MATH. SCAND. 24 (1969), 5–26

INFINITE DIMENSIONAL COMPACT CONVEX POLYTOPES

R. R. PHELPS

0. Introduction.

In this paper we extend the notion of "compact convex polytope" to infinite dimensional subsets of locally convex spaces. We want the extended class to include the Choquet simplexes [15] and, of course, to coincide with the usual notion for finite dimensional sets. (With the exception of Alfsen [1], the extensions made previously, e.g. Bastiani [3] and Maserick [13] [14], do not contain the infinite dimensional simplexes.) We would also like the extended class to possess many of the properties which are known to hold for finite dimensional polytopes and for simplexes. Now, there are two ways (dual to each other) of characterizing finite dimensional polytopes in terms of simplexes; this fact will lead to two different definitions in the infinite dimensional case. The first of these ways is the following:

(α) A finite dimensional compact convex set K is a polytope (that is, has finitely many extreme points) if and only if there exists a finite dimensional simplex S and an affine continuous map φ of S onto K.

(To prove the "only if" portion, simply take S to be the simplex of all probability measures on $\operatorname{ext} K$.) The dual characterization is the following (cf. [9]):

(β) A finite dimensional compact convex set K is a polytope if and only if there exists a finite dimensional simplex S and an affine variety M such that K is affinely equivalent to $S \cap M$.

(The connection of this with (α) comes from the fact that the polar body of a polytope is a polytope, the polar of a simplex is a simplex, and the adjoint of a surjection is an injection.)

Each of these two characterizations could be made the basis of our definition, by simply deleting the words "finite dimensional" wherever

Received May 15, 1968; revision May 31, 1968.

Work supported in part by the National Science Foundation, U.S.A.

they appear. This would be unsatisfactory, however; in the first case, *every* compact convex set K is the continuous affine image of the simplex of all probability measures on K, and in the second case, there are finite dimensional sets which are not polytopes yet are of the form $S \cap M$ for an infinite dimensional simplex S. These considerations have led us to the following definitions:

A compact convex set K is an α-polytope if there exists a simplex S and a continuous affine map φ of S onto K such that $\varphi^{-1}(x)$ is finite dimensional for each x in K.

A compact convex set K is a β-polytope if there exists a simplex S and a closed affine variety M of finite codimension such that K is affinely homeomorphic to $S \cap M$.

Incidental to his study of simplexes, Alfsen [1] defined and briefly discussed a class of "polyhedra" which is formally different from the class of α-polytopes, but which turns out to be the same.

The above definitions lead to two distinct classes of sets which share only some of the properties of finite dimensional polytopes. For instance, neither class can contain any infinite dimensional centrally symmetric sets, and as a consequence, both classes fail to be closed under finite intersections and finite vector sums. On the other hand, both classes share two of the important properties of simplexes. For instance, it is known [4] that a G_δ extreme point of a simplex is a "peak point" (point of strict maximum) for some continuous affine function; in fact, any closed G_δ face of a simplex is known [5] [8] [11] to be such a "peak set". Both classes of polytopes have this property; for α-polytopes, this follows from a nontrivial extension of the simplex result which has been proved by L. Asimow [2]. The proof for β-polytopes is easy for extreme points but somewhat harder for faces; the latter depends on a geometric lemma whose proof was kindly supplied us by A. Lazar.

Another property of simplexes which is true for both kinds of polytopes is the result, proved for simplexes by A. Lazar [11] [12] and D. A. Edwards [8], that any continuous affine functional on a closed face admits a continuous affine extension to the entire set. (In fact for simplexes, the extension can be chosen to be norm preserving, but – as pointed out by Lazar [11] – this aspect fails even for two dimensional quadrilaterals.) We show by example that this extension property can fail for sets which are not polytopes.

The remainder of the paper consists of four sections. In Section 1 we establish the necessary notation and definitions, Sections 2 and 3 are devoted to α- and β polytopes respectively, and Section 4 contains additional examples and remarks.

We wish to acknowledge helpful conversations on the subject matter of this paper with Professors H. H. Corson, V. L. Klee and A. Lazar.

1. Definitions.

Throughout this paper, when we refer to a compact convex set K, it will be understood to be a nonempty subset of some locally convex space. This insures that the Banach space $A(K)$ (supremum norm) of all continuous real valued affine functions on K has sufficiently many elements to separate points of K. Moreover, it then follows that K is affinely homeomorphic to the weak* compact convex set (which we again denote by K) of all linear functionals L in $A(K)^*$ for which $L(1)=1=\|L\|$. (For detailed proofs of these and other facts in this section, cf. [15].)

A *simplex* S is a compact convex set such that the cone of nonnegative functionals in $A(S)^*$ (of which S is a base) defines a lattice ordering on $A(S)^*$. This is equivalent [16] to the assertion that $A(S)^*$ is an abstract L-space, that is, $A(S)^*$ is a vector lattice, and $x \geqq 0$, $y \geqq 0$ in $A(S)^*$ implies $\|x+y\|=\|x\|+\|y\|$.

We denote by $P(K)$ the set of all Borel probability measures on K, and by $Q(K)$ (or simply by Q) the subset of $P(K)$ consisting of those measures which are maximal with respect to the ordering

$$\mu \succ \lambda \Leftrightarrow \mu(f) \geqq \lambda(f) \text{ for every continuous real valued convex function } f \text{ on } K.$$

If $\mu \in P(K)$, the resultant r_μ of μ is the unique point of K which satisfies $f(r_\mu)=\int_K f\,d\mu$ for each f in $A(K)$. When restricted to $Q(K)$, the resultant map r is still onto, and is one-to-one if and only if K is a simplex.

In the weak* topology of $C(K)^*$, the set $P(K)$ is a simplex. The set $Q(K)$ is "almost" a simplex, since the cone generated by it is lattice ordered, but $Q(K)$ is not generally weak* compact.

A subset F of a convex set K is called a *face* of K if F is convex and if $x,y \in F$ whenever $x,y \in K$ and $\lambda x+(1-\lambda)y \in F$ for some $0<\lambda<1$.

If K_1, K_2 are compact convex sets and if $\varphi\colon K_1 \to K_2$ is continuous and affine, then by the *linear extension* of φ we mean the map $\bar{\varphi}$ from $A(K_1)^*$ to $A(K_2)^*$ induced by φ [so $(\bar{\varphi}L)(f)=L(f\circ\varphi)$ if $L \in A(K_1)^*$, $f \in A(K_2)$]. When no confusion can result, we will denote $\bar{\varphi}$ simply by φ. (If we identify each K_i with its natural embedding in $A(K_i)^*$, then $A(K_i)^*$ is linearly generated by K_i and $\bar{\varphi}$ coincides with the linear extension of φ.) The map $\varphi\colon K_1 \to K_2$ will be said to have *finite dimensional kernel* provided its linear extension has finite dimensional kernel.

2. The class of α-polytopes.

Our first task in this section is to establish the equivalence between our definition and the one of Alfsen referred to in the Introduction. Let $K \subset A(K)^*$ be compact and convex, let Q be the set of maximal probability measures on K, and let E be the linear space (of signed measures) generated by Q. If r is the resultant map from Q onto K, then we can extend r to a linear map from E onto $A(K)^*$. Call K an *Alfsen polyhedron* if this extension has finite dimensional kernel.

THEOREM 2.1. *If K is a compact convex set, then the following assertions are equivalent:*

(i) *K is an α-polytope.*

(ii) *There exists a simplex S and a continuous affine map from S onto K having finite dimensional kernel.*

(iii) *The set $r^{-1}(x)$ is finite dimensional, for each x in K.*

(iv) *K is an Alfsen polyhedron, that is, the resultant map r has finite dimensional kernel.*

PROOF. It is obvious that (ii) implies (i) and that (iv) implies (iii). We prove that (i) implies (ii) as follows: Suppose that S is a simplex and $\varphi: S \to K$ a continuous affine surjection with $\varphi^{-1}(x)$ finite dimensional for each x in K. We regard φ as a linear map from $A(S)^*$ onto $A(K)^*$, and we wish to show that φ has finite dimensional kernel. We use the fact that $A(S)^*$ is a Banach lattice (in fact, an L-space), under the ordering defined by the cone R^+S generated by S. If $\{x_n\}$ is an infinite linearly independent sequence in $\varphi^{-1}(0)$, we can write $x_n = x_n^+ - x_n^-$ and assume without loss of generality that $\|x_n^-\| = 2^{-n}$. Let $x = \sum x_n^-$; this series converges in norm. Define

$$y_n = x + x_n = (\textstyle\sum_{k \neq n} x_k^-) + x_n^+ \geqq 0 \,.$$

Clearly, $\varphi(y_n) = \varphi(x)$ for each n, and $\{y_n\}$ is an infinite affinely independent set. We will obtain a contradiction if we show that $x \in S$ and $\{y_n\} \subset S$. Now, $S = \{y : y \geqq 0 \text{ and } \|y\| = 1\}$. Obviously, $x \geqq 0$, and

$$\|x\| = (x, 1) = \textstyle\sum (x_n^-, 1) = \sum 2^{-n} = 1 \,,$$

so $x \in S$. For f in $A(K)$, $\langle \varphi x_n, f \rangle = \langle x_n, f \circ \varphi \rangle$; taking $f = 1$, we see that

$$0 = \langle \varphi x_n, 1 \rangle = \langle x_n, 1 \rangle = \langle x_n^+, 1 \rangle - \langle x_n^-, 1 \rangle \,,$$

so $\langle x_n^+, 1 \rangle = \langle x_n^-, 1 \rangle = \|x_n^-\| = 2^{-n}$; it follows that $\langle y_n, 1 \rangle = 1$ and hence $y_n \in S$ for each n.

In order to prove that (iii) implies (iv), one uses the above argument;

it only requires the fact that the space E is closed (in the total variation norm) as a subspace of the space of all finite regular Borel signed measures on K. This follows easily from the fact (see, e.g. [15, p. 64]) that $\mu \in E$ if and only if $|\mu|(\bar{f}-f)=0$ for each f in $C(K)$.

Lemma 2.2 below shows that (i) implies (iv), while Lemma 2.4 will show that (iv) implies (ii).

LEMMA 2.2. *Let K_1 and K_2 be compact convex sets and let φ be a continuous affine map of K_1 onto K_2. Let T denote the induced map $\mu \to \mu \circ \varphi^{-1}$ of $P(K_1)$ into $P(K_2)$, and let r_i denote the resultant map from $P(K_i)$ onto K_i, $i=1,2$. Then*

(i) *T is a surjection.*
(ii) *$Q(K_2) \subset T[Q(K_1)]$.*
(iii) *$r_2(T\mu)=(r_1\mu)$ for each μ in $P(K_1)$.*
(iv) *$\{\lambda \in Q(K_2): r_2\lambda = x\} \subset T\{\mu \in Q(K_1): r_1\mu \in \varphi^{-1}x\}$, for each x in K_2.*

PROOF. (i) This is a standard result, using the monotone extension theorem and the fact that φ is a continuous surjection.

(ii) Note first that if λ and μ are in $P(K_1)$ and if $\lambda \succ \mu$, then $T\lambda \succ T\mu$. Indeed, if g is a continuous convex function on K_2, then $g \circ \varphi$ is continuous and convex on K_1, so that

$$(T\lambda)(g) = \lambda(g \circ \varphi) \geqq \mu(g \circ \varphi) = (T\mu)(g) .$$

Suppose, now, that $\nu \in Q(K_2)$. Choose μ in $P(K_1)$ such that $T\mu=\nu$. By a standard application of Zorn's lemma, there exists a maximal measure λ in $Q(K_1)$ with $\lambda \succ \mu$. Consequently, $T\lambda \succ T\mu = \nu$, and since ν is maximal, $T\lambda = \nu$.

(iii) It suffice to show that $h[r_2(T\mu)] = h[\varphi(r_1\mu)]$ for each affine continuous function h on K_2 (since these functions separate points of K_2), and this is immediate from the definitions of r_1, r_2 and T.

(iv) If $\lambda \in Q(K_2)$ and $r_2\lambda = x$, then by (ii), there exists μ in $Q(K_1)$ such that $T\mu = \lambda$. From (iii) it follows that

$$\varphi(r_1\mu) = r_2(T\mu) = r_2\lambda = x ,$$

which yields (iv).

COROLLARY 2.3. *If K is an α-polytope, then it is an Alfsen-polyhedron.*

PROOF. Suppose $\varphi(S)=K$, where S is a simplex and φ is a continuous affine map, with $\varphi^{-1}(x)$ finite dimensional for each x in K. Let r_2 denote the resultant map from $Q(K)$ onto K; by "(iii) implies (iv)" of Theorem 2.1, it suffices to show that $r_2^{-1}x$ is finite dimensional, for each x in K. If r_1 denotes the resultant map from $Q(S)$ onto S, then $r_1^{-1}(\varphi^{-1}x)$ is finite dimensional, and the desired result is immediate from (iv) of Lemma 2.2.

It remains to prove that (iv) implies (ii). If K is an Alfsen polyhedron, then it is the weak* continuous affine image of Q under the resultant map r, and the latter has finite dimensional kernel. The problem is that Q is not necessarily weak* compact. The next result shows, however, that there is a new topology under which Q is compact and r remains continuous.

LEMMA 2.4. *Suppose that K is a compact convex set and that the extended resultant map r (from the space E generated by $Q(K)$ onto the space $A(K)^*$) has finite dimensional kernel N. Then there exists a locally convex topology τ on E such that $Q(K)$ is τ-compact and r is τ-continuous.*

PROOF. Let q be the quotient map of E onto E/N and let p be *any* projection from E onto N which is continuous from the norm (total variation) topology on E to the norm topology on N. Define $t: E \to E/N \times N$ by $t\mu = (q\mu, p\mu)$; it is straightforward to verify that t is a linear bijection. Now, the map $s: \mu + N \to r\mu$ is a linear bijection between E/N and $A(K)^*$, so we can use it to carry the weak* topology of $A(K)^*$ to a (locally convex) topology on E/N. Thus, with this topology on E/N and with the norm topology on N, the product topology on $E/N \times N$ is locally convex and can be carried by t to a locally convex topology τ on E. In terms of nets, we see that $\mu_\alpha \to \mu$ in the τ-topology if and only if $r\mu_\alpha \to r\mu$ and $p\mu_\alpha \to p\mu$. In particular, r is τ-continuous. To see that $Q = Q(K)$ is τ-compact, we first note that τ is independent of which projection p we use. Indeed, if $p': E \to N$ is another such projection, and if τ' is the resulting topology on E, then by definition, (E, τ') is homeomorphic to $E/N \times N$, which is homeomorphic to (E, τ). Next, since Q is norm bounded and p is a bounded operator, the set pQ has compact closure in the finite dimensional space N. Also, the fact that s is a homeomorphism shows that $s^{-1}K$ is compact in the s-induced topology on E/N. It follows that $J = s^{-1}K \times pQ$ is compact in $E/N \times N$, so $t^{-1}J$ is τ-compact and clearly contains Q. It remains to show that Q is τ-closed. Suppose that $\mu_\alpha \in Q$, $\mu \in E$ and $\mu_\alpha \to \mu$ in the τ-topology. To show that $\mu \in Q$ we need only show that it is a probability measure. Now, $r\mu_\alpha \to r\mu$ and $rQ = K$ is closed, so $r\mu \in K$. This implies two things; first, there exists λ in Q with $r\lambda = r\mu$ (hence $\lambda - \mu \in N$). Second, using the definition of r and the affine constant function 1, we see that $1 = \langle r\mu, 1\rangle = \mu(K)$. It remains to show that $\mu \geqq 0$; suppose not. Since E is a linear subspace of $C(K)^*$, μ is not in the positive cone of the latter space, and hence there exists a function $f_1 \geqq 0$ in $C(K)$ with $\langle \mu, f_1\rangle < 0$. Also, writing $\mu_1 = \lambda - \mu$ we see that $\langle \mu_1, f_1 \rangle > 0$ and we can assume it equals 1. We can then choose $\mu_2, \ldots, \mu_n$ in N such that $\langle \mu_k, f_1 \rangle = 0$ for $k \geqq 2$ and $\mu_1, \mu_2, \ldots, \mu_n$ form a basis for N. We next choose $f_2, \ldots, f_n$ in $C(K)$ such that $\langle \mu_i, f_j \rangle = \delta_{ij}$, $i,j = 1,2,\ldots,n$. Finally, we define a norm-continuous projection p from E onto N by

$$p\nu = \Sigma_{k=1}^{n} \langle \nu, f_k \rangle \mu_k .$$

Since $p\mu_\alpha \to p\mu$, we have $0 \leqq \langle \mu_\alpha, f_1 \rangle \to \langle \mu, f_1 \rangle < 0$, a contradiction which completes the proof.

It is evident from Theorem 2.1 (ii) that if K is a finite dimensional α-polytope, then it is the continuous affine image of a finite dimensional simplex and hence is a polytope in the classical sense. On the other hand, if K is a finite dimensional polytope, then $\operatorname{ext} K$ is finite, so $Q(K)$ is the finite dimensional simplex of all probability measures on $\operatorname{ext} K$ and r is a continuous affine surjection with finite dimensional kernel. Thus, the class of finite dimensional α-polytopes coincides with the usual class of polytopes.

Proposition 2.5. *If K_1, K_2 are compact convex sets, if φ: $K_1 \to K_2$ is a continuous affine surjection with finite dimensional kernel (or with $\varphi^{-1}(x)$ finite dimensional for each x in K_2), and if K_1 is an α-polytope, then K_2 is an α-polytope.*

Proof. This follows easily from the definition or from Theorem 2.1 (ii), since a composition of two affine maps of the type involved is of the same type.

Proposition 2.6. *A closed face F of an α-polytope K is an α-polytope.*

Proof. If S is a simplex and φ: $S \to K$ is a continuous affine surjection, then $\varphi^{-1}(F)$ is a closed face of S, hence is itself a simplex. The result is now immediate.

Corollary 2.7. *If $\{K_\gamma\}$ is an arbitrary family of compact convex sets, and if the Cartesian product $K = \prod K_\gamma$ is an α-polytope, then each K_γ is an α-polytope.*

Proof. Given γ_0, choose $x \in \prod_{\gamma \neq \gamma_0} K$ with $x_\gamma \in \operatorname{ext} K_\gamma$, $\gamma \neq \gamma_0$. Let $K_0 = K_{\gamma_0} \times \{x\}$, so that K_0 is a compact subset of K. It is easily verified that K_0 is a face of K, hence is (by Proposition 6) an α-polytope. Furthermore, K_0 is obviously affinely homeomorphic to K_{γ_0}, so the latter is an α-polytope.

We show in Section 4 that the product of two simplexes need not be an α-polytope.

PROPOSITION 2.8. *If K is an infinite dimensional centrally symmetric compact convex set, then K is not an α-polytope.*

PROOF. We assume that 0 is a center of symmetry for K. By the Krein–Milman theorem, $\operatorname{ext} K$ is infinite. Since $x \in \operatorname{ext} K$ implies $-x \in \operatorname{ext} K$, each of the measures $\mu_x = \frac{1}{2}\varepsilon_x + \frac{1}{2}\varepsilon_{(-x)}$ is maximal (since it is supported by $\operatorname{ext} K$ [15]) and has the origin 0 as resultant. Thus $r^{-1}(0) \supset \{\mu_x : x \in \operatorname{ext} K\}$ and the latter set contains an infinite linearly independent subset, which implies that $r^{-1}(0) \cap Q(K)$ is infinite dimensional. By Theorem 2.1, then, K is not an α-polytope.

As we will see in Section 4, the above proposition leads to a number of examples which show that the usual operations which preserve finite dimensional polytopes fail to preserve α-polytopes, in particular, there exist simplexes S_1 and S_2 such that the compact convex set $\operatorname{conv}(S_1 \cup S_2)$ is not an α-polytope. We have an affirmative result of this type, however, if one of the sets is assumed to be finite dimensional, this is a consequence of the following proposition.

PROPOSITION 2.9. *Suppose that K is a compact convex subset of a locally convex space E and that B is a finite subset of E. If K is an α-polytope, then so is the convex hull K' of $K \cup B$.*

PROOF. It suffices to prove this when B is a single point $x_0 \notin K$ and then use induction. Suppose that $S \subset A(S)^*$ is a simplex and that $\varphi: S \to K$ is an affine continuous surjection with $\varphi^{-1}(x)$ finite dimensional for each x in K. Let $S' \subset A(S)^* \times R$ be the convex hull of $S \times \{0\}$ and $e = (0, 1)$; then S' is a simplex and every point in S' has a unique representation of the form $\lambda e + (1-\lambda)(s, 0)$, where $s \in S$ and $0 \leqq \lambda \leqq 1$. Define $\varphi' : S' \to K'$ by

$$\varphi'(\lambda e + (1-\lambda)(s, 0)) = \lambda x_0 + (1-\lambda)\varphi(s) .$$

This is easily seen to be a continuous affine surjection, so we need only show that if $x' \in K'$, then

$$\{\lambda e + (1-\lambda)(s, 0) : x' = \lambda x_0 + (1-\lambda)\varphi(s)\}$$

is finite dimensional. If $x' = x_0$, this is the single point e, so assume $x' \neq x_0$. Let J denote the segment obtained by intersecting K with the line through x' and x_0. (This might be a single point.) Since $x' = \lambda x_0 + (1-\lambda)\varphi(s)$ and since $0 \leqq \lambda < 1$, we have

$$\varphi(s) = \frac{x'}{1-\lambda} + \frac{(-\lambda)}{1-\lambda} x_0 \in J .$$

Thus,

$$\lambda e+(1-\lambda)(s,0) \in \operatorname{conv}[\{e\} \cup \varphi^{-1}(J)],$$

and the latter set is finite dimensional, that is, the inverse image of x' under φ' is contained in a finite dimensional set, which completes the proof.

As mentioned in the Introduction, it is known [8] [11] that any continuous affine functional on a closed face of a simplex admits a continuous affine extension (even norm preserving) to the entire simplex. Furthermore, it is impossible to find norm preserving extensions even for quadrilaterals in the plane. *Continuous* extensions exist, however, for α-polytopes (and β-polytopes), and the proof is far simpler than that for simplexes. (The proof for β-polytopes reduces to the theorem for simplexes.)

THEOREM 2.10. *Suppose that K is an α-polytope and that F is a closed face of K. If g is a continuous affine functional on F, then there exists a continuous affine functional f on K which extends g.*

PROOF. Choose a simplex S and a weak* continuous affine surjection $\varphi\colon A(S)^* \to A(K)^*$ for which $N=\varphi^{-1}(0)$ is finite dimensional. The set $F_1=\varphi^{-1}(F)$ is a closed face of S and $g_1=g\circ\varphi \in A(F_1)$. Let $M=R^+F_1-R^+F_1$ be the subspace generated by F_1; it is known [7] [11] that M is weak* closed, hence is a dual Banach space. It is easily checked that g_1 is weak* continuous on the unit ball $\operatorname{conv}(F_1\cup(-F_1))$ of M, hence (by the Krein–Šmulyan theorem) its linear extension (call it g_1) is weak* continuous on M. Let $N_1=N\cap M$ and choose a subspace N_2 of N such that $N=N_1+N_2$ and $N_1\cap N_2=\{0\}$. Clearly $g_1(N_1)=0$ and we can extend g_1 to a linear functional g_2 on $M+N$ by setting $g_2(N_2)=0$ and extending linearly. Since N_2 is finite dimensional, the set $g_2{}^{-1}(0)=g_1{}^{-1}(0)+N_2$ is weak* closed, so g_2 is weak* continuous on $M+N$. By the separation theorem (applied to the graph of g_2) we can extend g_2 to a weak* continuous linear functional f_1 on $A(S)^*$. Since $f_1(N)=0$, we can define f on K by $f(\varphi x)=f_1(x)$ for each x in S, and f is easily seen to be continuous on K.

We conclude this section by stating Asimow's theorem [2] on peak sets.

THEOREM 2.11 (Asimow). *If K is an α-polytope and if F is a closed face of K which is a G_δ set in K, then there exists f in $A(K)$ such that $f(F)=0$ and $f(x)>0$ if $x\in K\sim F$.*

3. The class of β-polytopes.

We first state the definition of β-polytope more precisely than was done in the Introduction.

DEFINITION. A compact convex set K is a *β-polytope* if there exists a simplex S and continuous affine functions $h_1, h_2, \ldots, h_n$ on S such that K is affinely homeomorphic to $\{x \in S: h_i(x) = 0,\ i = 1, 2, \ldots, n\}$.

It is obvious that by this definition, K is a β-polytope if and only if K is affinely homeomorphic to $M \cap S$, where M is a weak* closed subspace of finite codimension in $A(S)^*$ (and we identify the simplex S with its canonical embedding in $A(S)^*$). We will also refer to $M \cap S$ as a *finite codimensional slice* of S, so K is a β-polytope provided it is affinely homeomorphic to a finite codimensional slice of some simplex.

PROPOSITION 3.1. *A finite dimensional compact convex set is a β-polytope if and only if it is a polytope.*

PROOF. The "if" part is proved in [9, p. 71]; the idea of the proof was sketched in our Introduction. To prove the converse assertion, suppose that K is a finite dimensional β-polytope. Without loss of generality, we can assume that $K = M \cap S \subset A(S)^*$ for some simplex S and weak* closed subspace M of finite codimension in $A(S)^*$. Since K is finite dimensional, it has nonempty interior relative to the affine variety which it generates; let x be a relative interior point of K. By Caratheodory's theorem, x is a finite convex combination of extreme points of K, and by a theorem of Dubins [6] (see also [10]) each extreme point of K is a finite convex combination of extreme points of S. Thus, x is in the convex hull S_1 of a finite subset of $\operatorname{ext} S$. It is known that the convex hull of a finite number of closed faces of a simplex is again a simplex, a fact proved rather easily in this special case where the faces are points. Thus, S_1 is a finite dimensional simplex, and we need only show that $K = M \cap S_1$. Since $S_1 \subset S$, we have $M \cap S_1 \subset M \cap S = K$. Suppose, on the other hand, that $y \in K$, $y \neq x$. Since x is a relative interior point of K we can choose $z \in K$ and $0 < \lambda < 1$ so that $x = \lambda y + (1-\lambda)z$. Since $x \in S_1$ and the latter is a face, we have $y \in S_1$ and the proof is complete.

PROPOSITION 3.2. *If K_1 and K_2 are β-polytopes, then $K_1 \times K_2$ is a β-polytope.*

PROOF. For $i = 1, 2$ there exist simplexes S_i, weak* closed subspaces M_i of finite codimension in $A(S_i)^*$ and affine homeomorphisms $\varphi_i: K_i \to M_i \cap S_i$. Let $S \subset A(S_1)^* \times A(S_2)^*$ be the convex hull of $(S_1 \times \{0\}) \cup (\{0\} \times S_2)$, let $K = K_1 \times K_2$ and let $\varphi: K \to S$ be defined by

$$\varphi(x_1, x_2) = (\tfrac{1}{2}\varphi_1 x_1, \tfrac{1}{2}\varphi_2 x_2) = \tfrac{1}{2}(\varphi_1 x_1, 0) + \tfrac{1}{2}(0, \varphi_2 x_2)\,.$$

Define $f: A(S_1)^* \times A(S_2)^* \to R$ by $f(u, v) = \langle u, 1 \rangle$, and let $M = (M_1 \times M_2) \cap$

$f^{-1}(\frac{1}{2})$. Taking the product topology in every case, we will show that S is a simplex, that $M \cap S$ is of the form $\{(u,v) \in S: f_i(u,v)=0,\ i=1,\ldots,n\}$ where $f_1,\ldots,f_n \in A(S)$, and that φ is an affine homeomorphism between K and $M \cap S$.

It is clear that φ is affine, continuous and one-to-one, since each φ_i is. Furthermore, $\varphi_i(x_i) \in M_i$ implies $\frac{1}{2}\varphi_i(x_i) \in M_i$ for each i, so $\varphi(x_1,x_2) \in M_1 \times M_2$. Since $f[\varphi(x_1,x_2)] = \langle \frac{1}{2}\varphi_1 x_1, 1\rangle = \frac{1}{2}$, we have $\varphi(K) \subset M \cap S$. On the other hand, if $(u,v) \in M \cap S$, then there exist s_i in S_i and $0 \leqq \lambda \leqq 1$ with

$$(u,v) = \lambda(s_1,0)+(1-\lambda)(0,s_2) = \big(\lambda s_1,(1-\lambda)s_2\big) .$$

Since $f(u,v)=\frac{1}{2}$, we have $\lambda=\frac{1}{2}$. Now, $(u,v)=(\frac{1}{2}s_1,\frac{1}{2}s_2) \in M_1 \times M_2$ implies $s_i \in M_i \cap S_i = \varphi_i(K_i)$, so that $(u,v) \in \varphi(K)$ and $\varphi(K) = M \cap S$.

By hypothesis, there exist $h_1,\ldots,h_n$ in $A(S_1)$ and $g_1,\ldots,g_m$ in $A(S_2)$ such that

$$\varphi_1(K_1) = S_1 \cap \{x \in S_1 :\ h_j(x)=0, j=1,\ldots,n\}$$

and $\varphi_2(K_2)$ has a similar description in terms of $g_1,\ldots,g_m$. These functionals on $A(S_i)^*$ can be considered as functionals on $A(S_1)^* \times A(S_2)^*$ by letting $h_j(u,v)=h_j(u)$, $j=1,\ldots,n$ and $g_k(u,v)=g_k(v)$, $h=1,\ldots,m$. These are continuous in the product topology, hence on S. If $(x_1,x_2) \in K$, then $h_j[\varphi(x_1,x_2)] = h_j(\frac{1}{2}\varphi_1 x_1, \frac{1}{2}\varphi_2 x_2) = 0$ for each j; similarly, $g_k[\varphi(x_1,x_2)]=0$ and clearly $f[\varphi(x_1,x_2)]=\frac{1}{2}$. Conversely, if a point (u,v) of S satisfies the above set of equations, then $0=h_j(u,v)=h_j(u)$, $0=g_k(v)$ and $f(u,v)=\frac{1}{2}$, that is,

$$(u,v) \in (M_1 \times M_2) \cap f^{-1}(\tfrac{1}{2}) \cap S = M \cap S = \varphi(K) .$$

Thus, $\varphi(K)$ is a slice of S and it remains only to show that S is a simplex.

Following a suggestion by L. Asimow, we simplify our earlier proof of this fact by showing that S is a base for a lattice-ordered cone in $A(S_1)^* \times A(S_2)^*$. Indeed, it is easily verified that since each of the cones R^+S_i $(i=1,2)$ is lattice-ordered, then so is $R^+S_1 \times R^+S_2$. Furthermore, if (r_1s_1, r_2s_2) is a nonzero element of this cone, then $r=r_1+r_2>0$ and hence

$$(r_1s_1,r_2s_2) = r\,[r_1r^{-1}(s_1,0) + r_2r^{-1}(0,s_2)] \in R^+S ,$$

which completes the proof.

Proposition 3.3. *If K is symmetric and infinite dimensional, then K is not a β-polytope.*

Proof. Suppose that $K=-K$ and that $K=S \cap M$, where S is a simplex and M has finite codimension. We'll show that 0 admits two different representations as a convex combination of extreme points of S, contradicting the fact that S is a simplex. Indeed, if $x \in \operatorname{ext} K$, then by Dubins' theorem [6] [10], x is a convex combination of a finite subset $E_x \subset \operatorname{ext} S$, hence $0 = \frac{1}{2}x + \frac{1}{2}(-x)$ is a convex combination of $E_x \cup E_{(-x)}$. Since the latter is finite, its linear span N is finite dimensional, hence $N \cap K$ is a proper subset of K. Choose $y \in \operatorname{ext} K \sim N$; then 0 is a convex combination of $E_y \cup E_{(-y)} \subset \operatorname{ext} S$, and $y \notin N$ implies $E_y \cup E_{(-y)} \not\subset N$, so this representation of 0 differs from that obtained from x.

The first part of the next lemma is due entirely to A. Lazar, and we are grateful to him for his permission to include it here. It is the main step in proving that closed G_δ faces of β-polytopes are "peak sets" for affine continuous functionals.

LEMMA 3.4. *Suppose that K is a compact convex set, that $H \cap K$ is a closed slice of codimension one, and that F is a closed face of $H \cap K$. Then there exists a closed face F_1 of K such that $F = H \cap F_1$. If F is a G_δ in $H \cap K$, then F_1 is a G_δ in K.*

PROOF. We assume that $K \subset A(K)^*$ and that

$$H = \{x \in A(K)^* : f(x) = 0\}$$

for some f in $A(K)$. We can also assume that $\inf f(K) < 0 < \sup f(K)$. [Otherwise, if $\inf f(K) = 0$, say, then $H \cap K$ is a closed G_δ face of K and $F_1 = F$ will suffice.] We define F_1 to be all points x in K such that either $x \in F$ or there exist z in F and y in K such that $z = \lambda x + (1-\lambda)y$ for some λ with $0 < \lambda < 1$.

It is straightforward to show that $F_1 \cap H = F$ and that if $x \in F_1$, $x = \lambda u + (1-\lambda)v$ $(u, v \in K,\ 0 < \lambda < 1)$, then $u, v \in F_1$. It remains to prove that F_1 is closed and convex. Each of these makes use of the following sublemma:

If $x_1, x_2, \in F_1$, with $f(x_i) > 0$ and with corresponding points $y_1, y_2 \in K$, $z_1, z_2 \in F$ such that $z_i \in (x_i, y_i)$, then there exists $z \in F \cap (x_2, y_1)$.

Assume for the moment that this has been proved. To see that F_1 is convex, suppose that $x_1, x_2 \in F_1$ and $0 < \lambda < 1$; we want to show that

$$x = \lambda x_1 + (1-\lambda)x_2 \in F_1 .$$

This is obvious from the definition if $f(x_1)f(x_2) < 0$, so assume that $f(x_i) > 0$, say. With the notation in the sublemma, (x_2, y_1) intersects F, and consideration of the triangle with vertices x_1, x_2 and y_1 shows easily that $x \in F_1$. To see that F_1 is closed, suppose that $\{x_\alpha\}$ is a net in F_1, with $x_\alpha \to x \in K$; we want $x \in F_1$. Without loss of generality we may assume that $f(x_\alpha) > 0$ for each α. Pick some one x_α and call it x_1. Let

y_1 be the corresponding element of K with $f(y_1)<0$ such that (x_1,y_1) intersects F. The sublemma implies that for each α there exists z_α in $F\cap(x_\alpha,y_1)$, that is, there exists $0<\lambda_\alpha<1$ with

$$z_\alpha = \lambda_\alpha x_\alpha+(1-\lambda_\alpha)y_1 \quad \text{and} \quad 0 = \lambda_\alpha f(x_\alpha)+(1-\lambda_\alpha)f(y_1)\,.$$

Thus,

$$\lambda_\alpha=\frac{-f(y_1)}{f(x_\alpha)-f(y_1)} \to \frac{-f(y_1)}{f(x)-f(y_1)}=\lambda$$

and $0<\lambda\leqq 1$. It follows that $z_\alpha\to\lambda x+(1-\lambda)y_1=z$. Since $z_\alpha\in F$ and F is closed, we have $z\in F$. Thus, either $z=x\in F\subset F_1$, or $f(x)>0$ and $z\in(x,y_1)$, hence $x\in F_1$.

We now turn to the proof of the sublemma. Since $f(y_i)<0$, we can choose u in (x_1,y_2) and v in (x_2,y_1) such that $f(u)=0=f(v)$. Let C be the convex hull of x_1, y_1, x_2, y_2, and consider the two cases (I) C is two-dimensional and (II) C is three-dimensional. In case (I), let $z=v$. A consideration of the two possible cases (the segments $[x_1,y_1]$ and $[x_2,y_2]$ disjoint or intersecting) shows that z is either on a segment in F or is an endpoint of a segment in $H\cap K$ which has an interior point in F, so that the entire segment is in F. In case (II), the fact that $\inf f(C)< 0<\sup f(C)$ implies that $H\cap C$ is two-dimensional, wlth vertices u, v, z_1 and z_2. Since C is a tetrahedron, a sketch makes it clear that u,v and z_1,z_2 are opposing pairs of vertices of the quadrangle $H\cap C$, and the diagonals necessarily intersect at w, say. Since $[z_1,z_2]\subset F$, we know $w\in F$, and since F is a face, both u and v are in F. If we let $z=u$, the proof of the sublemma is complete.

It remains to prove the assertion concerning G_δ faces. Suppose first that

$$F \subset G \subset H\cap K, \quad G \text{ open in } H\cap K\,.$$

We can find an open convex subset $U\subset K$ with $F_1\subset U$ and $U\cap H\subset G$, as follows. For each x in $(H\cap K)\sim G$ there exists h in $A(H\cap K)$ with $h(x)<0<\inf h(F)$, by the separation theorem. Since $(H\cap K)\sim G$ is compact, we can find $h_1,\ldots,h_n$ in $A(H\cap K)$ such that

$$(H\cap K) \sim G \subset \textstyle\bigcup_1^n\{x\in H\cap K:\ h_i(x)<0\}\,.$$

If J is the convex hull of the union of the sets $\{x\in H\cap K:\, h_i(x)\leqq 0\}$, then J is compact, convex and contains $(H\cap K)\sim G$. Since F is a face, it is readily verified that it is disjoint from J, and hence F_1 is disjoint from J. By the separation theorem again, there exists h in $A(K)$ with $\sup h(J)<0<\inf h(F_1)$, and we can take $U=\{x\in K:\, h(x)>0\}$. Suppose now that

$$F = \textstyle\bigcap G_n, \quad \text{each } G_n \text{ open in } H\cap K\,.$$

Choose open convex sets U_n in K with $U_n \supset F_1$ and $U_n \cap H \subset G_n$, $n = 1, 2, \ldots$. We want to show that $F_1 = \bigcap U_n$, so suppose $x \in U_n$ for each n. Clearly $x \in F \subset F_1$ if $f(x) = 0$, so assume that $f(x) > 0$, say, and choose y in F_1 with $f(y) < 0$. Then

$$z = \alpha x + (1-\alpha)y \in H \quad \text{if} \quad \alpha = f(y)[f(y) - f(x)]^{-1};$$

since $0 < \alpha < 1$ and $y \in U_n$, we have $z \in U_n \cap H \subset G_n$ for each n. Thus, $z \in F \subset F_1$ and since F_1 is a face, we have $x \in F_1$.

It is now easy to prove that a closed G_δ face of a β-polytope is a "peak set."

THEOREM 3.5. *If K is a β-polytope and if F is a closed face of K which is a G_δ set in K, then there exists a continuous affine functional $f \geqq 0$ on K such that $F = \{x \in K : f(x) = 0\}$.*

PROOF. We can assume that $K = M \cap S$, where S is a simplex in $A(S)^*$ and M is a weak* closed subspace of finite codimension in $A(S)^*$. By Lemma 3.4 and an obvious induction argument, there exists a closed face F_1 of S which is a G_δ set in S and for which $F = M \cap F_1$. By the known theorem [5] [8] [11] for simplexes, there exists g in $A(S)$ with $g \geqq 0$ and $F_1 = \{x \in S\colon g(x) = 0\}$. Letting f be the restriction of g to K produces the desired result.

The next lemma is the main tool in proving that a continuous affine functional on a closed face of a β-polytope admits a continuous affine extension to the entire polytope (Theorem 3.7). It also allows us to show that a finite codimensional slice of a β-polytope is itself a β-polytope; this obvious-sounding result requires us to extend functionals from a slice of a simplex to the entire simplex. The hypothesis in the lemma that K_n is in no proper face of K is needed; as shown in Section 4, the result can fail (even for $n = 1$) if it is not met.

LEMMA 3.6. *Suppose that K is a compact convex set and that*

$$K_n = \{x \in K\colon h_i(x) = 0,\ i = 1, 2, \ldots, n\},$$

where $h_1, h_2, \ldots, h_n \in A(K)$. If K_n is contained in no proper closed face of K, then any continuous affine functional g on K_n has an extension to a continuous affine functional on K.

PROOF. Assume that $K \subset A(K)^*$ and let

$$M = \{x \in A(K)^*\colon h_i(x) = 0,\ i = 1, 2, \ldots, n\}.$$

We first show that M is the same as the linear span $[K_n]$ of K_n. For $k=1,2,\ldots,n$, let

$$K_k = \{x \in K\colon\ h_i(x)=0,\ i=1,2,\ldots,k\}\ ,$$

so that

$$K_n \subset K_{n-1} \subset \ldots \subset K_2 \subset K_1 \subset K_0 \equiv K\ .$$

Since K_n is not in any proper closed face of K_0, we can certainly assume that

$$\alpha_1 = \inf f_1(K_0) < 0 < \sup f_1(K_0) = \beta_1\ .$$

Furthermore,

$$\alpha_k = \inf f_k(K_{k-1}) < 0 < \sup f_k(K_{k-1}) = \beta_k\ ,$$

since otherwise K_n would be in a closed face of K_{k-1}, which in turn would be (as a consequence of Lemma 3.4) in a proper closed face of K. Thus, for each k we can choose u_k, v_k in K_{k-1} with

$$h_k(u_k) = \alpha_k < 0 < \beta_k = h_k(v_k)\ .$$

Suppose, now, that $0 \neq y \in M$; we must show $y \in [K_n]$. If e denotes the identically 1 function in $A(K)$, we can assume that $e(y)=0$. [Indeed, for any x in K_n, the function e vanishes on $y-e(y)x$, and if the latter is in $[K_n]$, then so is y.] It suffices to show that there exists x in K_n and $\alpha>0$ such that $x+\alpha y \in K_n$, since we could then write $y=\alpha^{-1}(x+\alpha y)-\alpha^{-1}x$. For later purposes we will show that if $\|y\|=1$, then there is a certain lower bound for α. (There is obviously no loss in generality in normalizing y.) The functional y on $A(K)\subset C(K)$ can be extended to a functional $\mu=\mu^+-\mu^-$ on $C(K)$ such that

$$1 = \|\mu\| = \mu^+(K)+\mu^-(K) \quad \text{and} \quad 0 = \mu^+(K)-\mu^-(K)\ .$$

By restricting $2\mu^+$ and $2\mu^-$ to $A(K)$ it follows that there exists x_0 in K_0 with x_0+2y in K_0. We next show that there exist x_1 in K_1 and $\lambda_1>0$ such that

$$x_1+2\lambda_1 y \in K_1\ .$$

Indeed, if $x_0 \in K_1$, take $\lambda_1=1$ and take $x_1=x_0$. If $x_0 \notin K_1$, then $h_1(x_0)\neq 0$; say $h_1(x_0)>0$. Let

$$\lambda_1 = \frac{\alpha_1}{\alpha_1-h_1(x_0)}; \quad \text{then } 0 < \frac{\alpha_1}{\alpha_1-\beta_1} \leqq \lambda_1 < 1$$

and with $x_1=\lambda_1 x_0+(1-\lambda_1)u_1$ we get the desired result. (Recall that $h_1(y)=0$.) If $h_1(x_0)$ had been negative, the analogous method would have yielded $(\beta_1-\alpha_1)^{-1}\beta_1$ as a lower bound for λ_1. Proceeding by induction, we will finally obtain $x_n \in K_n$ and $\lambda_1,\ldots,\lambda_n$ such that

$$x_n + 2\lambda_1\lambda_2 \ldots \lambda_n y \in K_n .$$

It is clear that there is a positive lower bound for the constant $\lambda_1\lambda_2 \ldots \lambda_n$, independent of y.

Returning to the function g on K_n, it is clear that it can be extended to a linear functional on $[K_n]$. In order to show that this extension (which we still call g) is weak* continuous on M, it suffices to show that it is weak* continuous on the unit ball of M. (This is a consequence of the Krein–Šmulyan theorem and the fact that M is the dual of a Banach space, namely, if N is the linear span of $h_1, \ldots, h_n$, then M is the dual of $A(K)/N$.] It is readily seen that for this it suffices to show continuity on those $y \in M$ satisfying $\|y\| \leqq 1$ and $e(y) = 0$. But the above argument shows there exists $m > 0$ such that $y = \alpha u - \alpha v$ for $u, v \in K_n$ and $0 \leqq \alpha \leqq m$. Using this fact, together with the weak* compactness of K_n and weak* continuity of g on K_n, we easily see that g is weak* continuous on M. Since M has finite codimension, it is also easy to extend g to a weak* continuous functional on $A(K)^*$, and this is, of course, defined by a function in $A(K)$ which is the desired extension of g to K.

THEOREM 3.7. *Suppose that K is a β-polytope and that F is a closed face of K. If g is a continuous affine functional on F, then g admits an extension to a continuous affine functional f on K.*

PROOF. Choose a simplex S and a weak* closed subspace M of finite codimension in $A(S)^*$ such that K is affinely homeomorphic to (hence may be identified with) $M \cap S$. By Lemma 3.4, there exists a face F_1 of S such that $F = M \cap F_1$. Let F_2 be the smallest closed face of F_1 which contains F. By Lemma 3.6, we can extend g to a continuous affine functional g_1 on F_2 (since $F = M \cap F_2$ is in no proper face of F_2) and by the Edwards–Lazar theorem [8] [11] we can extend g_1 from the face F_2 to the entire simplex S. We then let f be the restriction to K of this last extension.

PROPOSITION 3.8. *If K_1 is a finite codimensional slice of a β-polytope K, then K_1 is a β-polytope.*

PROOF. By hypotheses, there exist functions $f_1, f_2, \ldots, f_n$ in $A(K)$ such that $K_1 = \{x \in K\colon f_i(x) = 0,\ i = 1, 2, \ldots, n\}$. Furthermore, there exists a simplex S and functions $g_1, g_2, \ldots, g_m$ in $A(S)$ such that if

$$M = \{y \in A(S)^*\colon g_i(y) = 0,\ i = 1, 2, \ldots, m\},$$

then K is affinely homeomorphic to $S \cap M$; let $\varphi\colon K \to S \cap M$ denote this homeomorphism. The restriction of φ to K_1 is an affine homeomorphism, so it suffices to produce $h_1, h_2, \ldots, h_{m+n}$ in $A(S)$ such that

$$(*) \qquad \varphi(K_1) = \{y \in S\colon\ h_i(y)=0,\ i=1,2,\ldots,m+n\}\,.$$

The functions $f_i \circ \varphi^{-1}$, $i=1,2,\ldots,n$, are continuous on the β-polytope $\varphi(K)$ and hence, by the proof of Theorem 3.7 [$\varphi(K)$ is a face of itself] can be extended to continuous affine functions $h_1,\ldots,h_n$ on S. If we define $h_{n+k}=g_k$, $k=1,2,\ldots,m$, then it is immediate that (*) is satisfied and the proof is complete.

4. Examples and remarks.

Almost all of the examples in this section are based on the same set, so we first establish the notation which will be used throughout. In the space l_1 of absolutely summable real sequences $x=(x_n)_{n=1}^{\infty}$, we let

$$S = \{x\colon\ x_n \geqq 0 \text{ for each } n \text{ and } \textstyle\sum x_n = 1\}\,.$$

We consider l_1 to be the dual of the space c of all convergent real sequences $y=(y_n)_{n=1}^{\infty}$, with $y_1=\lim y_n$. (The duality being defined by $\langle x,y\rangle = \sum x_n y_n$.) Under the weak* topology defined on l_1 by c, the set S is a compact simplex and if δ_n is the sequence which equals 1 at n, 0 elsewhere, then $\operatorname{ext} S=\{\delta_n\}_{n=1}^{\infty}$. Note that $\operatorname{ext} S$ is weak* compact (in fact, $\delta_n \to \delta_1$).

As we showed earlier, neither α-polytopes nor β-polytopes can be infinite dimensional and centrally symmetric. This fact makes it easy to show that neither of these classes of polytopes is closed under many of the operations which preserve the class of finite dimensional polytopes. For instance, if $K_1=S$, $K_2=-S$ we see that (since it is centrally symmetric) $\operatorname{conv}(K_1 \cup K_2)$ is neither an α-polytope nor a β-polytope. The same conclusion holds for K_1+K_2, for the same reason. Similarly, although the set $[-1,1]\subset R$ is a one-dimensional simplex, the countable product of it with itself (as a subset of the countable product of lines) is centrally symmetric, hence not one of our polytopes. A final example of this kind is the following.

EXAMPLE 4.1. *There exist simplexes S_1 and S_2 such that $S_1 \cap S_2$ is centrally symmetric and infinite dimensional, hence not a polytope.*

PROOF. Let $x=(2^{-1}+2^{-2},2^{-3},2^{-4},2^{-5},\ldots)$ so that $x \in S$. Define $S_1=S-x$ and let $S_2=x-S=-S_1$. Both S_1 and S_2 are simplexes, and $S_1 \cap S_2$ is centrally symmetric. It is readily verified that for $n>1$, $2^{-n-1}(\delta_n-\delta_1)$ is in $S_1 \cap S_2$, so the latter is infinite dimensional and the proof is complete.

EXAMPLE 4.2. *There exists a simplex S and a point $u \notin S$ such that $K = \operatorname{conv}(S \cup \{u\})$ is not a β-polytope.*

PROOF. Let $u = (-1, 1, 2^{-1}, 2^{-2}, 2^{-3}, \ldots)$; then

$$K = \operatorname{conv}(S \cup \{u\}) = \operatorname{cl}\operatorname{conv}(\operatorname{ext} S \cup \{u\}) .$$

Since $\operatorname{ext} S$ is weak* closed, $\operatorname{ext} K \subset \operatorname{ext} S \cup \{u\}$, and using the fact that every point of K has the form $\lambda x + (1-\lambda)u$, $x \in S$, $\lambda \in [0,1]$, it follows easily that

$$\operatorname{ext} K = \operatorname{ext} S \cup \{u\} .$$

Suppose, now, that K is a β-polytope, so that $\varphi K = S' \cap M$ for some simplex S', affine variety M of finite codimension, and affine homeomorphism φ. For each $x \in \operatorname{ext} K$ there exists (just as in the proof of Proposition 3.1) a finite subset $E(x) \subset \operatorname{ext} S'$ such that $\varphi x \in \operatorname{conv} E(x)$. Consider the point

$$y = (0, 2^{-1}, 2^{-2}, 2^{-3}, \ldots) = \tfrac{1}{2}(\delta_1 + u) = \sum_{n=2}^{\infty} 2^{-(n-1)} \delta_n .$$

We have

$$\varphi(y) = \tfrac{1}{2}\varphi(\delta_1) + \tfrac{1}{2}\varphi(u) \in \operatorname{conv}[E(\delta_1) \cup E(u)]$$

and

$$\varphi(y) = \sum_{n=2}^{\infty} 2^{-(n-1)} \varphi(\delta_n) .$$

Each $\varphi(\delta_n)$ has the form $\varphi(\delta_n) = \sum_{k=1}^{m_n} \alpha_{n,k} z_{n,k}$ of a convex combination of elements of $E(\delta_n) \subset \operatorname{ext} S'$, so if we let

$$\mu = \sum_{n=2}^{\infty} 2^{-(n-1)} \left(\sum_{k=1}^{m_n} \alpha_{n,k} \varepsilon_{z_{n,k}} \right),$$

then μ is a probability measure which is supported by $\operatorname{ext} S'$ (hence is maximal) and represents $\varphi(y)$. Since

$$\varphi(y) \in \operatorname{conv}[E(\delta_1) \cup E(u)] \subset \operatorname{ext} S' ,$$

the uniqueness of maximal representing measures requires that $\{z_{n,k}\} \subset E(\delta_1) \cup E(u)$ and hence

$$\varphi(\delta_n) \in \operatorname{conv}[E(\delta_1) \cup E(u)] \quad \text{for } n = 2, 3, \ldots .$$

But this is impossible; the latter convex set is finite dimensional, and the $\varphi(\delta_n)$'s are affinely independent.

EXAMPLE 4.3. *There exists a simplex S and an interval I such that the product $S \times I$ is not an α-polytope. Moreover, there is an interval J in l_1 such that the vector sum $S + J$ also fails to be an α-polytope.*

PROOF. Let $I=[0,1]$ and let $x_0=(2^{-1},2^{-2},2^{-3},\ldots)\in S$. For each $n\geqq 2$, let

$$x_n = (2^{-1},2^{-2},\ldots,2^{-n+2},2^{-n+1},2^{-n+1},0,0,0,\ldots)$$
$$y_n = (2^{-1},2^{-2},\ldots,2^{-n+2},2^{-n+1},0,2^{-n},2^{-n-1},\ldots)\,.$$

Then $x_n,y_n\in S$ and $x_0=\frac{1}{2}x_n+\frac{1}{2}y_n$, $n=2,3,\ldots$. Thus, in $X\times I$ we have

$$(x_0,\tfrac{1}{2}) = \tfrac{1}{2}(x_n,1)+\tfrac{1}{2}(y_n,0), \qquad n=2,3,\ldots\,.$$

We will use this fact to show that $(x_0,\frac{1}{2})$ admits an infinite dimensional set of maximal representing measures. Since $\operatorname{ext}S=\{\delta_n\}_{n=1}^{\infty}$ and

$$\operatorname{ext}(S\times I) = \operatorname{ext}S\times\{0\}\cup\operatorname{ext}S\times\{1\}\,,$$

it is clear that each of the following measures $\mu_n(n=2,3,\ldots)$ represents $(x_0,\frac{1}{2})$ and is supported by $\operatorname{ext}(S\times I)$, hence is maximal. (We denote the unit mass at the extreme points $(\delta_n,1)$ and $(\delta_n,0)$ by $\varepsilon_{n,1}$ and $\varepsilon_{n,0}$, respectively.)

$$\mu_n = \sum_{k=1}^{n-1}2^{-k-1}\varepsilon_{k,1}+2^{-n}\varepsilon_{n,1}+\sum_{k=1}^{n-1}2^{-k-1}\varepsilon_{k,0}+\sum_{k=n+1}^{\infty}2^{-k}\varepsilon_{k,0}\,.$$

If we let

$$\mu_0 = \sum_{k=1}^{\infty}2^{-k-1}\varepsilon_{k,1}+\sum_{k=1}^{\infty}2^{-k-1}\varepsilon_{k,0}\,,$$

then it is clear that μ_0 also represents $(x_0,\frac{1}{2})$, so our proof will be complete if we show that the set $\{\mu_n-\mu_0\}_{n=2}^{\infty}$ is linearly independent. We have

$$\mu_n-\mu_0 = 2^{-n-1}\varepsilon_{n,1}-\sum_{k=n+1}^{\infty}2^{-k-1}\varepsilon_{k,1}-2^{-n-1}\varepsilon_{n,0}+\sum_{k=n+1}^{\infty}2^{-k-1}\varepsilon_{k,0}\,,$$

$n=2,3,\ldots$. Suppose that for some $N>0$ and real numbers $\alpha_2,\alpha_3,\ldots,\alpha_N$ we have $\mu=\sum_{n=2}^{N}\alpha_n(\mu_n-\mu_0)=0$. Since $S\times\{0\}$ and $S\times\{1\}$ are disjoint closed subsets of $S\times I$ it following that the restriction of μ to each of these sets is also zero; in particular, restricting μ to $S\times I$ yields

$$\sum_{n=2}^{N}\alpha_n\left(2^{-n-1}\varepsilon_{n,1}-\sum_{k=n+1}^{\infty}2^{-k-1}\varepsilon_{k,1}\right) = 0\,.$$

Since the extreme points $\{\delta_n\}_{n=2}^{\infty}$ of S are isolated in the weak* topology, we can find for each $n\geqq 2$ a function f_n in $C(S\times\{1\})$ which satisfies $f_n(\delta_k,1)=0$ unless $k=n$, and $f_n(\delta_n,1)=1$. Applying the above measure to f_2 yields $\alpha_2=0$. If we then apply it to f_3 we obtain $\alpha_3=0$, etc. By induction we conclude that each $\alpha_n=0$ and hence the set $\{\mu_n-\mu_0\}$ (and its translate $\{\mu_n\}$) is infinite dimensional; this shows that $S\times I$ is not an α-polytope. Next, let f denote the weak* continuous linear functional defined on l_1 by $f(x)=\sum x_n$ and choose $p \in l_1$ with $f(p)=1$. We let J be the interval $\{tp\colon 0 \leqq t \leqq 1\}$ and we define $\varphi\colon S \times I \to S+J$ by

$$\varphi(x,t) = x+tp, \quad x \in S,\ t \in [0,1] .$$

It is easily verified that φ is an affine continuous surjection, and will be a homeomorphism if it is one-to-one. But if $x+tp=y+sp$, then applying f we get $s=t$ and hence $x=y$. Thus, $S+J$ is affinely equivalent to $S \times I$, so it is not an α-polytope.

The next example shows that a finite dimensional slice of a simplex need not be a polytope.

EXAMPLE 4.4. *There exists a simplex S and a three-dimensional subspace M such that $S \cap M$ is not a polytope.*

PROOF. Let M be the subspace of l_1 generated by $u_1=(2^{-n})$, $u_2=(2 \cdot 3^{-n})$ and $u_3=(7 \cdot 2^{n-1} \cdot 9^{-n})$. Note that u_1, u_2, u_3 are in S and are linearly independent. The intersection $S \cap M$ consists precisely of all points x of the form $x=\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3$ where $\alpha_1+\alpha_2+\alpha_3=1$ and $x_n \geqq 0$ for all n. The latter condition is equivalent to $2^{-n}\alpha_1+2 \cdot 3^{-n}\alpha_2+7 \cdot 2^{n-1} \cdot 9^{-n}\alpha_3 \geqq 0$ for $n=1,2,3,\ldots$, and this in turn is equivalent to

$$(*) \qquad \alpha_1+2(\tfrac{2}{3})^n\alpha_2+(\tfrac{7}{2})(\tfrac{4}{9})^n\alpha_3 \geqq 0, \quad n=1,2,3,\ldots .$$

Since M (under the obvious mapping) is linearly homeomorphic to R^3, we see that $M \cap S$ is affinely homeomorphic to the subset C of R^3 consisting of all $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ which satisfy $\alpha_1+\alpha_2+\alpha_3=1$ and (*). This means that C is a base of the cone K in R^3 defined by (*), so C will have finitely many extreme points only if K has finitely many extreme rays. But the latter will be true only if the polar cone K^0 to K has finitely many extreme rays. Clearly, K^0 is generated by the elements

$$(1, 2(\tfrac{2}{3})^n, (\tfrac{7}{2})(\tfrac{4}{9})^n), \quad n=1,2,\ldots ,$$

and since these lie in the plane $\{\alpha\colon \alpha_1=1\}$, the cone K^0 has finitely many extreme rays only if the closed convex hull of this set of points has finitely many extreme points. But, these points lie on the parabola

$$\{\alpha=(\alpha_1,\alpha_2,\alpha_3)\ :\ \alpha_1=1,\ \alpha_3=(\tfrac{7}{4})\alpha_2{}^2\} ,$$

hence each of them is extreme in their closed convex hull, which shows that C must have infinitely many extreme points.

The next example is related to Theorem 2.10, Lemma 3.6 and Theorem 3.7.

EXAMPLE 4.5. *There exists a metrizable compact convex set K, a closed face F of K and a continuous affine functional on F which admits no continuous affine extension to K.*

PROOF. Let E be the space of all real sequences, in the topology of pointwise convergence. Let

$$F = \{x \in E : x_1 = 0 \text{ and } |x_n| \leqq 2^{-n},\ n = 2, 3, \ldots\}.$$

Let

$$F_1 = \{x \in E : x_1 = 1 \text{ and } |x_n| \leqq n^{-1},\ n = 2, 3, \ldots\}.$$

Both F and F_1 are compact and convex, and hence the convex hull K of $F \cup F_1$ is compact convex and metrizable (since E is metrizable). Since $F = \{x \in K : x_1 = 0\}$, it is clear that F is a closed face of K. If we define g on F by $g(x) = \sum x_n$, then g is readily seen to be continuous and affine. Suppose that f were a continuous affine extension of g to K. Let $\delta_1 = (1, 0, 0, 0, \ldots)$ and let

$$x^n = (0, 2^{-1}, 3^{-1}, \ldots, n^{-1}, 0, 0, 0, \ldots) \quad \text{if } n > 1 .$$

Clearly $\delta_1 \in K$, $2^{-n}x^n \in F$ and $\delta_1 + x^n \in K$. Furthermore, if $\alpha_n = (2^n + 1)^{-1}$, then $0 < \alpha_n < 1$ and

$$\alpha_n(\delta_1 + x^n) + (1 - \alpha_n)\cdot 0 = \alpha_n \delta_1 + (1 - \alpha_n) 2^{-n} x^n .$$

By applying f to both of these convex combinations we obtain

$$\begin{aligned} f(\delta_1 + x^n) &= f(\delta_1) + \alpha_n^{-1}(1 - \alpha_n) g(2^{-n} x^n) \\ &= f(\delta_1) + 2^n \textstyle\sum_{k=2}^{n} 2^{-n} k^{-1} \\ &= f(\delta_1) + \textstyle\sum_{k=2}^{n} k^{-1} . \end{aligned}$$

It is clear from this that $f(\delta_1 + x^n) \to \infty$ as $n \to \infty$, so that f is not even bounded on K.

We next consider the relationships between the class of all α-polytopes and the class of all β-polytopes. They both contain the simplexes and the finite dimensional polytopes, of course. The product $S \times I$ of two simplexes defined in Example 4.3 is (by virtue of Proposition 3.2) a β-polytope but not an α-polytope. Similarly, the set $\operatorname{conv}(S \cup \{u\})$ defined in Example 4.2 is an α-polytope (by Proposition 2.9) but not a β-polytope, so neither class contains the other. Finally, the next example show that the intersection of the two classes contains an infinite dimensional set which is not a simplex.

EXAMPLE 4.6. *There exists an infinite dimensional set which is both an α-polytope and a β-polytope, but not a simplex.*

PROOF. Define f on S by $f(x)=x_2+x_3-x_4-x_5$; then f is affine and continuous and hence $K=S\cap f^{-1}(0)$ is a β-polytope. Furthermore, the map $\varphi\colon S \to K$ defined by

$$\varphi(x) = (x_1, \tfrac{1}{2}(x_2+x_4), \tfrac{1}{2}(x_3+x_5), \tfrac{1}{2}(x_2+x_5), \tfrac{1}{2}(x_3+x_4), x_6, x_7, x_8, \ldots)$$

is easily seen to be a continuous affine surjection, so K is also an α-polytope. Finally, we see that

$$e_1 = \tfrac{1}{2}(\delta_2+\delta_4), \quad e_2 = \tfrac{1}{2}(\delta_3+\delta_5), \quad e_3 = \tfrac{1}{2}(\delta_2+\delta_5), \quad e_4 = \tfrac{1}{2}(\delta_3+\delta_4)$$

are distinct extreme points of K and

$$(0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, \ldots) = \tfrac{1}{2}(e_2+e_3) = \tfrac{1}{2}(e_4+e_5)$$

is an element of K having two different representations by probability measures on $\operatorname{ext} K$, so K is not a simplex.

REFERENCES

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. 15 (1964), 97–110.
2. L. Asimow, *Directed Banach space of affine functions*, Trans. Amer. Math. Soc. (to appear).
3. A. Bastiani, *Cones convexes et pyramides convexes*, Ann. Inst. Fourier (Grenoble) 9 (1959), 249–292.
4. N. Boboc and A. Cornea, *Cônes des fonctions continues sur un espace compact*, C. R. Acad. Sci. Paris 261 (1965), 2564–2567.
5. E. B. Davies, *Exposed sets in convexity theory.*
6. L. E. Dubins, *On extreme points of convex sets*, J. Math. Anal. Appl. 5 (1962), 237–244.
7. D. A. Edwards, *On the homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology*, Proc. Lond. Math. Soc. (3) 14 (1964), 399–414.
8. D. A. Edwards, *Minimum-stable wedges of semicontinuous functions*, Math. Scand. 19 (1966), 15–26.
9. B. Grünbaum, *Convex polytopes* (Pure and Applied Math. 16), Interscience Publ., New York, N.Y., 1967.
10. V. Klee, *On a theorem of Dubins*, J. Math. Anal. Appl. 7 (1963), 425–427.
11. A. J. Lazar, *On affine continuous functions on simplexes.* (Dittoed notes.)
12. A. J. Lazar, *Spaces of affine continuous functions on simplexes*, Trans. Amer. Math. Soc. 134 (1968), 503–525.
13. P. H. Maserick, *Convex polytopes in linear spaces*, Illinois J. Math. 9 (1965), 623–635.
14. P. H. Maserick, *Dual convex polytopes in Banach spaces*, J. Math. Anal. Appl. 19 (1967), 263–273.
15. R. R. Phelps, *Lectures on Choquet's theorem* (Van Nostrand Math. Studies 7), Princeton, N.J., 1966.
16. Z. Semadeni, *Free compact sets*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. 13 (1964), 141–146.

UNIVERSITY OF WASHINGTON, SEATTLE, WASH., U.S.A.