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CONVEX BODIES AND CONVEXITY ON GRASSMANN CONES XI
SUBLINEAR FUNCTIONS

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1.

A real valued function $f(x)$ defined on a non-empty set M of the n-dimensional affine space A^n is *sublinear* if

$$(1)\qquad f(x) \leqq \sum_{j=1}^{k} \lambda_j f(x_j) \qquad \text{for } x_j \in M,\ x = \sum_{j=1}^{k} \lambda_j x_j \in M,\ \lambda_j \geqq 0, \sum_{j=1}^{k} \lambda_j = 1 ,$$

where k is arbitrary. The function $f(x)$ is *sublinear of order m* if (1) holds for $k=m$ and hence $k \leqq m$.

If M is a convex set then sublinearity of order 2 means that $f(x)$ is a convex function and therefore has the two properties:

1) $f(x)$ is sublinear and

2) $f(x)$ is linearly bounded below, i.e., an affine (a not necessarily homogeneous linear) function $L(x)$ in A^n exists with $L(x) \leqq f(x)$ on M.

For arbitrary M the function $f(x)$ is called *convex* if it has the properties 1) and 2), which is equivalent to requiring that $f(x)$ can be extended to a convex function on the convex hull M^H of M, see [2].

Here we treat the basic questions with a new method and study in particular the implications of 1) alone. For open M^H it is shown in [2] that for an $f(x)$ which is linearly bounded below sublinearity of order $n+1$ implies sublinearity. Here we prove

THEOREM 1. *If $f(x)$ is sublinear of order $n+1$ then it is sublinear.*

THEOREM 2. *For some fixed t, $1 \leqq t \leqq n-1$, and each point $x \in M$ let a t-flat T_x through x exist such that $M \cap T_x = M^H \cap T_x$. If $f(x)$ is defined on M and sublinear of order $n-t+1$ then it is sublinear.*

Note that $M \cap T_x = M^H \cap T_x$ holds when $T_x \subset M$.

A simple observation which was overlooked in [2] shows that 1) implies 2) in all interesting cases.

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THEOREM 3. *If M contains a relative interior point of M^H and $f(x)$ is sublinear on M then $f(x)$ is convex.*

Thus it would be preferable to define convexity only for functions defined on M with this property. We refrain from doing so in order to conform with the other parts of this series.

If M is a union of lines through a fixed point then it satisfies this hypothesis as well as that of Theorem 2 with $t=1$. Therefore any $f(x)$ which is absolutely homogeneous of degree $p \geqq 1$ and sublinear of order n is convex.

The analogous question for positive homogeneous functions requires some case distinctions, see Corollaries 2, 3.

2.

For brevity we write $\varrho \in \Delta_m$ resp. $\varrho \in \Delta_m{}^+$ to indicate that ϱ is an m-tuple of reals $\varrho_1, \ldots, \varrho_m$ with $\sum_{i=1}^m \varrho_i = 1$ and $\varrho_i \geqq 0$ resp. $\varrho_i > 0$. The proofs will be based on the following

LEMMA. *In A^n let*

$$y = \sum_{j=1}^{k} \lambda_j y_j, \quad \lambda \in \Delta_k .$$

Then for a suitable r there are numbers $\theta \in \Delta_r{}^+$ and $\mu^i \in \Delta_k$, $i=1,\ldots,r$, such that

$$\sum_{i=1}^{r} \theta_i \mu_j^i = \lambda_j, \quad j = 1,\ldots,k , \tag{2}$$

and for each i

$$y = \sum_{j=1}^{k} \mu_j^i y_j \quad \text{and the } y_j \text{ with } \mu_j^i > 0 \text{ are independent} . \tag{3}$$

The set C in R^k of all $\sigma \in \Delta_k$ for which $y = \sum_{j=1}^k \sigma_j y_j$ is compact and convex, hence is the convex hull of its extreme points. In particular

$$\lambda = \sum_{i=1}^{r} \theta_i \mu^i, \quad \theta \in \Delta_r{}^+, \quad \mu^i \text{ an extreme point of } C .$$

We claim that the μ^i satisfy the assertion. Since $\mu^i \in C$ implies the equation in (3) we must show that the y_j with $\mu_j^i > 0$ are independent. For simplicity let $\mu_j^i > 0$ for $j \leqq s$ and $\mu_j^i = 0$ for $j > s$. If, for example,

$$y_s = \sum_{j=1}^{s-1} \varrho_j y_j, \quad \sum_{j=1}^{s-1} \varrho_j = 1 ,$$

then

$$\mu_j^i + \varepsilon\varrho_j \geqq 0, \quad j=1,\ldots,s-1, \quad \mu_s^i - \varepsilon \geqq 0 \text{ for } |\varepsilon| \leqq \varepsilon_0$$

with a suitable $\varepsilon_0 > 0$ and

$$y = \sum_{j=1}^{s-1} (\mu_j^i + \varepsilon\varrho_j) y_j + (\mu_s^i - \varepsilon) y_s .$$

But this means that μ^i is not an extreme point of C.—For this proof, which is shorter than the original one, I am indebted to W. Fenchel.

3.

Theorem 1 follows at once from the Lemma. For, if $y = \sum_{j=1}^k \lambda_j y_j$, $y \in M$, $y_j \in M$, $\lambda \in \Delta_k$, we introduce θ, μ^i according to the Lemma. For a given i there are at most $n+1$ positive μ_j^i. If $f(x)$ is sublinear of order $n+1$ we conclude using (2) that

$$f(y) = \sum \theta_i f(y) \leqq \sum_{i,j} \theta_i \mu_j^i f(y_j) = \sum \lambda_j f(y_j) .$$

This proof shows that Theorem 2 will follow if

$$f(y) \leqq \sum_{j=1}^{s+1} \lambda_j f(y_j) \tag{4}$$

for $y = \sum_{j=1}^{s+1} \lambda_j y_j \in M$, $y_j \in M$, $\lambda \in \Delta_{s+1}^+$ and independent y_j.

The hypothesis of Theorem 2 contains (4) for $s \leqq n-t$. Let $n-t < s$ ($\leqq n$). The y_j span a non-degenerate s-simplex S with y as interior point and S lies in an s-flat L. The t-flat T_y intersects L in a flat D with dimension

$$d \geqq s+t-n \geqq 1, \quad \text{whence} \quad s-d \leqq n-t . \tag{5}$$

For $d=t$ the flats T_y and L coincide and the assertion (4) follows from the convexity of $M \cap T_y$ and $n-t+1 \geqq 2$.

Let $d<t$. Then $T_y \cap S = D \cap S$ is a convex polyhedron, whose vertices $z_1,\ldots,z_q$ lie in $(s-d)$-faces of S and also in M, because $S \subset M^H$ and $M \cap T_y = M^H \cap T_y$. Then

$$y = \sum_{h=1}^{q} \mu_h z_h, \quad \mu \in \Delta_q, \tag{6}$$

and

$$f(y) \leqq \sum \mu_h f(z_h) \tag{7}$$

because $M \cap T_y$ is convex. Now, z_h lies on an $(s-d)$-face of S, so that

$$z_h = \sum_{j=1}^{s+1} \varrho_j^h y_j, \quad \varrho^h \in \Delta_{s+1}, \text{ with at most } s-d+1 \text{ positive } \varrho_j^h .$$

The hypothesis, (5), (6) and (7) yield

$$f(y) \leqq \sum_{h=1}^{q} \mu_h \sum_{j=1}^{s+1} \varrho_j^h f(y_j), \quad y = \sum_{j=1}^{s+1} \left(\sum_{h=1}^{q} \mu_h \varrho_j^h \right) y_j .$$

The λ_j in (4) are uniquely determined by y and the y_j, whence $\sum_h \mu_h \varrho_j^h = \lambda_j$, and (4) follows.

4.

Let $f(x)$ be defined on the arbitrary nonempty set M in A^n. For $x \in M^H$ put as in [2]

$$(8) \qquad g_f(x) = \inf_{x=\Sigma\lambda_j x_j} \sum_{j=1}^{k} \lambda_j f(x_j), \quad \lambda \in \Delta_k, \; x_j \in M ,$$

where k is arbitrary, possibly $g_f(x) = -\infty$. Applying the Lemma to x and x_j we find

$$\sum_{j=1}^{k} \lambda_j f(x_j) = \sum_{j=1}^{k} \sum_{i=1}^{r} \theta_i \mu_j^i f(x_j) \geqq \min_i \sum_j \mu_j^i f(x_j)$$

and therefore

$$(9) \quad g_f(x) = \inf_{x=\Sigma\lambda_j x_j} \sum_{j=1}^{k} \lambda_j f(x_j), \quad \lambda \in \Delta_k^+, \; x_j \in M, \; \textit{with independent } x_j .$$

In particular we see, as proved in [2], that it suffices in (8) to take $k = n+1$.

It is easy to see (comp. [2]) that $g_f(x)$ is convex if finite. On the other hand $g_f(x)$ is an extension of $f(x)$ if $f(x)$ is sublinear. To prove Theorem 3 it therefore suffices to show that $g_f(x)$ is finite if $f(x)$ is sublinear and M contains a relative interior point y of M^H.

Because of sublinearity $g_f(y) = f(y) > -\infty$. Let x be any other point of M^H. Since y is a relative interior point of M^H a $z \in M^H$ and $0 < \theta < 1$ with $y = (1-\theta)x + \theta z$ exist. If

$$x = \sum_{i=1}^{h} \lambda_i x_i, \quad x_i \in M_j, \; \lambda \in \Delta_h ,$$

$$z = \sum_{j=1}^{k} \mu_j z_j, \quad z_j \in M, \; \mu \in \Delta_k ,$$

then by sublinearity

$$f(y) \leqq (1-\theta) \sum_i \lambda_i f(x_i) + \theta \sum_j \mu_j f(z_j)$$

or

$$\sum_i \lambda_i f(x_i) \geqq (1-\theta)^{-1}[f(y)-\theta \sum_j f(z_j)] ,$$

so that $g_f(x)$ is finite.

5.

The hypothesis that M contains relative interior points of M^H is trivially satisfied in the frequently occurring case where M^H is open. It also holds when M is a union of t-flats through a fixed point w. For, assuming, without loss of generality, that $\dim M^H = n$, no supporting hyperplane of M at w exists, so that w is an interior point of M^H. Moreover M satisfies the hypothesis of Theorem 2. Therefore:

Corollary 1. *If M is a union of t-flats through a fixed point and $f(x)$ is defined on M and sublinear of order $n-t+1$, then $f(x)$ is convex.*

Theorem 4. *Let M be a union of rays with origin w and let $f(x)$ be defined on M and sublinear of order n.*

If M possesses a supporting hyperplane then $f(x)$ is sublinear but in general not convex. If M possesses no supporting hyperplane and

$$f(w) \leqq \sum_{j=1}^{n+1} \mu_j f(y_j) \quad \textit{for} \quad w = \sum_{j=1}^{n+1} \mu_j y_j,\ y_j \in M,\ \mu \in \Delta_{n+1}^{+} \tag{10}$$

and independent y_j, then $f(x)$ is convex. The condition (10) *cannot be omitted.*

Note that (10) holds when $f(w) = \min f(x)$.

A supporting plane of M passes through w so that in the first case w is not an interior point of a nondegenerate n-simplex with vertices in M. Because of the Lemma it suffices to prove (4) which holds for $s < n$ since we assume sublinearity of order n. Let $s=n$. Then y is an interior point of the n-simplex S_n with vertices $y_1,\ldots,y_{n+1}$. The ray from w through y intersects the boundary of S_n in two points z_1, z_2 which lie on $(n-1)$-faces of S_n and the proof proceeds as that of Theorem 2.

That $f(x)$ need not be convex (when M does not contain an interior point of M^H) is shown in Section 6.

In the second case w is an interior point of M. It suffices again to prove (4) for $s=n$. For $y=w$ this is (10). Let $y \neq w$ and use the previous notation. If w is not an interior point of S_n, then (4) follows as before.

If w lies in the interior of S_n the ray from w through y intersects an $(n-1)$-face of S_n in a point z, so that

$$z = \sum_{j=1}^{n+1} \varrho_j y_j, \quad \varrho_j \in \Delta_{n+1}, \text{ at least one } \varrho_j = 0\,.$$

We express w in the form (10) and obtain with a suitable $0 < \theta < 1$

$$\begin{aligned} y &= (1-\theta)w + \theta z = (1-\theta)\textstyle\sum_j \mu_j y_j + \theta \sum_j \varrho_j y_j\,, \\ f(y) &\leqq (1-\theta)f(w) + \theta f(z) \\ &\leqq (1-\theta)\textstyle\sum_j \mu_j f(y_j) + \theta \sum_j \varrho_j f(y_j) = \sum_j \lambda_j f(y_j)\,, \end{aligned}$$

because y and y_j determine the λ_j uniquely.

That $f(x)$ need not be convex without (10) is clear: Let M consist of the nonnegative x^i-axes R_i, $i = 1, \ldots, n$, and the ray R: $x^1 = \ldots = x^n$, $x^i \leqq 0$. Put $f(x) = 0$ on $\bigcup R_i$ and $f(x) = x^1$ on R.

The function $f(x)$ is *absolutely homogeneous of degree* $p > 0$ if $0 \in M \neq \{0\}$, $\alpha M = M$ for $\alpha \neq 0$ and $f(\alpha x) = |\alpha|^p f(x)$ for all α. For $p < 1$ the function is not convex on any line through 0 unless it vanishes; therefore $f(x)$ is not sublinear of order 2 unless $f(x) \equiv 0$.

The function $f(x)$ is *positive homogeneous of degree* $p > 0$ if $0 \in M \neq \{0\}$, $\alpha M = M$ for $\alpha > 0$ and $f(\alpha x) = \alpha^p f(x)$ for $\alpha \geqq 0$. For $p < 1$ the function is not sublinear of order 2 if it takes positive values and for $p > 1$ if it takes negative values. In most applications also $\alpha M = M$ for $\alpha < 0$, but not always $f(x) = f(-x)$. If $\alpha M = M$ for $\alpha \neq 0$ then $p < 1$ and sublinearity imply $f(x) \equiv 0$.

The most important special case of Corollary 1 is

COROLLARY 2. *$f(x)$ is convex if it is sublinear of order n and absolutely homogeneous of degree $p \geqq 1$ or is positive homogeneous of degree $p \geqq 1$ and $\alpha M = M$ also for $\alpha < 0$.*

Theorems 3, 4 imply

COROLLARY 3. *$f(x)$ is convex if it is sublinear of order n, positive homogeneous of degree $p > 0$ and M possesses a supporting hyperplane and contains an interior point of M^H.*

$f(x)$ is convex if it is sublinear of order n and positive homogeneous of degree $p > 1$ and M possesses no supporting plane.

$f(x)$ is convex if sublinear of order n, positive homogeneous of degree 1 and linearly bounded below.

In the second part we have $f(x) \geqq 0$ so that (13) holds. The third part contains Theorem (11) in [2, p. 7] where it is assumed that $M^H \setminus \{0\}$ (owing to a misprint [2] states M^H) is open. To prove it we observe that by hypothesis numbers $u_1, \ldots, u_n, c$ exist such that

$$u \cdot x - c = \sum_i u_i x_i - c \leqq f(x) \quad \text{on } M .$$

Then $u \cdot x \leqq f(x)$, because $u \cdot x_0 = f(x_0) - \delta$, $\delta > 0$, would imply

$$\alpha u \cdot x_0 - \alpha\delta = \alpha f(x_0) = f(\alpha x_0) \geqq u \cdot (\alpha x_0) - c \quad \text{for all } \alpha > 0 .$$

Therefore, if $0 = \sum_j \mu_j y_j$,

$$0 = u \cdot \sum_j \mu_j y_j = \sum_j \mu_j (u \cdot y_j) \leqq \sum_j \mu_j f(y_j)$$

and (10) holds. If M possesses a supporting hyperplane we apply the second part of Theorem 4 and that $f(x)$ is linearly bounded below.

6.

The order of sublinearity in the hypotheses of Theorems 1, 2 *and Corollaries* 1, 2, 3 *cannot be lowered*, although one might have expected the contrary in Corollaries 2, 3 because their proofs do not fully use the homogeneity of $f(x)$. We show this with some simple *examples*.

Let M_0^n consist of the vertices $x_1, \ldots, x_{n+1}$ and an interior point x of a nondegenerate simplex. Put $f(x_j) = 0$, $f(x) = 1$; then $f(x)$ is sublinear of order n on M_0^n but not sublinear. For Theorem 2 take any $(n-t)$-flat F and a t-flat T' intersecting F in a point. In F take a set

$$M_0^{n-t} = \{y_1, \ldots y_{n-t}, y\}$$

and let M_t^n consist of the t-flats T_i, T parallel to T' through y_i, y. Put $f(x) = 0$ on $\bigcup T_i$ and $f(x) = 1$ on T. Then $f(x)$ is sublinear of order $n-t$ but not $n-t+1$.

Further examples are based on the following observation. Let $f(y)$ be defined on a set M' in the hyperplane $x^n = 1$ in A^n and extend $f(y)$ to the cone M consisting of the rays from 0 through points of M' by

$$f(\tau y) = \tau f(y) \quad \text{for } \tau \geqq 0,\ y \in M' .$$

For $0 = \sum_{j=1}^k \lambda_j x_j$, $\lambda \in \Delta_k^+$ and $x_j \in M$ we have $x_j = 0$. If $0 \neq x = \sum_{j=1}^k \lambda_j x_j$, $\lambda \in \Delta_k$, $x \in M$, $x_j \in M$ and $x = \tau y$, $x_j = \tau_j y_j$, $y \in M'$, $y_j \in M'$ then

$$y = \sum_{j=1}^k \tau^{-1} \lambda_j \tau_j y_j, \qquad \sum_{j=1}^k \tau^{-1} \lambda_j \tau_j = 1 ,$$

because $y^n = y_j^n = 1$. From the definition of $f(x)$

$$f(x) \lesseqqgtr \sum_{j=1}^k \lambda_j f(x_j)$$

is equivalent resp. to

$$f(y) \gtreqqless \sum_{j=1}^k \tau^{-1} \lambda_j \tau_j f(y_j) .$$

Thus, if $f(x)$ is linear resp. sublinear of order k on M' then so is $f(x)$ on M.

We use this first to show that $f(x)$ need not be convex in the first part of Theorem 4. Take M' as an $(n-2)$-dimensional ellipsoid and define $f(x)$ on M' in any way such that $\inf f(y) = -\infty$. Then $f(y)$ is sublinear on M', but not linearly bounded below. The same holds for $f(x)$.

Next take as M' the set M_{t-1}^{n-1}, $t \geqq 1$; then $f(x)$ is sublinear of order $n-t$ but not $n-t+1$. Thus we obtain an example for Corollary 1, and for $t=1$, since $f(x)$ is positive homogeneous of degree 1, also for the last part of Corollary 3.

Let L_i be the x^i-axes and L the line $x^1 = \ldots = x^n$, $n > 2$. Put

$$M = L_1 \cup \ldots \cup L_n \cup L ,$$

$f(x) = 0$ on $\bigcup L_i$, $f(x) = |x^1|^p$, $p \geqq 1$, on L. Then $f(x)$ is sublinear of order $n-1$ but not n. This takes care of Corollary 2 and the second part of Corollary 3. Let M^* be the subset of M in $\{x \mid x^i \geqq 0\}$. Then the restriction of $f(x)$ to M^* settles the first part of Corollary 3 with $p \geqq 1$, and for $0 < p < 1$ we take $f(x) = -(x^i)^p$ on $L_i \cap M^*$, $f(x) = 0$ on $L \cap M^*$.

7.

The theory of functions which are convex on a nonconvex set was developed specifically for the case, where the space is the linear space V_r^n (also considered as affine space) of all r-vectors over A^n (considered as vector space) and M is the Grassmann cone G_r^n of all simple r-vectors R (compare [2]). The maximal dimension of a flat contained in G_r^n is

$$t_r^n = \max(r+1, n-r+1) ,$$

see [1, p. 300], and each point R of G_r^n lies on a t_r^n-flat through the origin. Since $\dim V_r^n = \binom{n}{r}$ we deduce from Corollary 1:

COROLLARY 4. *A function $f(R)$ which is defined on G_r^n and sublinear of order $g_r^n = \binom{n}{r} - \max(r, n-r)$ is convex.*

In the applications $f(R)$ is positive (frequently absolutely) homogeneous of degree 1. According to a previous observation this alone would not lead us to expect that g_r^n can be replaced by a smaller number. However, there are infinitely many t_r^n-flats in G_r^n through each R and G_r^n has in other respects a very special structure. If we denote by o_r^n the smallest integer such that sublinearity of order o_r^n guarantees convexity for a function on G_r^n which is positive homogeneous of degree 1, then most likely $o_r^n < g_r^n$. Possibly o_r^n has a smaller order of magnitude than

g_r^n, that is, $o_r^n/g_r^n \to 0$ when $n \to \infty$ and $\max(r, n-r)/n < \theta < 1$. For various questions in the theory of convex bodies it would be important to know the precise value of o_r^n.

In the simplest case $n=4$, $r=2$ it is known that $o_2^4 > 2$ (see [2, p. 21[3]]) and Corollary 4 gives $o_2^4 \leqq g_2^4 = 4$. It seems probable that $o_2^4 = 3$.

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