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ON A PROBLEM OF J. DIXMIER CONCERNING IDEALS IN A VON NEUMANN ALGEBRA

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In this note we give a negative answer to the following question raised by J. Dixmier [1, p. 22] (cf. also [2, Chap. III, § 1, Exerc. 10] and [4, Question 6 in the appendix]):

Let J be an ideal in a von Neumann algebra M. Can we conclude that the set $\{b \in J^+ \mid b \leqq a\}$ is right directed for every $a \in M^+$?

Since J is positively generated this is equivalent to the question: Let $a \in M^+$ and $b \in J_{sa}$ with $b \leqq a$; then, is there a $c \in J^+$ such that $b \leqq c \leqq a$? We give an example, which shows that if $a \notin J$ we cannot even conclude that there is a $c \in M^+ \setminus \{a\}$ such that $b \leqq c \leqq a$.

Let M be the algebra of bounded operators on a Hilbert space H with orthonormal basis $(\xi_n)_{n\in N}$ and J the ideal of compact operators on H. Let p be the projection on the subspace with orthonormal basis $(\xi_{2n})_{n\in N}$ and k the operator defined by

$$\begin{aligned} k\xi_{2n-1} &= n^{-1}\xi_{2n-1} + (n^{-1}-n^{-2})^{\frac{1}{2}}\xi_{2n} , \\ k\xi_{2n} &= (n^{-1}-n^{-2})^{\frac{1}{2}}\xi_{2n-1} - n^{-1}\xi_{2n} \end{aligned}$$

for all $n \in N$. Then k is compact and self-adjoint, $k+p$ is a non-compact projection and is a minimal upper bound of $\{0,k\}$ in M_{sa}. In fact, let $c \in M^+$ such that $k \leqq c \leqq k+p$. Then $0 \leqq k+p-c \leqq p$ and $0 \leqq k+p-c \leqq k+p$, hence $q \leqq p$ and $q \leqq k+p$, where q is the range projection of $k+p-c$. Since $pH \cap (k+p)H = \{0\}$, this implies $q=0$, hence $c=k+p$.

This example also shows that if Φ is a *-homomorphism from a C^*-algebra A onto a C^*-algebra B and if $a_1, a_2 \in A^+$, then we cannot conclude that

$$\Phi(\{a \in A^+ \mid a \leqq a_1,\, a \leqq a_2\}) = \{b \in B^+ \mid b \leqq \Phi(a_1),\, b \leqq \Phi(a_2)\} ,$$

not even in the case where $\Phi(a_1) = \Phi(a_2)$ (cf. [3, Prop. 5]). If A is the C^*-algebra generated by p and k, and Φ is the complex homomorphism defined by

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$$\Phi(a) = \lim_{n\to\infty}(a\xi_{2n}, \xi_{2n})$$

for all $a \in A$, then $\Phi(p) = \Phi(k+p) = 1$ but

$$\{a \in A^+ \mid a \leqq p, a \leqq k+p\} = \{0\}.$$

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