MATH. SCAND. 24 (1969), 102–110

IDENTITIES CONNECTING ELEMENTARY DIVISOR FUNCTIONS OF DIFFERENT DEGREES, AND ALLIED CONGRUENCES

D. B. LAHIRI

1. Introduction.

This paper deals with the function $\sigma_k(n)$, the sum of the kth powers of the divisors of n; this function is referred to as the elementary divisor function of degree k. There is a beautiful classical identity for the case $k=1$ where of course $\sigma_1(n)$ or $\sigma(n)$ appears. This relation [3, p. 212] may be stated as,

$$(1) \quad \sigma(n-0)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)- \\ -\sigma(n-12)-\sigma(n-15)+\ldots = 0 .$$

The identity is true with the understanding that $\sigma(0)=n$, and $\sigma(m)=0$ when $m<0$. The numbers $0,1,2,5,7,12,15,\ldots$ appearing in the successive terms of the series in (1) are the pentagonal numbers v given by

$$(2) \qquad v = \tfrac{1}{2}m(3m+1), \qquad m=0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots ,$$

so that the typical term is $\pm\sigma(n-v)$. The sign to be attached to $\sigma(n-v)$ is positive or negative according as $v=v_0$ or $v=v_1$ where

$$(3) \qquad v_0 = m(6m+1) \quad \text{and} \quad v_1 = (2m+1)(3m+1);$$

here also m stands for any integer positive, zero or negative as in (2).

It is natural to enquire whether analogous identities exist for divisor functions of higher degree. It is easily verified that replacement of σ_1 in (1) by just σ_k would not help in whatever manner $\sigma_k(0)$ might be defined; one may check by putting $n=3$, a case where the value of $\sigma_k(0)$ is irrelevant. One has therefore to look out for not so simple expressions involving $\sigma_k(n)$ for possible identities of more or less the same type as the classical one. Such expressions actually exist, and it will be noticed in the next section that the expressions obtained by the author for the successful replacement of $\sigma(n)$ in (1) by other functions not only involve $\sigma_k(n)$ the divisor function of degree k but also others

Received August 5, 1968.

of lower degree. Thus these identities connect the elementary divisor functions of different degrees, namely, $k=1,3,5,7,9$.

Although simple replacement of σ in (1) by σ_k does not give a valid identity yet it can be shown that a further replacement of the symbol $=$ by the symbol $\equiv$ yields valid albeit weaker relations, congruence relations for suitable moduli. Such congruences and others of an allied nature will be derived from the identical relations.

Additional identities and congruences of more or less similar nature including some which involve $\sigma_{11}(n)$ and $\sigma_{13}(n)$ also exist. It is hoped that these results will be published separately.

2. The main result.

We shall first explain certain summation notations. The symbol $\sum_{v=v_0}$ denotes summation over the pentagonal numbers $v=v_0$, that is, over

$$v = 0,5,7,22,26,\ldots ,$$

and similarly $\sum_{v=v_1}$ denotes summation over the pentagonal numbers $v=v_1$, that is, over

$$v = 1,2,12,15,35,40,\ldots .$$

Also we shall use sums of the type

$$\sum_{v} [\mp F(n-v)] , \tag{4}$$

where the summation is over all the pentagonal numbers v, with the understanding that the sign to be attached to the term $F(n-v)$ is positive or negative according as $v=v_0$ or $v=v_1$. In other words,

$$\sum_{v} [\mp F(n-v)] = \sum_{v=v_0} F(n-v) - \sum_{v=v_1} F(n-v) . \tag{5}$$

To shorten the expressions which we shall come across we shall in what follows write,

$$\sum_{n-v} [\mp F(m)] = \sum_{m=n-v} [\mp F(m)] = \sum_{v} [\mp F(n-v)] . \tag{6}$$

Thus in the first sum $\sum_{n-v}[\mp F(m)]$, m ranges over the values

$$n-0,\ n-1,\ n-2,\ n-5,\ n-7,\ n-12,\ n-15,\ \ldots ,$$

n being considered fixed.

Using the above summation notations the classical identity (1) may be restated in either of the following forms,

$$(7)\qquad \sum_{v} [\mp \sigma(n-v)] = 0, \qquad \textit{where } \sigma(0) = n;$$

$$(8)\qquad \sum_{n-v} [\mp \sigma(m)] = 0, \qquad \textit{where } \sigma(0) = n\,.$$

We now state the identities constituting our principal result.

THEOREM. *For any prescribed value of n we have the identity,*

$$\sum_{n-v} [\mp D(m)] = 0\,,$$

where $D(m)$ stands for any of the following expressions:

(T1) $\sigma(m)$,

(T2) $5\sigma_3(m) - 18m\,\sigma(m)$,

(T3) $7\sigma_5(m) - 150m\,\sigma_3(m) + 360m^2\sigma(m)$,

(T4) $5\sigma_7(m) - 294m\,\sigma_5(m) + 3780m^2\sigma_3(m) - 7560m^3\sigma(m)$,

(T5) $11\sigma_9(m) - 1350m\,\sigma_7(m) + 45360m^2\sigma_5(m) - 453600m^3\sigma_3(m) + {} + 816480m^4\sigma(m)$,

with the understanding that the terms $\sigma_k(0)$, $k = 1,3,5,7,9$, if they appear in the above identities (which happens only when n is a pentagonal number), are given by

(T1′) $\sigma(0) = n$,

(T2′) $5\sigma_3(0) = -12n^2 - n$,

(T3′) $7\sigma_5(0) = 192n^3 + 24n^2 + n$,

(T4′) $5\sigma_7(0) = -3456n^4 - 576n^3 - 36n^2 - n$,

(T5′) $11\sigma_9(0) = 331776n^5 + 69120n^4 + 5760n^3 + 240n^2 + 5n$;

and further $\sigma_k(m)$ is supposed to be vanishing when m is negative.

It may be pointed out that here as also elsewhere (excepting the last section 7) $m^\alpha\sigma_k(m) = 0$ when $m = 0$, $\alpha > 0$. It is also worth noting that $\sigma_k(0)$, $k > 0$, will not always be integral. It will also be recognized that the case (T1) with $D(m) = \sigma(m)$ gives the classical identity (8).

3. A lemma.

In connection with the lemma which we use to establish the above theorem we require the symbols (r,s), a_n, $p(n)$ and u_r, which are defined below.

The function $\Phi_{r,s}(x)$ has been defined by Ramanujan [4, p. 233] as

$$(9)\qquad \Phi_{r,s}(x) = \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \alpha^r\beta^s x^{\alpha\beta} = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n\,.$$

The author has slightly simplified this notation to $\Phi_{r,s}$ in [1]. However, in subsequent parts of this paper the author will follow the much simpler notation (r,s) which has been also used rather extensively in his paper [2]. Thus

$$(10) \qquad (r,s) = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n .$$

We define a_n by the 'pentagonal number' theorem of Euler,

$$(11) \qquad f(x) = \prod_{n=1}^{\infty} (1-x^n) = \sum_{-\infty}^{+\infty} (-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n .$$

The function $p(n)$ is the number of unrestricted partitions of n given by

$$(12) \qquad [f(x)]^{-1} = \sum_{n=0}^{\infty} p(n) x^n .$$

The function $u_{r,0}$ or simply u_r has been defined previously by the author [1] as,

$$(13) \qquad u_r = \left(\sum_{n=0}^{\infty} n^r a_n x^n\right) \left(\sum_{n=0}^{\infty} p(n) x^n\right) = \left(\sum_{n=0}^{\infty} n^r a_n x^n\right) \Big/ f(x) .$$

With the summation notation (4) explained in section 2 we can rewrite (11) and (13) as,

$$(14) \qquad f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{v} [\mp x^v] ,$$

$$(15) \qquad u_r = \sum_{v} [\mp v^r x^v] / \sum_{v} [\mp x^v] .$$

We shall now state our lemma.

LEMMA. *For each of the values* $r = 1,2,3,4,5$ *the function* u_r *is expressible as a linear function of* (a,b)*'s as follows.*

$$(\text{L1}) \qquad u_1 = -(0,1);$$

$$(\text{L2}) \qquad 12u_2 = 5(0,3) - 18(1,2) + (0,1);$$

$$(\text{L3}) \qquad 192u_3 = -7(0,5) + 150(1,4) - 10(0,3) - 360(2,3) + 36(1,2) - (0,1);$$

$$(\text{L4}) \qquad 3456u_4 = 5(0,7) - 294(1,6) + 21(0,5) + 3780(2,5) - 450(1,4) + 15(0,3) - 7560(3,4) + 1080(2,3) - 54(1,2) + (0,1);$$

$$(\text{L5}) \qquad 331776u_5 = -11(0,9) + 1350(1,8) - 100(0,7) - 45360(2,7) + 5880(1,6) - 210(0,5) + 453600(3,6) - 75600(2,5) + 4500(1,4) - 100(0,3) - 816480(4,5) + 151200(3,4) - 10800(2,3) + 360(1,2) - 5(0,1).$$

These relations have been previously established by the author in [1, p. 128], the only change being removal of the fractions appearing there, and writing u_r simply for $u_{r,0}$ and (r,s) simply for $\Phi_{r,s}$.

4. Proof of the theorem.

It can be seen without difficulty that the set of five relations given in the above lemma is equivalent to the following set

$$(16.1)\qquad -u_1 = (0,1);$$

$$(16.2)\qquad 12u_2+u_1 = 5(0,3)-18(1,2);$$

$$(16.3)\qquad -192u_3-24u_2-u_1 = 7(0,5)-150(1,4)+360(2,3);$$

$$(16.4)\qquad 3456u_4+576u_3+36u_2+u_1 = 5(0,7)-294(1,6)+3780(2,5)-7560(3,4);$$

$$(16.5)\qquad -331776u_5-69120u_4-5760u_3-240u_2-5u_1 = 11(0,9)-1350(1,8)+45360(2,7)-453600(3,6)+816480(4,5).$$

Making use of the relations (15) and (10) we obtain,

$$(17.1)\qquad -\sum_v [\mp vx^v]/\sum_v [\mp x^v] = \sum_{n=1}^{\infty} \sigma(n)x^n;$$

$$(17.2)\qquad \sum_v [\mp (12v^2+v)x^v]/\sum_v [\mp x^v] = \sum_{n=1}^{\infty} [5\sigma_3(n)-18n\sigma(n)]x^n;$$

$$(17.3)\qquad -\sum_v [\mp (192v^3+24v^2+v)x^v]/\sum_v [\mp x^v] = \sum_{n=1}^{\infty} [7\sigma_5(n)-150n\sigma_3(n)+360n^2\sigma(n)]x^n;$$

$$(17.4)\qquad \sum_v [\mp (3456v^4+576v^3+36v^2+v)x^v]/\sum_v [\mp x^v] = \sum_{n=1}^{\infty} [5\sigma_7(n)-294n\sigma_5(n)+3780n^2\sigma_3(n)-7560n^3\sigma(n)]x^n;$$

$$(17.5)\qquad -\sum_v [\mp (331776v^5+69120v^4+5760v^3+240v^2+5v)x^v]/\sum_v [\mp x^v] = \sum_{n=1}^{\infty} [11\sigma_9(n)-1350n\sigma_7(n)+45360n^2\sigma_5(n)-453600n^3\sigma_3(n)+816480n^4\sigma(n)]x^n.$$

Multiplying both sides of each of the above identities by $\sum_v [\mp x^v]$ and then equating the coefficients of x^n on both sides we establish the different cases of the theorem given in section 2. It will be noted that the coefficient of x^n on the new left hand side is zero when n is not a pentagonal number. However, when n is such a number v the coefficient of $x^n = x^v$ on that side for any particular case is then transferred to the right hand side with necessary changes in sign and branded as a suitable multiple of $\sigma_k(0)$ in such a manner that this transferred right hand coefficient becomes equal to $\mp D(0)$, the sign being positive or negative according as $n = v = v_0$ or $n = v = v_1$.

5. Divisibility properties of $\sum_{n-v} [\mp \sigma_k(m)]$.

We pass on now from the classical identity

$$(8) \qquad \sum_{n-v} [\mp \sigma(m)] = 0 \,, \quad \text{where} \quad \sigma(0) = n \,,$$

to congruences of similar form. These are easily derivable from our theorem, and are shown in the following corollary.

COROLLARY 1. *The following congruences hold for any prescribed value of n when the $\sigma_k(0)$'s are assumed to be as specified below,*

$$(\text{C1.1}) \qquad \sum_{n-v} [\mp \sigma_3(m)] \equiv 0 \pmod{2 \cdot 3^2},$$

where $\sigma_3(0) = -6n^2 + 7n$;

$$(\text{C1.2}) \qquad \sum_{n-v} [\mp \sigma_5(m)] \equiv 0 \pmod{2 \cdot 3 \cdot 5},$$

where $\sigma_5(0) = n$;

$$(\text{C1.3}) \qquad \sum_{n-v} [\mp \sigma_7(m)] \equiv 0 \pmod{2 \cdot 3 \cdot 7},$$

where $\sigma_7(0) = n$;

$$(\text{C1.4}) \qquad \sum_{n-v} [\mp \sigma_9(m)] \equiv 0 \pmod{2 \cdot 3^3 \cdot 5},$$

where $\sigma_9(0) = -54n^5 - 90n^3 + 120n^2 + 25n$.

The divisibility properties of $\sum_{n-v} [\mp \sigma_k(m)]$ are more interesting for the first and the last cases with $k = 3$ and 9 than for the other two. The congruences (C1.2) and (C1.3) are in a sense as deep as the classical identity (8). If this identity is assumed to be known then these two congruences follow from it fairly easily as parallel congruences hold on a term by term basis as indicated below:

$$(18) \qquad \sigma_5(m) \equiv \sigma(m) \pmod{2\cdot 3\cdot 5} \quad \text{as} \quad d^5 \equiv d \pmod{2\cdot 3\cdot 5},$$
$$(19) \qquad \sigma_7(m) \equiv \sigma(m) \pmod{2\cdot 3\cdot 7} \quad \text{as} \quad d^7 \equiv d \pmod{2\cdot 3\cdot 7};$$

moreover, $\sigma_5(0) = \sigma(0) = \sigma_7(0)$ each being equal to n. The congruences (C1.2) and (C1.3) can also be derived from cases (T3) and (T4) of the theorem, without any direct appeal to the classical identity. One must of course pay special attention to the value of $\mp D(0)$, and in that connection it is to be remembered that for all pentagonal numbers v,

$$(20) \qquad 2v^3 - v^2 - v \equiv 0 \pmod 5,$$
$$(21) \qquad 2v^4 - 2v^3 - v^2 + v \equiv 0 \pmod 7;$$

(20) and (21) are required respectively for the congruences (C1.2) and (C1.3).

The first and the last congruences of the corollary are deeper; an appeal to the classical identity can take us only to the extent of proving the congruences for the weaker moduli $2\cdot 3$ and $2\cdot 3\cdot 5$ instead of $2\cdot 3^2$ and $2\cdot 3^3\cdot 5$ as shown in the corollary. To prove the stronger results we need refer to the identities corresponding to cases (T2) and (T5) of the theorem.

6. Further congruences involving $\sum_{n-v}[\mp \sigma_k(m)]$.

There exist functions $F(m)$ such that for appropriate modulus $\sum_{n-v}[\mp \sigma_k(m)]$ and $\sum_{n-v}[\mp F(m)]$ belong to the same residue class whatever be the value of n. This is demonstrated by the following corollary (where (C. 2.3) is to be suitably interpreted when either side is fractional).

COROLLARY 2. *For all values of* n,

$$(\text{C2.1}) \qquad \sum_{n-v}[\mp \sigma_3(m)] \equiv -30 \sum_{n-v}[\mp m^2\sigma(m)] \pmod{2\cdot 3\cdot 5^2},$$

where $\sigma_5(0) = 6n^3 - 18n^2 + 43n$;

$$(\text{C2.2}) \qquad \sum_{n-v}[\mp \sigma_5(m)] \equiv -30 \sum_{n-v}[\mp m\sigma_3(m)] \pmod{2^3\cdot 3^2\cdot 5},$$

where $\sigma_5(0) = -24n^3 - 48n^2 + 103n$;

$$(\text{C2.3}) \qquad \sum_{n-v}[\mp \sigma_7(m)] \equiv 294\cdot\tfrac{1}{5} \sum_{n-v}[\mp m\sigma_5(m)] \pmod{2^2\cdot 3^3\cdot 7},$$

where $\sigma_7(0) = \frac{1}{5}(324n^4 - 576n^3 - 36n^2 - n)$;

$$(\text{C2.4}) \qquad \sum_{n-v}[\mp \sigma_9(m)] \equiv 8370 \sum_{n-v}[\mp m\sigma_7(m)] \pmod{2^4\cdot 3^4\cdot 5\cdot 7},$$

where $\sigma_9(0) = 1296n^5 + 2160n^4 - 3600n^3 + 20640n^2 + 16495n.$

The method used for establishing these congruences is similar to that used earlier for proving Corollary 1.

Congruences for $\sum_{n-v}[\mp \sigma_k(m)]$ for other interesting moduli are available for $k=7$ and 9. These are not quite as simple as the previous ones as can be seen from the next corollary. These are provable by the earlier methods.

COROLLARY 3. *For all values of n,*

(C3.1) $$\sum_{n-v}[\mp \sigma_7(m)] \equiv 126\sum_{n-v}[\mp m^2\sigma_3(m)]+42\sum_{n-v}[\mp m^3\sigma(m)] \pmod{2\cdot 3\cdot 7^2},$$

where $\sigma_7(0)=132n^4+120n^3-66n^2-59n$;

(C3.2) $$\sum_{n-v}[\mp \sigma_9(m)] \equiv 540\sum_{n-v}[\mp m^2\sigma_5(m)]+270\sum_{n-v}[\mp m^4\sigma(m)] \pmod{2\cdot 3^3\cdot 5^2},$$

where $\sigma_9(0)=216n^5+270n^4-90n^3+390n^2-245n$.

7. Congruences involving $\sum_{n-v}[\mp m^\alpha\sigma_k(m)]$ only.

In the sums appearing on the left hand sides of the congruences listed in Corollary 1 the coefficients of $\sigma_k(m)$ or $\sigma_k(n-v)$ are ± 1. This is also true of the classical identity (8) or (7) with $\sigma(m)$'s or $\sigma(n-v)$'s. It is therefore of some interest to note from the first congruence (C4.1) of the next corollary that there is a linear function, also of $\sigma(m)$'s or $\sigma(n-v)$'s but with other coefficients, which is divisible by 5 whatever be the value of n. Also speaking about linear functions with coefficients other than ± 1 we have another interesting congruence, the second one (C4.2). It should be specially noted that in stating Corollary 4 we have removed the earlier supposition that $m^\alpha\sigma_k(m)=0$ if $m=0$ for $\alpha>0$, (cf. section 2).

COROLLARY 4. *For all values of n*

(C4.1) $$\sum_{n-v}[\mp m\sigma(m)] \equiv 0 \pmod 5,$$

where $0\cdot\sigma(0)=-n^2+2n$;

(C4.2) $$\sum_{n-v}[\mp m\sigma_3(m)] \equiv \sum_{n-v}[\mp m^2\sigma(m)] \pmod 7,$$

where $0\cdot\sigma_3(0)=-n^3-n^2+2n,\ 0^2\cdot\sigma(0)=0$.

These congruences can be derived from the cases (T2) and (T3) of the theorem by following more or less the earlier methods.

REFERENCES

1. D. B. Lahiri, *On a type of series involving the partition function with applications to certain congruence relations*, Bull. Calcutta Math. Soc. 38 (1946), 125–132.
2. D. B. Lahiri, *On Ramanujan's function* $\tau(n)$ *and the divisor function* $\sigma_k(n)$. I, Bull. Calcutta Math. Soc. 38 (1946), 193–206.
3. I. Niven and H. S. Zuckerman, *An introduction to the theory of numbers*, John Wiley, New York, 1960.
4. S. Ramanujan, *Collected Papers*, Chelsea Publishing Company, New York, 1962.

INDIAN STATISTICAL INSTITUTE, CALCUTTA