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QUASI-MARTINGALES

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Introduction.

A supermartingale is said to have a Doob decomposition if it can be expressed as the sum of a martingale and a decreasing process. A discrete parameter supermartingale always has a Doob decomposition [3, p. 104]. The problem of the existence of a Doob decomposition of a continuous parameter supermartingale was settled by Meyer [3, p. 122]; a simple and natural approach to this problem can be found in [5]. A more general problem than that of the Doob decomposition, namely, the decomposition of a process into a sum of a martingale and a process whose paths are of bounded variation, has been studied by Fisk [1] and Orey [4]. Fisk settled the problem in case of processes with continuous paths and Orey generalised it to right continuous processes. One problem remained, that of characterizing all right-continuous quasi-martingales (or "F-processes" in the terminology of Orey). We shall solve this problem and at the same time give an elementary proof of Orey's result. Little more than the definition of martingales and properties of conditional expectations are needed to read Section 1, which essentially contains the solution of the problem. The remainder of the paper is concerned with the analogy between supermartingales and quasi-martingales. The term "quasi-martingales" is borrowed from Fisk.

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1.

We shall assume that we are given an increasing right-continuous family $F(t)$ of σ-fields. A process $X(t)$ is said to be adapted to $F(t)$ if $X(t)$ is $F(t)$-measurable for all t. All our processes will be assumed adapted to $F(t)$. The parameter set is always the set of non-negative real numbers. Unless otherwise mentioned all random variables considered are assumed to have finite expectations.

For definitions and properties of martingales etc. we refer to [3].

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Definition 1.1. We say that a process $X(t)$ has a Riesz decomposition if

$$X(t) = Y(t) + Z(t),$$

where $Y(t)$ is a martingale and $Z(t)$ a process such that

$$E(|Z(t)|) \to 0 \quad \text{as } t \to \infty .$$

The Riesz decomposition is essentially unique. If

$$X(t) = Y_1(t) + Z_1(t) = Y_2(t) + Z_2(t)$$

are two Riesz decompositions of $X(t)$, we must have

$$E(|Y_1(t) - Y_2(t)|) \to 0 \quad \text{as } t \to \infty .$$

However, $|Y_1(t) - Y_2(t)|$ being a submartingale, $E(|Y_1(t) - Y_2(t)|)$ is a non-decreasing function of t. It follows that $E(|Y_1(t) - Y_2(t)|) \equiv 0$, that is,

$$P(Y_1(t) = Y_2(t)) = 1 \quad \text{for every } t .$$

Definition 1.2. We shall call a process $X(t)$ a *quasi-martingale* iff there exists a constant M such that

$$\sup \textstyle\sum_{1 \leqq i \leqq n} E\big(|X(t_i) - E(X(t_{i+1})|F(t_i))|\big) \leqq M ,$$

where the supremum is taken over all finite sets $t_1 < t_2 \ldots < t_n$ of non-negative real numbers. If for a quasi-martingale $X(t)$ we have

$$\lim_{t \to \infty} E(|X(t)|) = 0 ,$$

we shall say that $X(t)$ is a *quasi-potential.*

Obvious quasi-martingales are the following: martingales; super-martingales $X(t)$ such that $\inf E(X(t)) > -\infty$; sub-martingales $X(t)$ such that $\sup E(X(t)) < \infty$. It is obvious that the sum and difference of two quasi-martingales are again quasi-martingales.

An ordinary potential is a quasi-potential. The difference of two potentials is a quasi-potential.

Theorem 1.1 (Riesz decomposition). *Every quasi-martingale can be written in essentially one way as the sum of a martingale and a quasi-potential.*

Proof. We have already seen that Riesz decompositions are essentially unique. We need only show the existence.

Let $s_1 < s_2 < \ldots < s_n < \ldots$ be any (fixed for the present) strictly increasing sequence with $\lim s_n = \infty$. Put

$$\Delta(i) = X(s_i) - E(X(s_{i+1})|F(s_i)) .$$

$X(\cdot)$ being a quasi-martingale implies $\sum_i E(|\Delta(i)|) \leqq M$. Let t be any fixed real number and let

$$Y_t(i) = E(X(s_i)|F(t)) .$$

For $s_i \geqq t$ we have

$$E(\Delta(i)|F(t)) = Y_t(i) - Y_t(i+1) .$$

Therefore

$$E(|Y_t(i) - Y_t(i+1)|) \leqq E(|\Delta(i)|)$$

which implies that

$$\sum_i E(|Y_t(i) - Y_t(i+1)|) < \infty .$$

It follows that

$$q(t) = \lim_{i\to\infty} Y_t(i)$$

exists almost surely and in L_1. The L_1-convergence of $Y_t(i)$ obviously implies

$$E(q(t)\,|\,F(s)) = \lim_{i\to\infty} E(Y_t(i)\,|\,F(s)) = \lim_{i\to\infty} E(X(s_i)\,|\,F(s)) = q(s) ,$$

that is, $q(t)$ is a martingale. Now we show that

$$E(|X(s_i) - q(s_i)|) \to 0 \quad \text{as } s_i \to \infty .$$

Given $\varepsilon > 0$ we can choose i_0 large enough so that

$$\sum_{j \geqq i_0} E(|\Delta(i)|) < \tfrac{1}{2}\varepsilon .$$

Let $k \geqq i_0$. Since $q(s_k)$ is the limit in L_1 of $Y_{s_k}(m)$ we can choose $m+1$ (depending on k) with

$$E(|q(s_k) - Y_{s_k}(m+1)|) < \tfrac{1}{2}\varepsilon .$$

We then have

$$(1.1) \qquad E(|X(s_k) - q(s_k)|) \leqq E(|X(s_k) - Y_{s_k}(m+1)|) + \tfrac{1}{2}\varepsilon .$$

Also, for $j \geqq k$,

$$E(\Delta(j)\,|\,F(s_k)) = Y_{s_k}(j) - Y_{s_k}(j+1)$$

so that

$$E(|\Delta(j)|) \geqq E(|Y_{s_k}(j) - Y_{s_k}(j+1)|) .$$

Hence

$$\begin{aligned} E(|X(s_k) - Y_{s_k}(m+1)|) &\leqq \textstyle\sum_{k \leqq j \leqq m} E(|Y_{s_k}(j) - Y_{s_k}(j+1)|) \\ &\leqq \textstyle\sum_{k \leqq j \leqq m} E(|\Delta(j)|) \leqq \sum_{i_0 \leqq j} E(|\Delta(j)|) \leqq \tfrac{1}{2}\varepsilon \end{aligned}$$

by the choice of i_0. Comparison of the last inequality with (1.1) shows that

$$\lim_{i\to\infty} E(|X(s_i)-q(s_i)|) = 0 .$$

Thus we have shown the following: To any sequence $s_1 < s_2 < \ldots$ strictly increasing to ∞ there corresponds a martingale $q(t)$ such that

$$E(|X(t)-q(t)|) \to 0 \quad \text{as } t \to \infty$$

along the sequence.

Consider the sequence $1,2,3,\ldots$. Let $Y(t)$ denote the corresponding martingale such that

$$E(|X(n)-Y(n)|) \to 0 \quad \text{as } n \to \infty .$$

We claim that

$$\lim_{t\to\infty} E(|X(t)-Y(t)|) = 0 .$$

Otherwise there would exist a strictly increasing sequence $t_1 < t_2 < \ldots$ increasing to ∞ such that

$$\lim_{i\to\infty} E(|X(t_i)-Y(t_i)|) \geqq \varepsilon$$

for some $\varepsilon > 0$. Superimpose this sequence with $1,2,\ldots$ and let the resulting sequence be $s_1 < s_2 < \ldots$, where $s_n \uparrow \infty$. Let $R(t)$ be a martingale such that

$$\lim_{i\to\infty} E(|X(s_i)-R(s_i)|) = 0 .$$

Now $Y(t)-R(t)$ is a martingale and so $E(|Y(t)-R(t)|)$ is a non-decreasing function of t. However, we have

$$E(|Y(n)-R(n)|) \leqq E(|X(n)-Y(n)| + E(|X(n)-R(n)|) .$$

The right hand side tends to zero by the choice of the martingales $Y(t)$, $R(t)$. Therefore

$$P(Y(t)=R(t)) = 1 \quad \text{for every } t .$$

This implies that $E(|X(s_i)-Y(s_i)|) \to 0$ as $s_i \to \infty$, and this is against the choice of the sequence k_i. To complete the proof, put $Z(t)=X(t)-Y(t)$. In the following $X(t)$ will denote a quasi-potential. We note that for every t and every sequence $t_1 < t_2 < \ldots$ strictly increasing to ∞ we have

$$\lim_{t_i\to\infty} E(X(t_i)\,|\,F(t)) = 0 \text{ a.s.} \tag{1.2}$$

and in L_1. That the limit exists has been shown in Theorem 1.1. The limit can only be zero since

$$\lim E(|X(t)|) = 0 \quad \text{as } t \to \infty .$$

Put for every $k=0,1,2,\ldots$, $n=0,1,2,\ldots$,

$$\Delta(k,n) = E\big(X(k2^{-n})-X((k+1)2^{-n})|F(k2^{-n})\big) . \tag{1.3}$$

By the definition of a quasi-potential we have

$$\sup_n \textstyle\sum_k E(|\varDelta(k,n)|) \leqq M < \infty . \tag{1.4}$$

It is easily shown, using (1.2), that

$$X(i2^{-n}) = E(\textstyle\sum_{k\geqq i}\varDelta(k,n)|F(i2^{-n})) . \tag{1.5}$$

Let the potentials $X_+^n(t)$ and $X_-^n(t)$ be defined by

$$X_+^n(t) = \textstyle\sum_{k\geqq[2^nt]+1}E(\varDelta^+(k,n)|F(t)) ,$$
$$X_-^n(t) = \textstyle\sum_{k\geqq[2^nt]+1}E(\varDelta^-(k,n)|F(t)) ,$$

where $[2^nt]$ denotes the largest integer smaller than or equal to 2^nt and $x^+=\max(x,0)$. We shall show that $X_+^n(t)$ is indeed a potential. That $X_-^n(t)$ is a potential follows similarly. We have if $s\leqq t$,

$$\begin{aligned} &E(X_+^n(t)\,|\,F(s)) \\ &\quad= \textstyle\sum_{k\geqq[2^nt]+1}E(\varDelta^+(k,n)\,|\,F(s)) \leqq \textstyle\sum_{k\geqq[2^ns]+1}E(\varDelta^+(k,n)\,|\,F(s)) \\ &\quad= X_+^n(s) . \end{aligned}$$

This shows that $X_+^n(t)$ in a supermartingale. Also

$$E(X_+^n(t)) = \textstyle\sum_{k\geqq[2^nt]+1}E(\varDelta^+(k,n))$$

is trivially right continuous and tends to zero as $t\to\infty$. Thus there certainly exists a right-continuous-with-left-limits modification of $X_+^n(t)$; however, we do not need this fact.

Next we shall prove that $X_+^n(t)\leqq X_+^{n+1}(t)$ a.s. If $i2^{-n}\leqq t<(i+1)2^{-n}$, we have

$$X_+^n(t) = \textstyle\sum_{k\geqq i+1}E(\varDelta^+(k,n)\,|\,F(t)) .$$

Suppose $2i2^{-(n+1)}\leqq t<(2i+1)2^{-(n+1)}$. Then

$$\begin{aligned} X_+^{n+1}(t) &= \textstyle\sum_{k\geqq 2i+1}E(\varDelta^+(k,n+1)\,|\,F(t)) \\ &= E(\varDelta^+(2i+1,n+1)\,|\,F(t))+\textstyle\sum_{k\geqq 2i+2}E(\varDelta^+(k,n+1)\,|\,F(t)) \\ &= E(\varDelta^+(2i+1,n+1)\,|\,F(t))+ \\ &\qquad+\textstyle\sum_{k\geqq i+1}E(\{\varDelta^+(2k,n+1)+\varDelta^+(2k+1,n+1)\}\,|\,F(t)) \\ &\geqq \textstyle\sum_{k\geqq i+1}E(\{\varDelta^+(2k,n+1)+\varDelta^+(2k+1,n+1)\}\,|\,F(t)) \\ &\geqq \textstyle\sum_{k\geqq i+1}E(\{\varDelta(2k,n+1)+\varDelta(2k+1,n+1)\}^+\,|\,F(t)) , \end{aligned}$$

because $(x+y)^+\leqq x^++y^+$,

$$= \textstyle\sum_{k\geqq i+1}E\big(E(\{\varDelta(2k,n+1)+\varDelta(2k+1,n+1)\}^+\,|\,F(2k2^{-(n+1)}))\;\big|\;F(t)\big) ,$$

because $F(2k2^{-(n+1)})\supset F(t)$ for $k\geqq i+1$,

$$\geqq \sum_{k\geqq i+1} E\big(\{E(\Delta(2\mathrm{k},n+1)+\Delta(2k+1,n+1)\,|\,F(2k2^{-(n+1)}))\}^+ \,\big|\, F(t)\big)\ ,$$

because $E(f^+|B) \geqq Ef\,|\,B)^+$,

$$= \sum_{k\geqq i+1} E(\Delta(k,n)^+\,|\,F(t)) = X_+^{\,n}(t)\ ,$$

because $E(\Delta(2k,n+1)+\Delta(2k+1,n+1)\,|\,F(2k2^{-(n+1)}))$ is equal to $\Delta(k,n)$. This finishes the proof in case $2i2^{-(n+1)} \leqq t < (2i+1)2^{-(n+1)}$. If $(2i+1)2^{-(n+1)} \leqq t < (i+1)2^{-n}$ the term $E(\Delta^+(2i+1,n+1)\,|\,F(t))$ does not occur in the expression for $X_+^{\,n+1}(t)$; except for this, the proof is identical. The proof that the $X_-^{\,n}(t)$ are increasing is absolutely similar. Define

$$X_+(t) = \sup_n X_+^{\,n}(t), \qquad X_-(t) = \sup_n X_-^{\,n}(t)\ .$$

If we had taken $X_+^{\,n}, X_-^{\,n}$ as right continuous, then X_+, X_- would necessarily be right continuous, by Theorem T16 p. 99 of [3]. In any case $X_+(t)$ is defined outside of a set of zero probability which may depend on t. The inequality (1.4) implies

$$\sup_n E(X_+^{\,n}(0)+X_-^{\,n}(0)) \leqq M\ .$$

Therefore $X_+(t)$ and $X_-(t)$ are non-negative supermartingales; (1.4) also implies that

$$\lim_{t\to\infty} E(X_+^{\,n}(t)+X_-^{\,n}(t)) = 0$$

uniformly in n, which in turn means that $X_+(t), X_-(t)$ are potentials.

If $r = i2^{-k}$ is a dyadic rational, we have from (1.5)

$$X(r)-(X_+^{\,n}(r)-X_-^{\,n}(r)) = \Delta(i\,2^{n-k},n)$$

so that

$$E\big(|X(r)-(X_+(r)-X_-(r))|\big) \leqq \liminf_{n\to\infty} E(|\Delta(i2^{n-k},n)|)$$
$$\leqq \liminf_{n\to\infty} E\left(\left|X(r)-X\left(r+\frac{1}{2^n}\right)\right|\right).$$

Now we note that $E(X_+(t))$ and $E(X_-(t))$ are right continuous functions of t; this is easily established. The last inequality together with this fact implies:

THEOREM 1.2. *If $X(t)$ is a quasi-potential and mean right continuous, that is, if*

$$\lim_{0<h\downarrow 0} E(|X(t+h)-X(t)|) = 0 \quad \textit{for every } t\ ,$$

then there exist two potentials $X_+(t), X_-(t)$ such that

$$E(|X(t)-X_+(t)+X_-(t)|) = 0 \quad \textit{for every } t\ .$$

2.

We shall now investigate the similarities between quasi-martingales and supermartingales. Note that every finite set of random variables with expectations is trivially a quasi-martingale. Lemma 2.1 below gives an inequality similar to the supermartingale inequalities.

LEMMA 2.1. *Let F_i be σ-fields and let X_i be adapted to F_i, $1 \leqq i \leqq n$. Assume that each X_i has finite expectation and put*

$$\Delta_i = X_i - E(X_{i+1} | F_i), \quad 1 \leqq i \leqq n-1, \quad \Delta_n = X_n ,$$
$$A_n(+) = \sum_{i=1}^{n} \Delta_i^+, \quad A_n(-) = \sum_{i=1}^{n} \Delta_i^- .$$

Then for every $\lambda > 0$,

$$(2.1) \quad \lambda P(\max X_i \geqq \lambda) \leqq \sum_{i=1}^{n-1} E(\Delta_i^+) + E(X_n | \max X_i \geqq \lambda) \leqq E(A_n(+)).$$

The inequality (2.1) is not as simple-minded as it looks. We shall see this in a moment. Since $(-x)^+ = x^-$ it follows at once from (2.1) (changing X_i to $-X_i$) that

$$(2.2) \quad \lambda P(\min X_i \leqq -\lambda) \leqq \sum_{i=1}^{n-1} E(\Delta_i^-) - E(X_n \colon \min X_i \leqq -\lambda)$$
$$\leqq E(A_n(-)) .$$

If the X_i form a supermartingale, then $\Delta_i = \Delta_i^+$ and $\Delta_i^- = 0$, and hence (2.1) and (2.2) reduce to (2.3) and (2.4) below:

$$(2.3) \quad \lambda P(\max X_i \geqq \lambda) \leqq E(X_1) - E(X_n) + E(X_n \colon \max X_i \geqq \lambda)$$
$$\leqq E(X_1) + E(X_n^-) ,$$

$$(2.4) \quad \lambda P(\min X_i \leqq -\lambda) \leqq -E(X_n \colon \min X_i \leqq -\lambda) \leqq E(X_n^-) .$$

Suppose now that $\eta_1, \eta_2, \ldots$ is a sequence of independent random variables with finite variances and vanishing expectations. Let $C_1, C_2, \ldots$ be a non-increasing sequence of positive constants. For an arbitrary but fixed integer N, put

$$X_i = C_{N+i}^2 (\eta_1 + \ldots + \eta_{N+i})^2, \quad 0 \leqq i \leqq n ,$$

and let F_i be the σ-field generated by $\eta_1, \ldots, \eta_{n+i}$. Independence clearly implies

$$\Delta_i = X_i - E(X_{i+1} | F_i)$$
$$= (C_{n+i}^2 - C_{N+i+1}^2)(\eta_1 + \ldots + \eta_{N+i})^2 - C_{N+i+1}^2 E(\eta_{N+i+1}^2), \quad 0 \leqq i \leqq n-1 ,$$

so that

$$\Delta_i^+ \leqq (C_{N+i}^2 - C_{N+i+1}^2)(\eta_1 + \ldots + \eta_{N+i})^2 .$$

An application of (2.1) yields

$$\begin{aligned}\lambda^2 P(\max_{N\leqq k\leqq N+n} C_k|\eta_1+\dots+\eta_k|\geqq\lambda) &= \lambda^2 P(\max X_i \geqq \lambda^2)\\ &\leqq \sum_{i=0}^{n-1}(C_{N+i}^2 - C_{N+i+1}^2)E(\eta_1{}^2+\dots+\eta_{N+i}^2) + C_{N+n}^2 E(\eta_1{}^2+\dots+\eta_{N+n}^2)\\ &= C_N{}^2 E(\eta_1{}^2+\dots+\eta_N{}^2) + \sum_{K=N+1}^{N+n} C_k{}^2 E(\eta_k{}^2)\,,\end{aligned}$$

which is the inequality of Hajek–Renyi [2, p. 242].

Proof of Lemma 2.1. For any $i \leqq n-1$, if we replace in

$$(*)\qquad \sum_{j=1}^{n-1}\Delta_j{}^+ + X_n$$

$\Delta_j{}^+$ by Δ_j for $j \geqq i$, the resulting expression is smaller than or equal to (*). Thus

$$E(\sum_{j=1}^{n-1}\Delta_j{}^+ + \Delta_n \mid F_i) \geqq \sum_{k\leqq i-1}\Delta_k{}^+ + X_i \geqq X_i\,.$$

If B_i denotes the event that i is the first integer for which $X_i \geqq \lambda$, we get

$$\begin{aligned}\lambda P(\max X_i \geqq \lambda) = \lambda \sum_i P(B_i) &\leqq \sum_i E(X_i\colon B_i)\\ &\leqq \sum_i E(E(\sum_{k=1}^{n-1}\Delta_k{}^+ + \Delta_n|F_i)\colon B_i)\\ &= \sum_i E(\sum_{k=1}^{n-1}\Delta_k{}^+ + \Delta_n\colon B_i)\\ &\leqq \sum_{k=1}^{n-1} E(\Delta_k{}^+) + E(X_n\colon \cup B_i)\,.\end{aligned}$$

For definitions of the stopping times T and the related σ-fields $F(T)$ we refer to [3, pp. 65–74]. It is known that for a right continuous process $Z(t)$ the function $Z(T)$ is measurable for every stopping time T with respect to the σ-field $F(T)$.

Definition. Let $\mathscr{T}$ denote the collection of all stopping times. A right continuous process $Z(t)$ is said to belong to the class (DL) iff for each non-negative integer N the collection $\{Z(T\wedge N), T\in\mathscr{T}\}$ is uniformly integrable; it is said to belong to the class (D) iff the collection

$$\{Z(T\wedge N), T\in\mathscr{T}, N \text{ any non-negative integer}\}$$

is uniformly integrable.

A supermartingale $X(t)$, such that $E(X(t))$ is a right continuous function of t, has a right-continuous-with-left-limits modification [3, p. 95]. The results of the last section therefore show that a mean-right-continuous quasi-martingale always has a right-continuous-with-left-limits modification. We shall henceforth assume, unless otherwise stated, that all processes considered are right continuous and have left limits at every time point.

Definition. A right continuous process $A(t)$ with almost surely non-decreasing sample paths and $P(A(0)=0)=1$, is called an *increasing process*. An increasing process is called *integrable* iff $\sup E(A(t))<\infty$; it is called *natural* iff

$$E(\textstyle\int_0^s Y(t)dA(t)) = E(\int_0^s Y(t-)dA(t))$$

for every bounded, positive martingale $Y(t)$ and every $s \geqq 0$. The proof of the following theorem can be found in [**3**, pp. 105–122]:

Theorem 2.1 (Meyer). *A supermartingale $X(t)$ has a Doob decomposition*

$$X(t) = M(t) - A(t) , \tag{2.5}$$

where $M(t)$ is a martingale and $A(t)$ an increasing process, if and only if $X(t)$ belongs to the class (DL). *There then exists a decomposition for which the process $A(t)$ is natural, and this decomposition is unique.*

Remark. Suppose $X(t)$ is a non-negative supermartingale of class (DL). Then $M(t)$ is obviously non-negative and $E(A(t)) \leqq E(M(t)) = E(M(0)) = E(X(0))$. Therefore $A(t)$ is natural integrable increasing. Consequently, $X(t)$ belongs to the class (D) iff $M(t)$ is a uniformly integrable martingale.

Definition. A process $M(t)$ is called a local martingale iff there exists an increasing sequence of stopping times T_n such that

1. $P(T_n \leqq n)=1$, $P(T_n \to \infty)=1$;
2. $M(t \wedge T_n)$ is a uniformly integrable martingale for each n.

The following theorem is due to K. Itô and S. Watanabe [6].

Theorem 2.2. *Let $X(t)$ be a non-negative supermartingale. Then we can write*

$$X(t) = M(t) - A(t) , \tag{2.6}$$

where $M(t)$ is a local martingale and $A(t)$ a natural integrable increasing process. This decomposition is unique.

Proof. Since $X(t) \geqq 0$ is a supermartingale, we have

$$P(\sup X(t) < \infty) = 1 . \tag{2.7}$$

Define the stopping times T_n by

$$T_n = \{\inf(t\colon X(t) \geqq n)\} \wedge n .$$

Then $T_n \leqq T_{n+1}$ and (2.7) implies that $P(T_n \to \infty) = 1$. Put $X_n(t) = X(t \wedge T_n)$. Then $X_n(t)$ is a supermartingale of class (D) because

$$0 \leqq X_n(t) \leqq X(T_n) \vee n .$$

By the Remark following Theorem 2.1 we can write

$$X_n(t) = M_n(t) - A_n(t) , \tag{2.8}$$

where $M_n(t)$ is a uniformly integrable martingale and $A_n(t)$ is a natural integrable increasing process. Also $X_{n+1}(T_n \wedge t) = X_n(t)$. Since $M_{n+1}(t)$ is a uniformly integrable martingale, so is $M_{n+1}(T_n \wedge t)$. $A_{n+1}(T_n \wedge t)$ being the stopped process (at time T_n) of a natural process is itself natural [3, p. 112]. The uniqueness of the decomposition implies that

$$\begin{aligned} M_{n+1}(T_n \wedge t) &= M_n(t) \quad \text{for all } t , \\ A_{n+1}(T_n \wedge t) &= A_n(t) \quad \text{for all } t . \end{aligned}$$

The processes $M(t), A(t)$ defined by

$$\begin{aligned} M(t) &= M_n(t) \quad \text{if} \quad t \leqq T_n , \\ A(t) &= A_n(t) \quad \text{if} \quad t \leqq T_n , \end{aligned}$$

are therefore well-defined. Clearly $M(t)$ is a local martingale and $A(t)$ is increasing. Since

$$\begin{aligned} E(A(t)) &= \lim_{n\to\infty} E(A(t) \colon T_n \geqq t) \\ &= \lim_{n\to\infty} E(A_n(t) \colon T_n \geqq t) \\ &\leqq \lim_{n\to\infty} (A_n(t)) \\ &\leqq \lim_{n\to\infty} E(X_n(0)) = \lim_{n\to\infty} E(X(0)) = E(X(0)) , \end{aligned}$$

we see that $A(t)$ is integrable. If $Y(t)$ is any positive bounded martingale we have

$$\begin{aligned} E(\textstyle\int_0^t Y(s)dA(s)) &= \lim_{n\to\infty} E(\textstyle\int_0^t Y(s)dA(s) \colon T_n \geqq t) \\ &= \lim_{n\to\infty} E(\textstyle\int_0^t Y(s)dA_n(s) \colon T_n \geqq t) \\ &= \lim_{n\to\infty} E(\textstyle\int_0^t Y(s-)dA_n(s) \colon T_n \geqq t) \\ &= \lim_{n\to\infty} E(\textstyle\int_0^t Y(s-)dA(s) \colon T_n \geqq t) \\ &= E(\textstyle\int_0^t Y(s-)dA(s)) , \end{aligned}$$

proving that $A(t)$ is a natural process. The uniqueness of the decomposition is established similarly.

A non-negative local martingale $M(t)$ is necessarily a supermartingale. Indeed if $s \leqq t$ and $\mathfrak{A} \in F(s)$,

$$
\begin{aligned}
E(M(t) : \mathfrak{A}) &= \lim_{n\to\infty} E(M(t) : \mathfrak{A}, T_n \geqq t) \\
&= \lim_{n\to\infty} E(M(t \wedge T_n) : \mathfrak{A}, T_n \geqq t) \\
&\leqq \lim_{n\to\infty} E(M(t \wedge T_n) : \mathfrak{A}, T_n \geqq s) \\
&= \lim_{n\to\infty} E(M(s \wedge T_n) : \mathfrak{A}, T_n \geqq s) \quad \text{(since } M(t \wedge T_n) \text{ is a martingale)} \\
&= E(M(s) : \mathfrak{A}) .
\end{aligned}
$$

Therefore the difference of two positive local martingales is necessarily a quasi-martingale. The results of Section 1 and Theorem 2.2 therefore lead to

THEOREM 2.3. *A right continuous process $X(t)$ is a quasi-martingale if and only if it has a generalised Doob decomposition*

$$X(t) = Y(t) + M_{\text{loc}}(t) - B(t) ,$$

where $Y(t)$ is a martingale, $M_{\text{loc}}(t)$ is the difference of two non-negative local martingales, and $B(t)$ is the difference of two natural integrable increasing processes. This decomposition is unique.

REMARK. Being the difference of two integrable increasing processes, $B(t)$ has paths of bounded variation and the expected total variation of $B(t)$ is finite.

3.

Certain results not obvious from the definition of a quasi-martingale or the fact that it is the difference of two super-martingales follow from the decomposition proved in Theorem 2.3. Our starting point in this section is the decomposition

$$X(t) = M(t) - B(t) \tag{3.1}$$

of a quasi-martingale into a local martingale and a natural process with expected total variation finite. This decomposition is unique. We have

THEOREM 3.1. *Let $X(t)$ be a quasi-martingale (right continuous and with left limits) with the decomposition* (3.1). *Then*

1. *$X(t)$ belongs to the class* (DL) *if and only if $M(t)$ is a martingale; it belongs to the class* (D) *if and only if $M(t)$ is a uniformly integrable martingale.*
2. *Let for each natural integer k the stopping time R_k be defined by*

$$R_k = \{\inf(t\colon |X(t)| \geqq k)\} \wedge k\,.$$

Then $X(t)$ *belongs to the class* (D) *if and only if* $\{X(R_k)\}$ *is uniformly integrable. Thus, if* $X(t)$ *has continuous sample paths, then*

$$X(t) \in (\mathrm{D}) \iff kP(\sup_t |X(t)| \geqq k) \to 0 \text{ as } k \to \infty\,.$$

PROOF. 1. $M(t)$ being a local martingale, there exists a sequence T_n of stopping times such that $M(T_n \wedge t)$ is a uniformly integrable martingale for each n. If $X(t)$ belongs to the class (DL) so does $M(t)$ and therefore $\{M(T_n \wedge t)$, all integers $n\}$ is uniformly integrable for each t. If $s \leqq t$ and $B \in F(s)$ we have

$$\begin{aligned} E(M(t)\colon B) &= \lim_{n\to\infty} E(M(t\wedge T_n)\colon B, T_n \geqq t) \\ &= \lim_{n\to\infty} E(M(t\wedge T_n)\colon B, T_n \geqq s) - \\ &\quad - \lim_{n\to\infty} E(M(t\wedge T_n)\colon B,\ s \leqq T_n < t)\,. \end{aligned}$$

The second term tends to zero by uniform integrability and the fact that $T_n \to \infty$ a.s., and hence

$$E(M(t)\colon B) = \lim_{n\to\infty} E(M(s\wedge T_n)\colon B, T_n \geqq s) = E(M(s)\colon B)\,.$$

If further $X(t)$ belongs to (D), $M(t)$ must clearly be uniformly integrable.

2. A local martingale stopped at a bounded stopping time is easily seen to be a local martingale. $X(R_k \wedge t)$ belongs to the class (D) since

$$|X(R_k \wedge t)| \leqq k \vee |X(R_k)|\,.$$

So does $M(R_k \wedge t)$; it must therefore be a uniformly integrable martingale. Hence $M(R_k)$ is a martingale with respect to $F(R_k)$. The uniform integrability of $X(R_k)$ implies that of $M(R_k)$. It follows that $M(t)$ is a uniformly integrable martingale and this is true iff $X(t)$ belongs to the class (D). Here we have used the easily proved fact that R_k tends to infinity with probability one; this follows, for instance, from the decomposition given in Theorem 2.3.

THEOREM 3.2. *Let* $X(t)$ *be a quasi-potential and suppose* $|X(t)| \leqq C$ *where* C *is a constant. Then in the decomposition*

$$X(t) = E(B(\infty)\,|\,F(t)) - B(t) = M(t) - B(t)\,, \tag{3.2}$$

where $B(t)$ *is natural, we have*

$$E(B(\infty)^2) \leqq 2CE(V)\,,$$

where V *is the total variation of* $B(t)$.

PROOF. The decomposition (3.2) clearly exists. Put

$$\begin{aligned} \Delta(k,n) &= X(k2^{-n}) - E\big(X((k+1)2^{-n})/F(k2^{-n})\big) \\ &= E\big(B((k+1)2^{-n})/F(k2^{-n})\big) - B(k2^{-n}) , \\ B(\infty,n) &= \textstyle\sum_k \Delta(k,n) . \end{aligned}$$

The naturalness of $B(t)$ implies that the $B(\infty,n)$ converge to $B(\infty)$ weakly in L_1. We have

$$\begin{aligned} E(|B(\infty,n)|) &\leqq \textstyle\sum_k E(|\Delta(k,n)|) \\ &\leqq \textstyle\sum_k E\big(|B((k+1)2^{-n}) - B(k2^{-n})|\big) \\ &\leqq E(V) . \end{aligned}$$

Define

$$\begin{aligned} B(k+1,n) &= \textstyle\sum_{j\leqq k} \Delta(j,n) , \\ M(k,n) &= E\big(B(\infty,n)|F(k2^{-n})\big) . \end{aligned}$$

It is easy to verify that

$$B(k,n) + X(k2^{-n}) = M(k,n) .$$

If $X(t)$ is square-summable so is $\Delta(k,n)$ for all k,n. The same is true of $B(k,n)$. Assume now that $|X(t)| \leqq C$. Using the fact that $B(k+1,n)$ is $F(k2^{-n})$-measurable we can easily verify that

$$\begin{aligned} &E\Big(\textstyle\sum_{k\leqq N-1}\big(B(k+1,n)-B(k,n)\big)\big(X((k+1)2^{-n})+X(k2^{-n})\big)\Big) \\ &\quad = 2E\big(M(N,n)B(N,n)\big) - E\big(B^2(N,n)\big) \\ &\quad = E\big(M(N,n)B(N,n)\big) + E\big(B(N,n)X(N2^{-n})\big) \\ &\quad = E\big(M(N,n)^2\big) - E\big(M(N,n)X(N2^{-n})\big) + E\big(B(N,n)X(N2^{-n})\big) \\ &\quad = E\big(M(N,n)^2\big) - E\big(X(N2^{-n})^2\big) . \end{aligned}$$

This implies that

$$\begin{aligned} E\big(M(N,n)^2\big) &\leqq E\big(X(N2^{-n})^2\big) + E\big(\textstyle\sum_{k\leqq N-1}|\Delta(k,n)|\,2C\big) \\ &\leqq 2CE(V) + E\big(X(N2^{-n})^2\big) . \end{aligned}$$

As $N \to \infty$ we get, since $X(t) \to 0$ a.s. as $t \to \infty$,

$$E\big(B(\infty,n)^2\big) \leqq 2CE(V) .$$

$B(\infty)$ is the weak limit of $B(\infty,n)$. Therefore

$$E\big(B(\infty)^2\big) \leqq 2CE(V) .$$

We are now justified in making the following calculations which lead to the "Energy formula":

$$\begin{aligned} &E\left(\int_0^\infty (X(t)+X(t-))\,dB(t)\right) \\ &\quad = E\left(\int_0^\infty (M(t)+M(t-))\,dB(t)\right) - E\left(\int_0^\infty (B(t)+B(t-))\,dB(t)\right) \\ &\quad = 2E(B(\infty)^2) - E(B(\infty)^2) = E(B(\infty)^2)\,. \end{aligned}$$

THEOREM 3.3. *Let $X(t)$ be a continuous quasi-martingale. Then in the decomposition* (3.1), *$M(t)$ and $B(t)$ are continuous.*

PROOF. Let $R_k = \{\inf (t: |X(t)| \geqq k\} \wedge k$. Then R_k increases to ∞ a.s. Putting $X_k(t) = X(R_k \wedge t)$, we have $|X_k(t)| \leqq k$. The process $X_k(t)$ is the sum of a martingale and a quasi-potential. This martingale is bounded by k (see the proof of the Riesz-decomposition). It follows that the quasi-potential is bounded by, say, $2k$. Hence by Theorem 3.2, the natural process is square summable. We are therefore justified in making the following calculations. Write

$$X_k(t) = M_k(t) - B_k(t)\,.$$

We then have

$$E\left(\int_0^\infty (X_k(t) - X_k(t-))\,dB_k(t)\right) = E\left(\int_0^\infty (B_k(t-) - B_k(t))\,dB_k(t)\right)\,.$$

The left side is zero by continuity and the right side is the expectation of the sum of squares of the jumps of $B_k(t)$. It follows that $B_k(t)$ is continuous and hence $M_k(t)$ is continuous. By the uniqueness,

$$M_k(t) = M(t \wedge R_k), \qquad B_k(t) = B(t \wedge R_k)\,.$$

REMARK. If $X(t)$ belongs to the class (D) and is continuous, Theorem 3.3 gives Fisk's result.

REFERENCES

1. D. L. Fisk, *Quasi-martingales*, Trans. Amer. Math. Soc. 120 (1965), 369–389.
2. B. V. Gnedenko, *Theory of probability*, Chelsea, New York, 1962.
3. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham · Toronto · London, 1966.
4. S. Orey, *F-processes*, Proc. Fifth Berkeley Symp. on Stat. and Prob. II_1, 301–313.
5. K. Murali Rao, *On decomposition theorems of Meyer*, Math. Scand. 24 (1969), 66–78.
6. S. Watanabe and K. Itô, *Transformations of Markov processes by multiplicative functionals*, Ann. Inst. Fourier (Grenoble) 15 (1965), 13–30.

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