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SUMS AND INTERSECTIONS OF LEBESGUE SPACES

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To begin with, consider the following problem in the calculus of variations:

Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be n numbers in the open interval $(0,1)$. We want to minimize the expression

$$N = \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} h_i(x)\,dx \right]^{\alpha_i}, \tag{1}$$

where $h_1,h_2,\ldots,h_n$ vary through all those non-negative functions in $L^1(-\infty,\infty)$ such that the sum

$$g = \sum_{i=1}^{n} [h_i(x)]^{\alpha_i} \tag{2}$$

remains fixed.

In this paper we shall prove, among other things, that this variational problem has a solution, that is, the infimum of N is actually attained. To fix our idea, we restrict ourselves to the case where $n=2$. The discussions of the general case will be completely parallel.

We use [3] and [2] as our chief references in real analysis and harmonic analysis respectively.

Let us change our notation to write $\alpha_i=1/q_i$ and $[h_i(x)]^{\alpha_i}=g_i(x)$, $i=1,2$. Then $g_i \in L^{q_i}$, and (2) becomes

$$g = g_1 + g_2 \,. \tag{3}$$

It is quite clear that the infimum of g is not changed if we allow g_i to take complex values so long as we insert absolute value signs under the signs of integration in (1):

$$N = \|g_1\|_{q_1} + \|g_2\|_{q_2} \,. \tag{4}$$

This suggests that we introduce the sum of the Lebesgue spaces L^{q_1} and L^{q_2}, that is, the set of all functions g which are expressible in the form (3). In fact we have

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THEOREM 1. *Let* $S=S_{q_1,q_2}$ *be the set of all complex-valued functions* g *of the form* (3) *with* $g_i \in L^{q_i}$, *where* $1 \leqq q_i \leqq \infty$, $i=1,2$. *Then* S *is a Banach space if we supply it with the norm*

$$\|g\| = \inf N , \tag{5}$$

where N *is defined by* (4) *and the infimum is taken over all decompositions of* g *in the form* (3).

PROOF. There is no difficulty in verifying that S is a complex linear space and that (5) defines a semi-norm on it. To prove that (5) indeed gives a norm, assume that $\|g\|=0$. Then there exist sequences $g_1(n)$ in L^{q_1} and $g_2(n)$ in L^{q_2} such that

$$g = g_1(n)+g_2(n) ,$$

and

$$\lim \|g_1(n)\|_{q_1} = \lim \|g_2(n)\|_{q_2} = 0 .$$

As the last fact implies that both $g_1(n)$ and $g_2(n)$ converge to 0 in measure, $g=0$ a.e. Hence (5) defines a norm.

It remains to be proved that S is complete under this norm. Thus let $g(n) \in S$ be such that $\sum_{n=1}^{\infty} \|g(n)\| < \infty$. Select $g_1(n) \in L^{q_1}$ and $g_2(n) \in L^{q_2}$ such that $g(n)=g_1(n)+g_2(n)$ and

$$\|g_1(n)\|_{q_1} + \|g_2(n)\|_{q_2} \leqq \|g(n)\| + 2^{-n} . \tag{6}$$

It follows from (6) that both $\sum_{n=1}^{\infty} \|g_1(n)\|_{q_1}$ and $\sum_{n=1}^{\infty} \|g_2(n)\|_{q_2}$ converge. Since L^{q_1} and L^{q_2} are complete, $\sum_{n=1}^{\infty} g_1(n)$ and $\sum_{n=1}^{\infty} g_2(n)$ exist in L^{q_1} and L^{q_2} respectively. Denote their sums by g_1 and g_2 respectively, and set $g=g_1+g_2$. Then

$$\|g - \textstyle\sum_{n=1}^{k} g(n)\| \leqq \|g_1 - \sum_{n=1}^{k} g_1(n)\|_{q_1} + \|g_2 - \sum_{n=1}^{k} g_2(n)\|_{q_2} .$$

Hence $\sum_{n=1}^{\infty} g(n)=g$ in S. This completes the proof.

One consequence of this theorem is that the variational problem posed at the outset would be solved if we can show that the norm (5) of S is attained by a certain decomposition of g in the form (3) for $1<q_i<\infty$, $i=1,2$. This will be done in Corollary 1.

Next we like to identify the dual space of S. This will be done in the case where neither q_1 nor q_2 is ∞. Let p_1 and p_2 be defined by

$$\frac{1}{p_i}+\frac{1}{q_i} = 1, \qquad i=1,2 , \tag{7}$$

with the usual convention of arithmetic on the symbol ∞. Then for any bounded linear functional T on S, the restrictions of T to L^{q_i} are bounded linear functionals on L^{q_i}. Hence there exist functions $f_i \in L^{p_i}$ such that

$$Tg_i = \int_{-\infty}^{\infty} g_i(x) f_i(x)\, dx$$

for all $g_i \in L^{q_i}$. In particular, if $g \in L^{q_1} \cap L^{q_2}$, we have

$$Tg = \int_{-\infty}^{\infty} g(x) f_1(x)\, dx = \int_{-\infty}^{\infty} g(x) f_2(x)\, dx .$$

Since $L^{q_1} \cap L^{q_2}$ includes the characteristic functions of all sets with finite measure, this implies that $f_1 = f_2$ a.e. Call their common value f. Then $f \in D$ where

$$D = D_{p_1, p_2} = L^{p_1} \cap L^{p_2} .$$

If $g = g_1 + g_2$ is a decomposition of g in the form (3), then

$$Tg = \int_{-\infty}^{\infty} g_1(x) f(x)\, dx + \int_{-\infty}^{\infty} g_2(x) f(x)\, dx ,$$

that is,

$$Tg = \int_{-\infty}^{\infty} g(x) f(x)\, dx . \tag{8}$$

Conversely, it is quite evident that, for any $f \in D$, (8) defines a bounded linear functional on S.

Next we would like to calculate $\|T\|$ where T is defined by (8). Decompose g according to (3) as $g = g_1 + g_2$. Then

$$|Tg| \leqq \|g_1\|_{q_1} \|f\|_{p_1} + \|g_2\|_{q_2} \|f\|_{p_2} \leqq N \max(\|f\|_{p_1}, \|f\|_{p_2}) .$$

Hence

$$|Tg| \leqq \|g\| \max(\|f\|_{p_1}, \|f\|_{p_2}) .$$

Hence $\|T\| \leqq \mathrm{Max}(\|f\|_{p_1}, \|f\|_{p_2})$. We shall prove that equality actually holds here. This is trivially true if $f = 0$ a.e. Otherwise assume without loss of generality that $\|f\|_{p_1} \geqq \|f\|_{p_2}$. Let a number ε in $(0, \|f\|_{p_1})$ be given. Then there exists a function $g \in L^{q_1} \subset S$, not a.e. 0, such that

$$\left| \int_{-\infty}^{\infty} g(x) f(x)\, dx \right| \geqq \|g\|_{q_1} (\|f\|_{p_1} - \varepsilon) .$$

Since $g=g+0$ is a decomposition of g of the type (3), it follows from (8) and (5) that

$$\|T\|\,\|g\| \geqq |Tg| \geqq (\|f\|_{p_1}-\varepsilon)\|g\|_{q_1} \geqq (\|f\|_{p_1}-\varepsilon)\|g\| .$$

Hence

$$\|T\| \geqq \|f\|_{p_1} - \varepsilon = \max(\|f\|_{p_1}, \|f\|_{p_2}) - \varepsilon .$$

Since ε is arbitrary, it follows that

$$\|T\| = \max(\|f\|_{p_1}, \|f\|_{p_2}) .$$

Thus we have proved the following two theorems:

THEOREM 2. *Let p_1 and p_2 be two numbers in $(1,\infty]$ and let $D=D_{p_1,p_2}=L^{p_1}\cap L^{p_2}$. Then D is a Banach space if we supply it with the norm*

$$\|f\| = \mathrm{Max}(\|f\|_{p_1}, \|f\|_{p_2}) , \qquad f \in D .$$

THEOREM 3. *If q_1 and q_2 are numbers in $[1,\infty)$, then the conjugate space of $S=S_{q_1,q_2}$ is isometrically isomorphic to $D=D_{p_1,p_2}$, where p_i and q_i are related by* (7) *and the operation of $f \in D$ on $g \in S$ is given by* (8).

Actually, Theorem 2 remains valid even if we allow p_1 and p_2 to vary in $[1,\infty]$. The proof of this fact is similar to (and simpler than) that of Theorem 1, and, accordingly, will be omitted.

Our next task is to find the conjugate space of D when p_1 and p_2 are in $[1,\infty)$. This is given by

THEOREM 4. *If p_1 and p_2 are numbers in $[1,\infty)$, then the conjugate space of $D=D_{p_1,p_2}$ is isometrically isomorphic to $S=S_{q_1,q_2}$, where p_i and q_i are related by* (7) *and the operation of $g \in S$ on $f \in D$ is given by*

$$T(f) = \int_{-\infty}^{\infty} f(x)\,g(x)\,dx . \tag{9}$$

PROOF. Clearly for every $g \in S$, the functional T defined by (9) is linear. Further, if $g=g_1+g_2$, $g_i \in L^{q_i}$, is a decomposition of g in the form (3), then

$$\begin{aligned} |Tf| &\leqq \left|\int_{-\infty}^{\infty} f(x)\,g_1(x)\,dx\right| + \left|\int_{-\infty}^{\infty} f(x)\,g_2(x)\,dx\right| \\ &\leqq \|f\|_{p_1}\,\|g_1\|_{q_1} + \|f\|_{p_2}\,\|g_2\|_{q_2} \\ &\leqq \|f\|(\|g_1\|_{q_1}+\|g_2\|_{q_2}) . \end{aligned}$$

But this implies that T is bounded and $\|T\| \leqq \|g\|$.

Because D contains the characteristic functions of all sets with finite measures, it follows from (9) that the correspondence $g \to T$ is one-to-one. We have still to show that this correspondence is onto and that $\|T\| = \|g\|$.

To do so, consider the Banach space $L^{p_1} \oplus L^{p_2}$ with the norm

$$\|(f_1, f_2)\| = \max(\|f_1\|_{p_1}, \|f_2\|_{p_2}) .$$

Then D can be embedded in this space as its diagonal, that is, for $f \in D$, define $\varphi(f) = (f, f)$. Then φ maps D isometrically into $L^{p_1} \oplus L^{p_2}$. Now let T be a bounded linear functional on D. Then, by the Hahn–Banach theorem, $T \circ \varphi^{-1}$ has a norm preserving extension to $L^{p_1} \oplus L^{p_2}$. As the conjugate space of $L^{p_1} \oplus L^{p_2}$ is clearly $L^{q_1} \oplus L^{q_2}$ with the norm

$$\|(g_1, g_2)\| = \|g_1\|_{q_1} + \|g_2\|_{q_2}, \qquad g_i \in L^{q_i},\ i = 1, 2 ,$$

there are functions $g_1 \in L^{q_1}$ and $g_2 \in L^{q_2}$ such that

$$T(f) = \int_{-\infty}^{\infty} f(x)\, g_1(x)\, dx + \int_{-\infty}^{\infty} f(x)\, g_2(x)\, dx .$$

Define $g = g_1 + g_2$. Then $g \in S$ and $T(f)$ is given by (9) for this g. As the norm of T is the same as its extension, we have

$$\|T\| = \|g_1\|_{q_1} + \|g_2\|_{q_2} \geqq \|g\| . \tag{10}$$

But we proved before that $\|T\| \leqq \|g\|$. Hence we get $\|T\| = \|g\|$, and the theorem is proved.

It follows from (10) that

$$\|g\| = \|T\| = \|g_1\|_{q_1} + \|g_2\|_{q_2} . \tag{11}$$

Hence,

COROLLARY 1. *For each $g \in S$, there exist functions $g_1 \in L^{q_1}$ and $g_2 \in L^{q_2}$ such that $g = g_1 + g_2$ and that*

$$\|g\| = \|g_1\|_{q_1} + \|g_2\|_{q_2} .$$

This Corollary implies that the variational problem stated at the beginning of this paper always has a solution, as we have mentioned before.

COROLLARY 2. *If p_1 and p_2 are in $(1, \infty)$, then $D = D_{p_1, p_2}$ is reflexive. If q_1 and q_2 are in $(1, \infty)$, then $S = S_{q_1, q_2}$ is reflexive.*

Although Theorem 3 does not cover the case where $D = D_{1,p}$ with p in $(1, \infty]$, it is still true that D is a conjugate space in this case. More precisely, we have

Theorem 5. *Let q be a number in $[1, \infty)$ and let C_0 denote the set of all continuous functions on the real line which vanish at infinity. Denote by $\Sigma = \Sigma_q$ the set of all functions on the real line which can be written as*

$$g = g_1 + g_2 , \tag{12}$$

where $g_1 \in C_0$ and $g_2 = L^q$. For $g \in \Sigma$, define

$$\|g\| = \inf(\sup|g_1| + \|g_2\|_q) ,$$

where the infimum is taken over all decompositions of g in the form (12). Then Σ becomes a Banach space with respect to this norm. Further, the conjugate space of Σ is isometrically isomorphic to $D = D_{1,p}$ with

$$\frac{1}{p} + \frac{1}{q} = 1 ,$$

the operation of $f \in D$ on $g \in \Sigma$ being given by (8).

Proof. That Σ is a Banach space is proved by the same argument, *mutatis mutandis*, as that for Theorem 1. That the conjugate space of Σ is isometrically isomorphic to D is proved by the same argument, likewise *mutatis mutandis*, as that for Theorem 3. The only non-trivial modification made here occurs when we want to show that every bounded linear functional T on Σ is given by (8) for some function $f \in D$. This is done in the following manner: Since $C_0 \subset \Sigma$, the restriction of T on C_0 defines a bounded linear functional on C_0. Hence there is a complex bounded Radon measure ν on $(-\infty, \infty)$ such that

$$Tg = \int_{-\infty}^{\infty} g(x)\, d\nu(x), \qquad g \in C_0 .$$

Similarly, since $L^q \subset \Sigma$, there is a function $f \in L^p$ such that

$$Tg = \int_{-\infty}^{\infty} g(x) f(x)\, dx, \qquad g \in L^q .$$

In particular, if g is a continuous function with compact support, then both formulas are valid, and

$$\int_{-\infty}^{\infty} g(x)\, d\nu(x) = \int_{-\infty}^{\infty} g(x) f(x)\, dx .$$

This implies that ν is absolutely continuous and

$$d\nu(x) = f(x)\, dx .$$

Therefore, $f \in L^1$. Hence $f \in D$. Now if g is an arbitrary function in Σ, let $g = g_1 + g_2$ be a decomposition of g in the form of (12). Then

$$\begin{aligned} Tg = Tg_1 + Tg_2 &= \int_{-\infty}^{\infty} g_1(x)\, d\nu(x) + \int_{-\infty}^{\infty} g_2(x) f(x)\, dx \\ &= \int_{-\infty}^{\infty} g(x) f(x)\, dx , \end{aligned}$$

which is (8). This proves the theorem.

There are a few nooks and corners which should be cleared up before we go on. First, if $1 \leqq q_1 < q_2 < q_3 \leqq \infty$, then any function $g_2 \in L^{q_2}$ can be represented as $g_1 + g_3$ with $g_1 \in L^{q_1}$ and $g_3 \in L^{q_3}$. In fact, one way to define g_1 and g_3 is

$$g_1(x) = \begin{cases} g_2(x) & \text{if } |g_2(x)| > 1, \\ 0 & \text{if } |g_2(x)| \leqq 1, \end{cases} \qquad g_3 = g_2 - g_1 .$$

Hence if we define $S = L^{q_1} + L^{q_2} + L^{q_3}$ with the norm

$$\|g\| = \inf(\|g_1\|_{q_1} + \|g_2\|_{q_2} + \|g_3\|_{q_3}) ,$$

where the infimum is taken over all decompositions $g = g_1 + g_2 + g_3$, $g_i \in L^{q_i}$, then $S = S_{q_1, q_3}$, as sets, and a simple application of the open mapping theorem implies that this norm of g in S is equivalent to its norm in S_{q_1, q_3}. A similar reasoning applies to the space $D = L^{p_1} \cap L^{p_2} \cap L^{p_3}$. Further, this generalizes to any number of indices p_i and q_i. The variational problem for (1) when $n > 2$ is solved, however, by the previous technique which we used for $n = 2$.

Secondly, the solution of our variational problem is in general not unique. Thus, let g be the characteristic function of the unit interval $[0,1]$. Then for any q in $(1,\infty)$, $\|g\|_q = 1$. If $g = g_1 + g_2$ where $g_1 \in L^{q_1}$, $g_2 \in L^{q_2}$, $q_1 < q_2$, and both g_1 and g_2 are supported in $[0,1]$, then $\|g_2\|_{q_2} \geqq \|g_2\|_{q_1}$. Hence

$$\|g_1\|_{q_1} + \|g_2\|_{q_2} \geqq \|g_1\|_{q_1} + \|g_2\|_{q_1} \geqq \|g\|_{q_1} = 1 .$$

Hence

$$\|g\| = 1 = \|g\|_{q_1} + \|0\|_{q_2} = \|\tfrac{1}{2}g\|_{q_1} + \|\tfrac{1}{2}g\|_{q_2} .$$

Thus, both $(g_1, g_2) = (g, 0)$ and $(g_1, g_2) = (\frac{1}{2}g, \frac{1}{2}g)$ play the roles of solutions of the variational problem.

Thirdly, for $g \in S$ we define

$$\text{(13)} \qquad |||g||| = \inf \max(\|g_1\|_{q_1}, \|g_2\|_{q_2}) ,$$

where the infimum is taken over all decompositions $g = g_1 + g_2$ with $g_i \in L^{q_i}$, $i = 1, 2$; and for $f \in D$, we define

$$\text{(14)} \qquad |||f||| = \|f\|_{p_1} + \|f\|_{p_2} .$$

Then (13) and (14) provide norms equivalent to the original norms on S and D respectively. Further, Theorems 3 and 4 remain valid with these norms. Other equivalent norms are also feasible, and to each of these equivalent norms on S there corresponds a solvable variational problem.

Fourthly, the assumption that the underlying measure in our definitions of S and D is the Lebesgue measure on $(-\infty, \infty)$ is made only for the simplicity of exposition. In fact, we can use any measure space $(X, \mathscr{B}, \mu)$ in our definitions of S and D. Theorems 1 and 2 remain valid in this general setting. Theorem 3 is true in general if neither q_1 nor q_2 is 1, and is true even for these indices if μ is σ-finite. Similarly, Theorem 4 is true in general if neither p_1 nor p_2 is 1, and is true even for these indices if μ is σ-finite. Theorem 5 does not make sense unless there is some sort of topology on X, and it becomes a valid theorem if X is a locally compact space and μ is a Radon measure on it.

From now on we shall consider a locally compact group G and its left Haar measure μ. In this case the set $D = D_{1,p}$, $p > 1$, has some additional structure:

LEMMA. *$D = D_{1,p} = L^1(G, \mu) \cap L^p(G, \mu)$ is a dense left ideal in L^1, where $L^1 = L^1(G, \mu)$ is considered as a Banach algebra with convolution*

$$f * g(x) = \int_G f(y)\, g(y^{-1}x)\, d\mu(y)$$

as multiplication.

PROOF. D is a left ideal because L^1-functions operate boundedly linearly by convolutions from the left. D is dense in L^1 because it contains all the continuous functions on G with compact supports.

THEOREM 6. *D with its own norm is a Banach algebra under convolution. Further, D is commutative if and only if G is abelian.*

PROOF. Since D is known to be a Banach space and an ideal in L^1, the first statement will follow if we show that $\|f*g\| \leqq \|f\| \, \|g\|$ for all $f, g \in D$. Indeed,

$$\|f*g\| = \max(\|f*g\|_1, \|f*g\|_p) \leqq \|f\|_1 \, \|g\| \leqq \|f\| \, \|g\| .$$

Hence we have the first statement. The second statement follows from a similar statement for L^1 and the density of D in L^1.

We are going to study the ideal theory in $D = D_{1,p}$ where $1 < p < \infty$. For a closed right ideal $I_1 \subset L^1$ define

$$\delta(I_1) = I_1 \cap D . \tag{15}$$

Then clearly $\delta(I_1)$ is a closed right ideal in D.

THEOREM 7. *By (15) is defined a one–one mapping δ from the set of all closed right ideals in L^1 onto the set of all closed right ideals in $D = D_{1,p}$, $1 < p < \infty$. Further, $\delta(I_1)$ is a two-sided ideal in D if and only if I_1 is a two-sided ideal in L^1.*

PROOF. It is known that there is a net u_α of continuous functions with compact supports such that

$$\|f - f*u_\alpha\|_r \to 0 \quad \text{for each } f \in L^r, \qquad 1 \leqq r < \infty .$$

We shall prove our theorem by dint of this net.

First, let $\delta(I_1) = I$ and let J_1 be the L^1-closure of I. Clearly $J_1 \subset I_1$. On the other hand, if $f \in I_1$, then each $f*u_\alpha \in I_1$, since $u_\alpha \in L^1$ and I_1 is a right ideal in L^1. Also $f*u_\alpha \in D$ since $u_\alpha \in D$ and D is a left ideal in L^1. Thus $f*u_\alpha \in I$. As $f*u_\alpha \to f$ in L^1, we get $f \in J_1$. This proves that $I_1 = J_1$. In particular, δ is a one–one mapping.

Next let I be a closed right ideal in D and let I_1 be the L^1-closure of I. We want to show that $I = \delta(I_1)$. For this, take $f \in \delta(I_1)$. Then there is a sequence $f_n \in I$ such that $f_n \to f$ in L^1. Since convolution is continuous,

$$f_n * u_\alpha \to f * u_\alpha \quad \text{as} \quad n \to \infty$$

both in L^1 and in L^p, hence in D. As I is a right ideal and as $u_\alpha \in D$, $f_n * u_\alpha \in I$. Hence $f*u_\alpha \in I$ for each α, since I is closed. Finally $f*u_\alpha \to f$ both in L^1 and in L^p, hence in D. This yields that $f \in I$. Hence $I = \delta(I_1)$. This means that the mapping δ is onto.

It is quite clear that if I_1 is two-sided, then $\delta(I_1)$ is also two-sided. Conversely, if $\delta(I_1)$ is two-sided, then I_1, being its L^1-closure, is also two-sided, by the density of D and the joint continuity of convolution in L^1.

One consequence of Theorem 7 is that the maximal ideals of L^1 and those of D correspond to each other. It says nothing, however, of the correspondence between their regular maximal ideals. We prove now that this holds for any *abelian* group G.

THEOREM 8. *Suppose G is abelian; then for each p in $(1,\infty]$, the maximal ideal space of $D=D_{1,p}$ is homeomorphic to the dual group $\hat{G}$ of G.*

PROOF. Let f be a non-zero element of D. Then for each integer $n>1$ we have

$$\|f^n\| = \max(\|f^{n-1}*f\|_1, \|f^{n-1}*f\|_p) \leqq \|f\|_1^{n-1}\|f\| .$$

Extracting nth roots on both sides and then letting $n \to \infty$, we get

$$\varrho(f) \leqq \|f\|_1 , \tag{16}$$

where $\varrho(f)$ denotes the spectral radius of f. Clearly (16) also holds for $f=0$.

Now let F be a multiplicative linear functional on D. Then $|F(f)| \leqq \varrho(f) \leqq \|f\|_1$ for each $f \in D$. Since D is dense in L^1, F has a unique extension to a multiplicative linear functional on L^1. As the maximal ideal space of L^1 can be identified to $\hat{G}$, there exists an element $\xi \in \hat{G}$ such that

$$F(f) = \int_G \overline{(x,\xi)}\, f(x)\, d\mu(x) \tag{17}$$

for each $f \in D$. Conversely, if $\xi \in \hat{G}$, the functional F on D defined by (17) is multiplicative and linear. Also, different ξ's determine different F's because D is dense in L^1.

Another consequence of the density of D in L^1 is that the Gelfand topologies of $\hat{G}$ as maximal ideal spaces of D and of L^1 coincide. Theorem 8 is therewith proved.

Combining Theorems 7 and 8, we see that if G is abelian and if $1<p<\infty$, then Wiener's Tauberian theorem holds for $D=D_{1,p}$. Also, spectral synthesis fails for $D_{1,p}$, where $1<p<\infty$, unless G is discrete, which is a trivial case because then $D_{1,p}=L^1$.

If G is compact, then $D = L^p$. Hence Theorems 7 and 8 hold for the L^p algebras of compact groups.

Theorems 6, 7 and 8 are suggested by some similar results in a previous paper [1].

ADDED IN PROOF. The editor brought to our attention the paper *The structure space of a left ideal*, Math. Scand. 14 (1964), 90–92, by G. K. Pedersen, where it is proved that if D is a left ideal in a ring L, then to each maximal regular right ideal I in D there corresponds a maximal regular right ideal J in L such that $J \cap D = I$. Furthermore there is a homeomorphism between the right structure space of D and the (open) set of right primitive ideals in L not containing D.

Now in our situation where $L = L^1$ is an involutive algebra and D is a dense left ideal, this yields that the structure space of D is homeomorphic to the structure space of L, the latter being the kernels in L of irreducible unitary representations of the group G. When G is abelian, this result combined with Theorem 8 gives that the Gelfand topology coincides with the Jacobson (hull-kernel) topology on the maximal ideal space of D.

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