MATH. SCAND. 24 (1969), 66–78

ON DECOMPOSITION THEOREMS OF MEYER

K. MURALI RAO

Introduction.

It is well known that a discrete parameter supermartingale is the sum of a martingale and a decreasing process. Meyer [1, p. 104] gave necessary and sufficient conditions under which a continuous parameter supermartingale is the sum of a martingale and a "natural decreasing process": a process with a very special, important property. This concept of a natural process, however, is quite untractable without the aid of a theory of stop rules which is in itself important and interesting. Nowhere in Meyer's proof there is any indication of a way from the discrete case to the continuous case. We shall give a proof of Meyer's theorem which is both elementary and short. We shall show that Meyer's decomposition can be obtained by passage to the limit from the discrete case. Our proof clarifies the relation between Doob decomposition and Meyer's decomposition. The natural process will appear as the continuous analogue of the process occurring in the Doob decomposition in the discrete case. It will be noticed that our proof that the natural process corresponding to a regular potential is continuous, is also simpler than that of Meyer.

The author is indebted to professor K. Itô for his interest and encouragement.

1.

For definitions of terms used we refer to [1]. In the following Ω stands for a probability space. If $X_n, n \geqq 0$, is a supermartingale relative to an increasing sequence F_n of σ-fields, it is well known [1, p. 104] that we can write, in exactly one way,

$$X_n = M_n - A_n ,$$

where M_n is a martingale relative to F_n, $A_0 = 0$, $A_n \leqq A_{n+1}$, and A_{n+1} is F_n-measurable, for $n \geqq 0$. We call this the Doob-decomposition of the supermartingale X_n. If moreover X_n is non-negative, then the

Received March 10, 1968.

expectation of A_n remains bounded by $E(M_0)$ and therefore $A_\infty = \lim A_n$ (which exists without any restriction on X_n) has a finite expectation. Following Meyer we call a non-negative supermartingale X_n for which $\lim E(X_n) = 0$ a *potential*. Thus, if X_n is a potential, $M_n = A_n + X_n$ is a uniformly integrable martingale since both A_n and X_n are uniformly integrable: A_n because $0 \leqq A_n \leqq A_\infty$ and $E(A_\infty) < \infty$, and X_n because $X_n \geqq 0$ and $E(X_n) \to 0$. The martingale convergence theorem implies that $M_\infty = \lim M_n$, $X_\infty = \lim X_n$ both exist and $M_n = E(M_\infty | F_n)$. Since X_n is a potential, Fatou's lemma implies that $X_\infty = 0$ almost surely, that is, that $A_\infty = M_\infty$ a.s. Thus:

LEMMA 1. *A potential X_n can be written in the form*

$$X_n = E(A_\infty | F_n) - A_n, \quad n \geqq 0,$$

where $A_0 = 0$, and A_{n+1} is F_n-measurable.

It is immediately seen that

$$A_{n+1} - A_n = E(X_n - X_{n+1} | F_n), \quad n \geqq 0.$$

We also note that if T is a stopping time, then [1, p. 90]

$$X_T = E(A_\infty | F_T) - A_T \quad \text{a.s.}$$

It is natural to ask if Lemma 1 extends to continuous parameter supermartingales. Since the representation $X_n = M_n - A_n$ is not unique without the condition that A_{n+1} be F_n-measurable, one should also seek a continuous analogue of this restriction. To this end let us reformulate the condition that A_{n+1} be F_n-measurable. Assuming that $E(A_\infty) < \infty$, it is not difficult to show that A_{n+1} is F_n-measurable iff for all bounded martingales Y_n we have

$$E\left(\textstyle\sum_1^\infty Y_{k-1}(A_k - A_{k-1})\right) = E(Y_\infty A_\infty),$$

where $Y_\infty = \lim Y_n$. In the continuous parameter case the left hand side of the last equation should correspond to $E\left(\int_0^\infty Y(s-)\,dA(s)\right)$.

In the sequal, $F(t)$ denotes an increasing right continuous family of σ-fields. We assume that all processes considered are adapted to $F(t)$ [1, p. 65] and that, with probability one, they are right continuous and have left limits at every time point. An increasing process $A(t)$ with $A(0) = 0$ and $E(A(\infty)) < \infty$, where $A(\infty) = \lim_{t\to\infty} A(t)$, is called a *natural integrable increasing* process if for every bounded positive martingale $Y(t)$ we have

$$E\left(\int_0^\infty Y(s-)\,dA(s)\right) = E\left(A(\infty)\,Y(\infty)\right)$$

where as usual $Y(\infty)=\lim_{t\to\infty} Y(t)$. As in the discrete case, a potential $X(t)$ is by definition a non-negative supermartingale for which $\lim_{t\to\infty} E(X(t))=0$. Now we can state Meyer's theorem which answers the above questions completely.

THEOREM (Meyer). *A potential $X(t)$ can be decomposed in the form*

$$X(t) = E(A(\infty)\,|\,F(t)) - A(t), \quad t \geqq 0,$$

where $A(0)=0$, and $A(t)$ is increasing, iff the family $\{X(T)\}$ of random variables is uniformly integrable for all stopping times T.

In this case $A(t)$ can be taken to be a natural integrable increasing process.

A decomposition with $A(t)$ natural is unique.

PROOF. For each natural integer n

$$X(i2^{-n}), \quad i=0,1,2,\ldots,$$

is a discrete potential. We can therefore, by Lemma 1, write

$$X(i2^{-n}) = E(A(\infty, n) \mid F(i2^{-n})) - A(i2^{-n}), \quad i \geqq 0,$$

where $A(i2^{-n},n)$ is $F((i-1)2^{-n})$-measurable, $i \geqq 1$. Here

$$A(\infty,n) = \lim_{i\to\infty} A(i2^{-n},n).$$

Suppose we know that the sequence $A(\infty,n)$ is uniformly integrable. A necessary and sufficient condition for this will be given later. There exists a function $A(\infty)$ such that $A(\infty,n)$ tends weakly to $A(\infty)$ along a subsequence, say $\mathfrak{A}$ [1, p. 20]. Denote by $M(t)$ a right continuous modification of the martingale $E(A(\infty)\,|\,F(t))$ [1, p. 95]. If $r \leqq s$ are dyadic rationals then from a certain n on, $A(r,n)$ and $A(s,n)$ make sense and $A(r,n) \leqq A(s,n)$, that is,

$$E(A(\infty,n)\,|\,F(r)) - X(r) \leqq E(A(\infty,n) \mid F(s)) - X(s).$$

The operation of conditional expectation being continuous in the weak topology as n tends to ∞ along $\mathfrak{A}$, we get

$$M(r) - X(r) \leqq M(s) - X(s) \quad \text{a.s.}$$

Put $A(t)=M(t)-X(t)$. $A(t)$ is right continuous and almost surely increasing on the dyadic rationals and hence everywhere. Since $X(t) \to 0$ as $t \to \infty$,

$$\lim_{t\to\infty} A(t) = A(\infty).$$

We now show that the process $A(t)$ is natural, that is, that for every bounded martingale $Y(t)$ whose paths have almost sure left limits we have

$$E(\int_0^\infty Y(s-)dA(s)) = E(A(\infty)Y(\infty)),$$

where $Y(\infty)=\lim_{t\to\infty} Y(t)$. By Lebesgue's bounded convergence theorem,

$$E(\int_0^\infty Y(s-)dA(s)) = \lim_{n\to\infty}\sum_{i=0}^{\infty} E\Big(Y(i2^{-n})\big(A((i+1)2^{-n})-A(i2^{-n})\big)\Big).$$

Because $Y(i2^{-n})$ is $F(i2^{-n})$-measurable, we have

$$\begin{aligned}\sum_{i=0}^{\infty} E&\Big(Y(i2^{-n})\big(A((i+1)2^{-n})-A(i2^{-n})\big)\Big)\\ &= \sum_{i=0}^{\infty} E\Big(Y(i2^{-n})E\big(A((i+1)2^{-n}) - A(i2^{-n})\,|\,F(i2^{-n})\big)\Big)\\ &= \sum_{i=0}^{\infty} E\Big(Y(i2^{-n})E\big(X(i2^{-n}) - X((i+1)2^{-n})\,|\,F(i2^{-n})\big)\Big)\\ &= \sum_{i=0}^{\infty} E\Big(Y(i2^{-n})\big(A((i+1)2^{-n},n)-A(i2^{-n},n)\big)\Big),\end{aligned}$$

since $X(t)+A(t)$ is a martingale, and by the definition of $\{A(i2^{-n},n)\}$. Also

$$E\big(Y(i2^{-n})A((i+1)2^{-n},n)\big) = E\big(Y((i+1)2^{-n})A((i+1)2^{-n},n)\big),$$

since $A\big((i+1)2^{-n},n\big)$ is $F(i2^{-n})$-measurable. This means that for all n,

$$\sum_{i=0}^{\infty} E\Big(Y(i2^{-n})\big(A((i+1)2^{-n})-A(i2^{-n})\big)\Big) = E\big(A(\infty,n)\,Y(\infty)\big).$$

One need only take limits along the sequence $\mathfrak{A}$.

The above equation actually shows that $A(\infty,n)$ converges weakly to $A(\infty)$ as $n\to\infty$. In fact, if $Y(\infty)$ is an $F(\infty)$-measurable random variable, $F(\infty)$ being the σ-field generated by all $F(t)$, $t\geqq 0$, and $Y(t)$ a right continuous-with-left-limits modification of the martingale $E\big(Y(\infty)\,|\,F(t)\big)$, the left hand side of the last equation tends, as $n\to\infty$, to $E(\int_0^\infty Y(s-)dA(s))$ which we know is equal to $E\big(Y(\infty)A(\infty)\big)$. This means that

$$E\big(Y(\infty)A(\infty,n)\big) \to E\big(Y(\infty)A(\infty)\big) \quad \text{as } n\to\infty,$$

that is, that $A(\infty,n)\to A(\infty)$ weakly. This in turn shows that two natural processes defining the same potential must be identical almost surely. Indeed let $A(t), B(t)$ be natural and define $X(t)$. The process $X(t)$ determines $A(\infty,n)$ and we see from the above-said that $A(\infty,n)$ converges weakly. It can have but one weak limit. Therefore $A(\infty)=B(\infty)$ a.s. We also know that

$$X(t) = E(A(\infty)\,|\,F(t)) - A(t) = E(B(\infty)\,|\,F(t)) - B(t)\,.$$

This completes the proof.

Following Meyer we say that a potential $X(t)$ belongs to the class D if $\{X(T)\}$ is uniformly integrable for all stopping times T, finite or not, where we define $X(T)=0$ whenever $T=\infty$.

LEMMA 2. *Let $X(t)$ be a potential and let $A(\infty,n)$ be defined as before. A necessary and sufficient condition that $\{A(\infty,n)\}$ be uniformly integrable is that $X(t)$ belongs to D.*

PROOF. If the $A(\infty,n)$ are uniformly integrable, we can write

$$X(t) = E(A(\infty)\,|\,F(t)) - A(t) = M(t) - A(t) \text{ (say)}$$

where $A(t)$ increases. Then

$$X(T) \leqq M(T) = E(A(\infty)\,|\,F(T))$$

[1, p. 106], and it is easy to see from this that the family $\{X(T)\}$ is uniformly integrable.

Conversely, suppose that $X(t)$ belongs to D. For every $\lambda>0$ and every integer n, define (observe that $\inf\emptyset=\infty$)

$$T_{n,\lambda} = \inf\{i2^{-n} : A((i+1)2^{-n},n)>\lambda\}\,,$$

where, as before, $A(i2^{-n},n)$ is the increasing process appearing in the Doob decomposition of the discrete potential $\{X(i2^{-n},n);\ i=0,1,2,\ldots\}$. It is clear that $T_{n,\lambda}$ is a stopping time since $A(i2^{-n},n)$ is $F((i-1)2^{-n})$-measurable. Also $A(\infty,n)>\lambda$ iff $T_{n,\lambda}<\infty$. We have

$$X(T_{n,\lambda}) = E(A(\infty,n)\,|\,F(T_{n,\lambda})) - A(T_{n,\lambda},n)$$

which implies

$$\begin{aligned}(1)\qquad & E(A(\infty,n) : A(\infty,n)>\lambda) \\ &\quad = E(A(T_{n,\lambda},n) : T_{n,\lambda}<\infty) + E(X(T_{n,\lambda}) : T_{n,\lambda}<\infty) \\ &\quad \leqq \lambda P(A(\infty,n)>\lambda) + E(X(T_{n,\lambda}) : T_{n,\lambda}<\infty)\,,\end{aligned}$$

since $A(T_{n,\lambda},n)\leqq\lambda$ by the definition of $T_{n,\lambda}$. This means that

$$E(A(\infty,n)-\lambda : A(\infty,n)>2\lambda) \leqq E(X(T_{n,\lambda}) : T_{n,\lambda}<\infty)$$

or that

$$(2)\qquad 2\lambda P(A(\infty,n)>2\lambda) \leqq 2E(X(T_{n,\lambda}) : T_{n,\lambda}<\infty)\,.$$

Replacing λ by 2λ in (1) we get

$$\begin{aligned} E(A(\infty,n) &: A(\infty,n) > 2\lambda) \\ &\leqq 2\lambda P(A(\infty,n) > 2\lambda) + E(X(T_{n,2\lambda}) : T_{n,2\lambda} < \infty) \\ &\leqq 2E(X(T_{n,\lambda}) : T_{n,\lambda} < \infty) + E(X(T_{n,2\lambda}) : T_{n,2\lambda} < \infty) \end{aligned}$$

by (2). Finally,

$$\lambda P(T_{n,\lambda} < \infty) \leqq E(A(\infty,n)) = E(X_0)$$

since $A(o)=0$ so that the set $(A(\infty,n) > \lambda)$ has small probability for all large λ. The fact that $\{X(T)\}$ is uniformly integrable for all stopping times T implies that $X(T)$ has small expectation over sets of small probability, uniformly for all stopping times [1, p. 17], q.e.d.

2.

In this section we shall deduce some other results of Meyer. It is clear from Section 1 that the concept of a natural process is relevant already in the discrete case. As a further confirmation of this fact we shall prove

THEOREM 2.1. *Let $X(t)$ be a potential and let*

$$X(t) = E(A(\infty) | F(t)) - A(t) ,$$

with $A(t)$ natural; then

$$E(A(\infty)^2) = E\left(\int_0^\infty (X(t) + X(t-)) dA(t)\right) .$$

PROOF. Suppose first that $E(A(\infty)^2) < \infty$ and let $M(t)$ denote the martingale $E(A(\infty) | F(t))$. We then have

$$\begin{aligned} &E\left(\int_0^\infty M(t) dA(t)\right) \\ &\quad = \lim_{k\to\infty} E\left(\sum_{i=0}^\infty M((i+1)k^{-1})\left(A((i+1)k^{-1}) - A(ik^{-1})\right)\right) \\ &\quad = \lim_{k\to\infty} \sum_{i=0}^\infty E\left(M((i+1)k^{-1})\left(A((i+1)k^{-1}) - A(ik^{-1})\right)\right) \\ &\quad = \lim_{k\to\infty} \sum_{i=0}^\infty E\left(M((i+1)k^{-1})A((i+1)k^{-1}) - M(ik^{-1})A(ik^{-1})\right) \\ &\quad = \lim_{k\to\infty} E(M(\infty)A(\infty)) = E(A(\infty)^2) \end{aligned}$$

since

$$E\left(M((i+1)k^{-1})A(ik^{-1})\right) = E(M(ik^{-1})A(ik^{-1})) .$$

We have therefore

$$
\begin{aligned}
&E\big(\textstyle\int_0^\infty (X(t)+X(t-))\,dA(t)\big) \\
&\quad = E\big(\textstyle\int_0^\infty (M(t)+M(t-))\,dA(t)\big) - E\big(\textstyle\int_0^\infty (A(t)+A(t-))\,dA(t)\big) \\
&\quad = 2E(A^2(\infty)) - E(A^2(\infty)) = E(A(\infty)^2)
\end{aligned}
$$

because, $A(t)$ being natural,

$$E(\textstyle\int_0^\infty M(t-)\,dA(t)) = E(M(\infty)A(\infty)) = E(A(\infty)^2)\,,$$

and

$$
\begin{aligned}
&E\big(\textstyle\int_0^\infty (A(t)+A(t-))\,dA(t)\big) \\
&\quad = \lim_{k\to\infty} E\Big(\textstyle\sum_{i=0}^\infty \big(A((i+1)k^{-1})+A(ik^{-1})\big)\big(A((i+1)k^{-1})-A(ik^{-1})\big)\Big) \\
&\quad = \lim_{k\to\infty} E\Big(\textstyle\sum_i \big(A((i+1)k^{-1})^2 - A(ik^{-1})^2\big) = E\big(A(\infty)^2\big)\Big)\,.
\end{aligned}
$$

Conversely suppose $E\big(\int_0^\infty (X(t)+X(t-))\,dA(t)\big)<\infty$. From the abovesaid it follows that we need only show $E\big(A(\infty)^2\big)<\infty$. Retaining the notation of Section 1 it is obviously sufficient to show that $E\big(A(\infty,n)^2\big)<\infty$ (taking weak limits does not increase norms). Now the fact that $E(\int_0^\infty X(t-)\,dA(t))<\infty$ obviously implies that for all large n

$$E\Big(\textstyle\sum_{i=0}^\infty X(i2^{-n})\big(A((i+1)2^{-n})-A(i2^{-n})\big)\Big) < \infty\,,$$

that is,

$$E\Big(\textstyle\sum_i X(i2^{-n})\big(A((i+1)2^{-n},n)-A(i2^{-n},n)\big)\Big) < \infty\,.$$

Thus it is sufficient to show the following: If $X(n)$ is a discrete potential, if

$$X(n) = E\big(A(\infty)\,|\,F(n)\big)-A(n)$$

is its Doob-decomposition (this means that $A(n+1)$ is $F(n)$-measurable), and if

$$\textstyle\sum_{n=0} E\big(X(n)(A(n+1)-A(n))\big) < \infty\,,$$

then $E\big(A(\infty)^2\big)<\infty$. Since $E\big(X(n+1)\,|\,F(n)\big)\leqq X(n)$, we have

$$\textstyle\sum_n E\big(X(n+1)(A(n+1)-A(n))\big) \leqq \sum_n E\big(X(n)(A(n+1)-A(n))\big)\,.$$

For any integer N, let $A_N(n)=\min(N,A(n))$ and write

$$X_N(n) = E\big(A_N(\infty)\,|\,F(n)\big) - A_N(n)\,.$$

Since $A_N(n)$ is a bounded "natural" process, we have

$$
\begin{aligned}
E\big(A_N(\infty)^2\big) &= \textstyle\sum_n E\big((X_N(n+1)+X_N(n))(A_N(n+1)-A_N(n))\big) \\
&\leqq \textstyle\sum_n E\big((X(n+1)+X(n))(A_N(n+1)-A_N(n))\big) \\
&\leqq E\big((X(n+1)+X(n))(A(n+1)-A(n))\big)\,.
\end{aligned}
$$

We then need only let $N\to\infty$.

Before proving another result of Meyer we need a definition.

Definition. Let T be a stopping time. We shall denote by $F(T-)$ the σ-field generated by $F(0)$ and the class of all events of the form $E\cap(T>t)$ for all $E\in F(t)$ and for all $t\geqq 0$.

Lemma 2.1. *The following statements hold:*

1° $F(T-)\subset F(T)$.

2° $S\leqq T \Rightarrow F(S-)\subset F(T-)$. *For any stopping times* S,T *and for any set* $E\in F(S)$, *the event* $E\cap(S<T)$ *is in* $F(T-)$.

3° *If* S_n *increases towards* T, *then* $F(T-)$ *is the* σ*-field generated by* $\bigcup_n F(S_n-)$. *Therefore, according to* 1°, $F(T-)$ *is contained in the* σ*-field generated by* $\bigcup_n F(S_n)$.

Proof.

1° Easy.

2° Since $S\leqq T$ we have $(S>t)=(S>t)\cap(T>t)$. If $E\in F(t)$, then also $E_1=E\cap(S>t)\in F(t)$ and $E\cap(S>t)=E_1\cap(T>t)\in F(T-)$ by definition. For any $E\in F(S)$ we have

$$E\cap(S<T) = \bigcup E\cap(S<r)(r<T)\,,$$

where the union is over all rationals. By the definition of $F(S)$, each of the events $E\cap(S<r)$ is in $F(r)$ and therefore $E\cap(S<r)\cap(r<T)$ is in $F(T-)$ for each r.

3° Obviously $(T>t)=\bigcup_n(S_n>t)$.

Lemma 2.2. *If* $A(t)$ *is a natural increasing process then for every stopping time* $T, A(T)$ *is* $F(T-)$*-measurable.*

Proof. Let us retain the notation of Section 1. It is immediately verified that

$$E\big(A(\infty,n)\,|\,F(t)\big) = E\big(A(\infty)-A((k+1)2^{-n})\mid F(t)\big)+A\big((k+1)2^{-n},n\big)$$

if $k2^{-n}\leqq t<(k+1)2^{-n}$. Let T_n denote the stopping time defined by

$$T_n = (i+1)2^{-n} \quad \text{if} \quad i2^{-n}\leqq T<(i+1)2^{-n}\,.$$

Then the above relation can be generalised to

$$E\big(A(\infty,n)\,|\,F(T)\big) = E\big(A(\infty)-A(T_n)\,|\,F(T)\big)+A(T_n,n).$$

Since $A(\infty,n)\to A(\infty)$ weakly so does $E\big(A(\infty,n)\,|\,F(T)\big)$ tend weakly to $E\big(A(\infty)\,|\,F(T)\big)$. By the right continuity, $E\big(A(T_n)\,|\,F(T)\big)$ tends to $A(T)$

in L_1. This means that $A(T_n, n)$ tends weakly to $A(T)$. Using the fact that $A((i+1)2^{-n}, n)$ is $F(i2^{-n})$-measurable it is easy to show that $A(T_n, n)$ is $F(T)$-measurable. The same then is true for the weak limit $A(T)$.

Lemma 2.1 and Lemma 2.2 together imply

COROLLARY 2.1 (Meyer). *If $A(t)$ is a natural process and S_n is a sequence of stopping times increasing to the stopping time T, then $A(T)$ is measurable with respect to the σ-field generated by $\bigcup_n F(S_n)$.*

REMARK. For any two stopping times T, S and any $E \in F(S)$ the event $E \cap (S < T)$ is in $F(T-)$. Therefore if $P(S<T)=1$ we have $F(S) \subset F(T-)$. Suppose a process $A(t)$ is such that for every increasing sequence S_n of stopping times converging to $T, A(T)$ is measurable with respect to the σ-field generated by $\bigcup F(S_n)$. Then for any accessible stopping time [1, p. 130] $A(T)$ is necessarily $F(T-)$-measurable. However, if T is totally inaccessible [1, p. 130] and if the process $A(t)$ is natural, then $P(A(T)=A(T-))=1$ [1, p. 135] so that again $A(T)$ is $F(T-)$-measurable. Thus Lemma 2.2 is also a consequence of Corollary 2.1. If $P(t)$ is the Poisson process and T is the time of first jump, it is easily shown that $P(T)$ (which is 1 or 0) is $F(T-)$-measurable; it is known that T is totally inaccessible.

3. The case of regular potentials.

Let $A(t)$ be any increasing process, natural or not. Define the submartingales $A_n(t)$ by

$$A_n(t) = E\big(A((k+1)2^{-n}) \,|\, F(t)\big) \quad \text{if} \quad k2^{-n} \leqq t < (k+1)2^{-n}.$$

It is clear that the $A_n(t)$ decrease and that $P(A_n(t) \to A(t)) = 1$ for all t. It is assumed, as usual, that $A_n(t)$ are right continuous and have left limits at all points. For every stopping time T we have

$$A_n(T) = E\big(A(\Phi_n(T)) \,|\, F(T)\big)$$

where the function $\Phi_n(t)$ is defined by

$$\Phi_n(t) = (k+1)2^{-n} \quad \text{if} \quad k2^{-n} \leqq t < (k+1)2^{-n}.$$

This is a consequence of the optional sampling theorem. For every ε let the stopping times $T_{n,\varepsilon}$ be defined by

$$T_{n,\varepsilon} = \inf\{t : A_n(t) - A(t) \geqq \varepsilon\};$$

note that $\inf \emptyset = \infty$.

Clearly $T_{n,\varepsilon} \leqq T_{n+1,\varepsilon}$. Put $T_\varepsilon = \lim T_{n,\varepsilon}$. We claim that

$$E(A(T_\varepsilon) - A(T_{n,\varepsilon})) \geqq \varepsilon P(T_\varepsilon < \infty) \quad \text{for all } n .$$

This is an easy consequence of the following obvious relations:

$$E\big(A(\Phi_n(T_\varepsilon)) - A(T_\varepsilon)\big) \to 0 \quad \text{as } n \to \infty ,$$
$$A(\Phi_n(T_\varepsilon)) \geqq A(\Phi_n(T_{n,\varepsilon})) ,$$
$$E\big(A(\Phi_n(T_\varepsilon)) \mid F(T_{n,\varepsilon})\big) \geqq E\big(A(\Phi_n(T_{n,\varepsilon})) \mid F(T_{n,\varepsilon})\big) = A_n(T_{n,\varepsilon}) ,$$
$$A_n(T_{n,\varepsilon}) - A(T_{n,\varepsilon}) \geqq \varepsilon \quad \text{if} \quad T_{n,\varepsilon} < \infty .$$

Now suppose in addition that $A(t)$ is natural, $E(A(\infty)^2) < \infty$, and that $P(A(r) = A(r-)) = 1$ for every dyadic rational r. Since $A_n(t)$ is a martingale in the range $(k2^{-n}, (k+1)2^{-n})$, the assumptions on $A(t)$ imply, using the notation

$$\Delta_{nk} = \{t : k2^{-n} \leqq t < (k+1)2^{-n}\} ,$$

that

$$\begin{aligned} E(\textstyle\int_{\Delta_{nk}} A_n(t)\,dA(t)) &= E(\textstyle\int_{\Delta_{nk}} M_k(t)\,dA(t)) \\ &= E(\textstyle\int_{\Delta_{nk}} M_k(t)\,dA(t)) \\ &= E\big(A((k+1)2^{-n})\big(A((k+1)2^{-n}) - A(k2^{-n})\big)\big) \\ &= E(\textstyle\int_{\Delta_{nk}} M_k(t-)\,dA(t)) \\ &= E(\textstyle\int_{\Delta_{nk}} M_k(t-)\,dA(t)) \\ &= E(\textstyle\int_{\Delta_{nk}} A_n(t-)\,dA(t)), \end{aligned}$$

where $M_k(t)$ denotes the martingale $E\big(A((k+1)2^{-n}) \mid F(t)\big)$. Summation over k then yields

$$E(\textstyle\int_0^\infty A_n(t-)\,dA(t)) = \sum_k E\big(A((k+1)2^{-n})\big(A((k+1)2^{-n}) - A(k2^{-n})\big)\big) .$$

Thus we get

$$E\big(\textstyle\int_0^\infty (A(t) - A(t-))\,dA(t)\big) = \lim_{n\to\infty} E\big(\textstyle\int_0^\infty (A_n(t-) - A(t-))\,dA(t)\big) .$$

Also, putting $T = T_{n,\varepsilon}$,

$$\begin{aligned} &E\big(\textstyle\int_0^\infty (A_n(t-) - A(t-))\,dA(t)\big) \\ &= E\big(\textstyle\int_0^T (A_n(t-) - A(t-))\,dA(t)\big) + E\big(\textstyle\int_{(T,\infty)} (A_n(t-) - A(t-))\,dA(t)\big) \\ &\leqq \varepsilon E(A(T_{n,\varepsilon})) + E(\textstyle\int_{(T,\infty)} A_n(t-)\,dA(t)) \\ &\leqq \varepsilon E(A(T)) + E\big(A(\infty)(A(\infty) - A(T))\big), \end{aligned}$$

by the definition of $T = T_{n,\varepsilon}$ and because

$$E(\textstyle\int_{(T,\infty)} A_n(t-)\,dA(t)) \leqq E(\int_{(T,\infty)} B(t-)\,dA(t))$$
$$= E\big(A(\infty)(A(\infty) - A(T))\big),$$

where $B(t)$ is the martingale $B(t) = E(A(\infty)\,|\,F(t))$.

THEOREM 3.1. *For any increasing process* $A(t)$

$$E(A(T_\varepsilon) - A(T_{n,\varepsilon})) \geqq \varepsilon P(T_\varepsilon < \infty) \quad \textit{for all } n\,.$$

If further $A(t)$ *is a natural increasing process such that* $E(A(\infty)^2) < \infty$, $P(A(r) = A(r-)) = 1$ *for every dyadic rational and* $P(T_\varepsilon < \infty) = 0$ *for every* ε, *then* $A(t)$ *is a continuous process.*

PROOF. Only the last statement needs proof. Indeed we have for every ε

$$E\big(\textstyle\int_0^\infty (A(t) - A(t-))\,dA(t)\big) \leqq \varepsilon E(A(\infty))\,.$$

The left hand side is therefore zero and this means that $A(t)$ is almost surely continuous.

DEFINITION. Let $X(t)$ be an arbitrary potential. We shall say that $X(t)$ is *regular* if for every sequence T_n of stopping times increasing to T we have

$$\lim_{n\to\infty} E(X(T_n)) = E(X(T))\,.$$

From Theorem T 20, p. 102 of [1], we immediately deduce that a regular potential automatically satisfies the hypotheses of Meyer's theorem, and hence every regular potential has a Doob decomposition. Meyer proves that the natural process occurring in the Doob decomposition of a regular potential is continuous. His proof makes use of the fact that a natural process is the weak limit of continuous processes.

THEOREM 3.2. *Let* $X(t)$ *be a regular potential. Then the corresponding natural process is continuous.*

PROOF. We shall prove this assuming that $X(t)$ is bounded. The general case then follows as in [1, p. 127]. Write

$$X(t) = E(A(\infty)\,|\,F(t)) - A(t)\,.$$

By Theorem 2.1,

$$E(A(\infty)^2) < \infty\,.$$

The regularity of $X(t)$ implies that $E(X(t))$ is continuous in t, that is, that $E(A(t))$ is continuous in t. Thus $P(A(r)=A(r-))=1$ for every dyadic rational r. Retaining the notation used in Theorem 3.1 we have $E(X(T_\varepsilon))=\lim E(X(T_{n,\varepsilon}))$, and hence $E(A(T_\varepsilon)-A(T_{n,\varepsilon}))\to 0$. This means that $P(T_\varepsilon<\infty)=0$ for every ε. An appeal to Theorem 3.1 completes the proof.

Theorem 3.1 can be reformulated in terms of potentials. Let $X(t)$ be a potential and let $A(t)$ denote the corresponding natural increasing process. Let the potentials $X_n(t)$ be defined as

$$X_n(t) = E\big(X(\Phi_n(t))\,|\,F(t)\big)\,,$$

where the functions $\Phi_n(t)$ have the same meaning as before. It is clear that $X_n(t)$ increases and

$$P\big(\lim X_n(t)=X(t)\big) = 1 \quad \text{for all } t\,.$$

We can indeed show that $X_n(T)\to X(T)$ a.s. for every stopping time T. For every $\varepsilon>0$ and every integer n, define the stopping times $S_{n,\varepsilon}$ as follows (observe that $\inf\emptyset=0$):

$$S_{n,\varepsilon} = \inf\big(t:\ X(t)-X_n(t)\geqq\varepsilon\big)\,,$$

We then have

THEOREM 3.3. *Assume that $X(t)$ is a potential bounded by a constant c. Then*

$$E\big(\textstyle\int_0^\infty (X(t-)-X(t))\,dA(t)\big) \leqq \varepsilon E\big(A(S_{n,\varepsilon})\big) + cE\big(A(\infty)-A(S_{n,\varepsilon})\big)$$

for all n. Therefore, if $P(S_{n,\varepsilon}\uparrow\infty)=1$ for all $\varepsilon>0$, and $P(A(r)=A(r-))=1$ for all dyadic rationals, then the natural process $A(t)$ is continuous.

The proof is almost identical with that of Theorem 3.1. Note that the left hand side of the last inequality is equal to

$$E\big(\textstyle\int_0^\infty (A(t)-A(t-))\,dA(t)\big)\,.$$

REMARKS. Suppose $A(t)$ is a natural process. Returning to the notation of Section 1 we can very easily show that

$$E\big((A(\infty)-A(\infty,n))^2\big) = \textstyle\sum_k E(B(k,n)^2)\,,$$

where

$$B(k,n) = A((k+1)2^{-n}) - A(k2^{-n}) - E\big(A((k+1)2^{-n})-A(k2^{-n})\,|\,F(k2^{-n})\big)\,.$$

Clearly

$$E(B(k,n)^2) \leqq E\Big(\big(A((k+1)2^{-n})-A(k2^{-n})\big)^2\Big)\,.$$

Also $\sum_k \big(A((k+1)2^{-n}) - A(k2^{-n})\big)^2$ tends to the sum of squares of the jumps of $A(t)$. Therefore, if $A(t)$ is continuous, $A(\infty, n)$ tends in L_2 to $A(\infty)$. This suggests that $A(\infty, n)$ tends in L_1 to $A(\infty)$ if and only if $A(t)$ is natural. We have not succeeded in deciding if this is so.

REFERENCE

1. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham · Toronto · London, 1966

UNIVERSITY OF GÖTEBORG, SWEDEN