be a left ideal of R and K/I a submodule of R/I. Define

$$(K/I)^{j} = \left(\sum_{i=1}^{n} Ke_{ij}+I_n\right)\Big/ I_n .$$

Then, if K/I is a simple R-submodule of R/I, $(K/I)^{j}$ is a simple R_n-submodule of R_n/I_n.

Now assume R is a T-ring and let $L=T^{1}(R)$. Since L is essential in R, L_n is essential in R_n by Lemma 3.6 of [1]. If S is a minimal left ideal of R, then S^{j} is a minimal left ideal of R_n and clearly L_n is generated by

$$\{S^{j} \mid j=1,\ldots,n,\ S \text{ a minimal left ideal of } R\} .$$

By 1), $L_n=R_n\text{-soc}(R_n)$ and so $T^{1}(R_n)=(T^{1}(R))_n$.

Assume inductively that $T^{\beta}(R_n)=(T^{\beta}(R))_n$ for all ordinals $\beta<\alpha$. If α is a limit ordinal, then

$$T^{\alpha}(R_n) = \bigcup_{\beta<\alpha} T^{\beta}(R_n) = \bigcup_{\beta<\alpha} (T^{\beta}(R))_n = (T^{\alpha}(R))_n .$$

If $\alpha=\delta+1$ is not a limit ordinal, we let

$$I = T^{\delta}(R), \quad J = T^{\alpha}(R),$$

and proceed as in the case $\alpha=1$. The submodule J_n/I_n of R_n/I_n is generated by the submodules $(K/J)^j$ where K/I is simple in R/I and these submodules are simple in R_n/I_n by 2). By a slight modification of Lemma 3.6 of [1], J_n/I_n is essential in R_n/I_n and so

$$T^{\alpha}(R_n)/T^{\delta}(R_n) = (T^{\alpha}(R))_n/T^{\delta}(R_n) .$$

Hence $T^{\alpha}(R_n)=(T^{\alpha}(R))_n$ and the induction is complete.

Since R is a T-ring, $T^{\alpha}(R)=R$ for some ordinal α by Proposition 2.1. But then

$$T^{\alpha}(R_n) = (T^{\alpha}(R))_n = R_n,$$

and so R_n is a T-ring. This proves a).

If S is a simple submodule of a module M and if R-soc(M) is generated by the simples S_δ, $\delta \in D$, then $S \approx S_\delta$ for some $\delta \in D$. Using this remark and Corollary 2.2 we see that any simple R_n-module is isomorphic to $(K/I)^j$ where $I=T^{\alpha}(R)$ for some ordinal α and K/I is simple in R/I. Note that K/I is R-isomorphic to the R-module $e_{11}(K/I)^j$ under the map

$$k+I \rightarrow ke_{1j}+I_n .$$

Now using the fact that a ring satisfies (*) if and only if every simple module is either injective or projective, we see by Corollary 2.3 of [10] that R satisfies (*) if and only if R_n satisfies (*). This proves b).

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