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THE EXISTENCE AND UNICITY OF BEST APPROXIMATIONS

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I. Introduction. Terminology and notation.

One of the central problems in approximation theory may be described thus: a linear subspace P (whose elements are termed "approximants") is prescribed in a normed linear space E, and for each f in E we seek its best approximations in P. The latter are the elements p in P for which $\|f-p\| = \operatorname{dist}(f,P)$. If at least one such best approximation exists for each f in E, then P is called an *existence* space, or briefly, an $\mathscr{E}$-space. If each f in E possesses at most one best approximation in P, then P is called a *unicity* space or a $\mathscr{U}$-space. Subspaces having both properties are called $\mathscr{E}\mathscr{U}$-spaces. In the literature they are sometimes called Čebyšev-subspaces. It is desirable to have general theorems which describe the $\mathscr{E}$-, the $\mathscr{U}$-, and the $\mathscr{E}\mathscr{U}$-spaces and to have concrete results of this type in the normed linear spaces of greatest interest in analysis, such as $C[X]$ and $L_1[X]$. There is also some interest in theorems which characterize best approximations, as such theorems are basic in studying the unicity problem.

These matters have been studied in a general setting by R. C. James [6], K. Fan and I. Glicksberg [1], R. R. Phelps [11], and by Ivan Singer in a long sequence of papers of which we mention two here [17], [18]. In the particular spaces C and L_1 the most recent investigations are those of A. L. Garkavi [2], [3], S. Ya. Havinson [5], B. R. Kripke and T. J. Rivlin [9], R. R. Phelps [11], [12], [13], V. Pták [14], and Ivan Singer [18], [19]. The present paper continues these studies. We are especially interested in obtaining characterizations of the $\mathscr{E}$-, the $\mathscr{U}$-, and the $\mathscr{E}\mathscr{U}$-subspaces of $C[X]$ and $L_1[X]$ by intrinsic properties of the approximating functions. In particular, we have sought theorems in which the

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zero sets of the approximants play a dominant role. The classical theorem of Haar (Corollary 14 below) affords a model: An n-dimensional subspace P of $C[X]$ is a $\mathcal{U}$-space if and only if each $p \in P \setminus \{0\}$ has a zero set containing at most $n-1$ points. Several new theorems of this type are given below.

In Section II a number of results concerning general linear spaces is collected. In many instances these theorems are elementary yet useful; some proofs are given in outline only.

Section III is devoted to approximation in $C[X]$. A dimension-free version of Haar's Theorem, similar in spirit to one given by Phelps in [12], is proved (Theorem 10).

Section IV concerns the Lebesgue space $L_1[X]$ over an arbitrary measure space. The $\mathcal{U}$-subspaces of $L_1[X]$ are characterized in a manner fully analogous to the characterization in $C[X]$ (Theorem 21).

Section V concerns the space $C[X] \cap L_1[X]$ endowed with the L_1-norm. The $\mathcal{U}$-spaces are again characterized (Theorems 22–24). The $\mathcal{U}$-spaces of finite dimension or finite codimension are studied especially (Propositions 28ff.). Of special interest is a characterization of the finite-codimensional $\mathcal{U}$-spaces which possess a continuous metric projection (Theorem 35). The paper concludes with a list of open problems.

A summary of notation is given here. E denotes a linear space, often topologized or normed. E^* denotes the linear space of continuous linear functionals on E. X denotes a set, often furnished with topological or measure structure. C or $C[X]$ denotes the space of bounded, continuous, real-valued functions on X, normed by $\|f\|_\infty = \sup|f(x)|$. C_1 or $C_1[X]$ is the subspace of C consisting of f which satisfy $\int|f| < \infty$, renormed by $\|f\|_1 = \int|f|$. For a set A, $\bar{A}$ is the closure, $\operatorname{co}A$ is the convex hull of A, $\overline{\operatorname{co}}A$ is the closure of $\operatorname{co}A$, $\operatorname{ext}A$ is the set of extreme points of A, and $\operatorname{card}A$ is the cardinal number of A. If f is a function then

$$Z(f) = \{x : f(x) = 0\}, \quad S(f) = \{x : |f(x)| < \|f\|_\infty\},$$
$$\operatorname{crit}(f) = \{x : |f(x)| = \|f\|_\infty\}, \quad D(f) = \{x : f \text{ is discontinuous at } x\}.$$

If P is a linear subspace of E then $P^\perp = \{\varphi \in E^* : \varphi(p) = 0 \text{ for all } p \in P\}$. If E is normed, then

$$\operatorname{dist}(f,P) = \inf\{\|f-p\| : p \in P\}, \quad P^\circ = \{f \in E : \|f\| = \operatorname{dist}(f,P)\},$$
$$S = \{f \in E : \|f\| \leqq 1\}, \quad S^* = \{\varphi \in E^* : \|\varphi\| \leqq 1\},$$

and E/P is normed by putting $|||f+P||| = \operatorname{dist}(f,P)$.

II. General theorems.

In the following lemma, E is a locally convex topological space, and E^* is its conjugate space, topologized with the $\sigma(E^*,E)$-topology. Let P be any linear subspace of E, and let B denote an equicontinuous subset of E^*. A functional Δ is defined on E by the equation $\Delta(f) = \sup\{\varphi(f) : \varphi \in B\}$. Finally, for each $f \in E$, we define $B_f = \{\varphi \in \bar{B} : \varphi(f) = \Delta(f)\}$.

1. Lemma. *For an element f of E the following are equivalent:*

(1) $\Delta(f) \leqq \Delta(f-p)$ *for all* $p \in P$.

(2) $P^\perp \cap \overline{\text{co}}\, B_f$ *is nonempty.*

Proof. Assume (2) and select $\varphi \in P^\perp \cap \overline{\text{co}}\, B_f$. Then there is a net $\varphi^\alpha \in \text{co}\, B_f$ such that $\varphi^\alpha \to \varphi$. Each φ^α is of the form $\varphi^\alpha = \sum_i \lambda_i^\alpha \varphi_i^\alpha$, with $\lambda_i^\alpha > 0$, $\sum_i \lambda_i^\alpha = 1$, and $\varphi_i^\alpha \in B_f$. Thus for arbitrary $p \in P$ we have

$$\begin{aligned}\Delta(f-p) &\geqq \varphi(f-p) = \varphi(f) = \lim \varphi^\alpha(f) \\ &= \lim \textstyle\sum_i \lambda_i^\alpha \varphi_i^\alpha(f) = \lim \sum_i \lambda_i^\alpha \Delta(f) = \Delta(f).\end{aligned}$$

For the converse, suppose that (2) is false. The set $P^\perp$ is closed and convex, and the set $\overline{\text{co}}\, B_f$ is compact and convex, by Theorem 18.5 of [8]. By a standard separation theorem [8], these two sets can be strictly separated by a continuous linear functional, L. Since the conjugate of E^* under the topology $\sigma(E^*,E)$ is E [15, p. 33], there exists an element $g \in E$ such that $L(\varphi) = \varphi(g)$ for all $\varphi \in E^*$. Hence g has the property

$$\min\{\varphi(g) : \varphi \in \overline{\text{co}}\, B_f\} > \max\{\varphi(g) : \varphi \in P^\perp\}.$$

If $g \notin \bar{P}$, then by the same separation theorem there would exist an element $\varphi \in P^\perp$ and $\varphi(g) > 0$. Since $\lambda\varphi$ (for $\lambda > 0$) has the same properties, it would follow that $\max\{\varphi(g) : \varphi \in P^\perp\} = +\infty$. Since this is impossible, $g \in \bar{P}$. By Theorem 18.5 of [8], $\overline{\text{co}}\, B_f$ is equicontinuous. Hence the functional $\theta(h) = \min\{\varphi(h) : \varphi \in \overline{\text{co}}\, B_f\}$ is continuous. Since $\theta(g) > 0$, there exists an element $p \in P$ such that $\theta(p) > 0$. Let $B_1 = \{\varphi \in B : \varphi(p) > \frac{1}{2}\theta(p)\}$ and $B_2 = B \setminus B_1$. Then $\sup\{\varphi(f) : \varphi \in B_2\} = \Delta(f) - \varepsilon$ for some $\varepsilon > 0$. Let $\lambda > 0$. Then for $\varphi \in B_1$ we have

$$\varphi(f - \lambda p) \leqq \Delta(f) - \tfrac{1}{2}\lambda\theta(p),$$

while for $\varphi \in B_2$ we have

$$\varphi(f - \lambda p) \leqq \Delta(f) - \varepsilon + \lambda|\varphi(p)|.$$

Hence for appropriate λ we obtain $\Delta(f - \lambda p) < \Delta(f)$.

Apropos the hypotheses of Lemma 1, we note that if E is a barrelled linear topological space, then every $\sigma(E^*,E)$-bounded set in E^* is equicontinuous.

In the next result the setting is the same as for Lemma 1, except that P is now finite-dimensional.

2. LEMMA. *For $f \in E$ the following are equivalent*:

(1) $\Delta(f) \leqq \Delta(f-p)$ *for all* $p \in P$.
(2) $P^{\perp} \cap \operatorname{co} B_f$ *is nonempty.*

PROOF. The implication (2) $\Rightarrow$ (1) follows from Lemma 1. For the converse, assume (1). By Lemma 1 there is an element φ_0 in $P^{\perp} \cap \overline{\operatorname{co}} B_f$. Hence there is a net $\varphi^{\alpha} \in \operatorname{co} B_f$ such that $\varphi^{\alpha} \to \varphi_0$. Put $C_f = \{\varphi \mid P : \varphi \in B_f\}$. $\varphi | P$ denotes the restriction of φ to P. Then $\varphi^{\alpha} | P \in \operatorname{co} C_f$ and $\varphi^{\alpha} | P \to \varphi_0 | P = 0$. Since C_f is a compact subset of the finite-dimensional space P^*, $\overline{\operatorname{co}} C_f = \operatorname{co} C_f$. Hence $0 \in \operatorname{co} C_f$, and $P^{\perp} \cap \operatorname{co} B_f$ is nonvoid.

In the next result, E is a pseudonormed linear space, and P is a linear subspace of E. The unit sphere of E^* is denoted by S^*. For any $f \in E$, we write

$$A_f = \{\varphi \in S^* : \varphi(f) = \|f\|\}, \quad N_f(g) = \sup\{\varphi(g) : \varphi \in A_f\}.$$

The set of extreme points of a convex set K is denoted by $\operatorname{ext} K$. Observe that $\operatorname{ext} A_f \subset \operatorname{ext} S^*$.

The equivalence of (1) and (2) in this theorem is a result of I. Singer [17].

3. PROPOSITION. *For $f \in E$ the following are equivalent*:

(1) $\|f\| \leqq \|f-p\|$ *for all* $p \in P$.
(2) $A_f \cap P^{\perp}$ *is nonempty.*
(3) *For each $p \in P$ there exist $\varphi_i \in \operatorname{ext} A_f$ and $\lambda_i > 0$ such that*

$$\sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \left| \sum_{i=1}^{n} \lambda_i \varphi_i(p) \right| < 1 .$$

(4) $N_f(f) \leqq N_f(f-p)$ *for all* $p \in P$.
(5) $N_{f-p}(f) \leqq N_{f-p}(f-p)$ *for all* $p \in P$.

PROOF. (1) $\Rightarrow$ (2). This follows from Lemma 1 by taking B to be S^* and observing that B_f is a convex set which is compact in the $\sigma(E^*,E)$-topology.

(2) $\Rightarrow$ (3). Let φ_0 be an element of $A_f \cap P^{\perp}$. By the Krein–Milman Theorem [8, p. 131] φ_0 is in the $\sigma(E^*,E)$-closure of $\operatorname{co} \operatorname{ext} A_f$. If $p \in P$,

then the set $U=\{\varphi \in E^*: |\varphi(p)|<1\}$ is a $\sigma(E^*,E)$-neighborhood of 0. Thus there is a point ψ in $(\varphi_0+U)\cap \operatorname{co}\operatorname{ext}A_f$. We can write $\psi=\sum_i\lambda_i\varphi_i$, with $\lambda_i>0$, $\sum_i\lambda_i=1$, $\varphi_i\in \operatorname{ext}A_f$, and $|(\psi-\varphi_0)(p)|<1$. Equivalently, $|\sum_i\lambda_i\varphi_i(p)|<1$.

(3) $\Rightarrow$ (4). For each $p\in P$ and $\varepsilon>0$ there exists $\varphi=\sum_i\lambda_i\varphi_i$ with $\varphi_i\in\operatorname{ext}A_f$, $\lambda_i>0$, $\sum_i\lambda_i=1$, and $|\varphi(p/\varepsilon)|<1$. Hence $\varphi(f)=\|f\|$ and $\varphi\in A_f$. Thus

$$N_f(f-p) \geqq \varphi(f-p) = \|f\|-\varphi(p) > N_f(f)-\varepsilon .$$

Hence $N_f(f-p)\geqq N_f(f)$.

(4) $\Rightarrow$ (5). $N_{f-p}(f)\leqq N_f(f)\leqq N_f(f-p)\leqq N_{f-p}(f-p)$.

(5) $\Rightarrow$ (1). If (1) is false, then for some $p\in P$, $\|f-2p\|<\|f-p\|<\|f\|$. Let $\varphi\in A_{f-p}$. Then $\|f-p\|=\varphi(f-p)=\frac{1}{2}\varphi(f)+\frac{1}{2}\varphi(f-2p)$. Since $\varphi(f-2p)<\|f-p\|$, it follows that $\varphi(f)>\|f-p\|$. Hence $N_{f-p}(f)>N_{f-p}(f-p)$, in denial of (5).

In the next result, E is a normed linear space and P is any linear subspace of E. The symbols A_f and N_f have their former significance, and $p^{\perp}$ denotes $\{\varphi\in E^*: \varphi(p)=0\}$.

4. Proposition. *Let f be an element of E having 0 for a best approximation in P. For $p\in P$ the following conditions are equivalent:*

(1) $\|f\|=\|f-p\|$.
(2) $p^{\perp}\cap\operatorname{ext}A_{f-p}$ *is nonempty.*
(3) $N_{f-p}(f)=N_{f-p}(f-p)$.
(4) $A_{f-p}\cap P^{\perp}=A_f\cap P^{\perp}$.

Proof. (1) $\Rightarrow$ (2). If (1) is true, then $\frac{1}{2}p$ is also a best approximation to f. By the Krein–Milman Theorem there exists an element φ in $\operatorname{ext}A_{(f-\frac{1}{2}p)}$. Thus

$$\|f\| = \tfrac{1}{2}\|f\|+\tfrac{1}{2}\|f-p\| \geqq \tfrac{1}{2}\varphi(f)+\tfrac{1}{2}\varphi(f-p) = \varphi(f-\tfrac{1}{2}p) = \|f-\tfrac{1}{2}p\| = \|f\| .$$

Since $|\varphi(f)|\leqq\|f\|$ and $|\varphi(f-p)|\leqq\|f-p\|=\|f\|$, it follows that $\varphi(f)=\|f\|$ and that $\varphi(f-p)=\|f\|$. Thus $\varphi(p)=0$ and $\varphi(f-p)=\|f-p\|$. Now if $\varphi=\frac{1}{2}\varphi_1+\frac{1}{2}\varphi_2$, with $\varphi_i\in A_{f-p}$, then from the above we have

$$\tfrac{1}{2}\varphi_1(f-\tfrac{1}{2}p)+\tfrac{1}{2}\varphi_2(f-\tfrac{1}{2}p) = \varphi(f-\tfrac{1}{2}p) = \|f-\tfrac{1}{2}p\| .$$

Since $|\varphi_i(f-\frac{1}{2}p)|\leqq\|f-\frac{1}{2}p\|$, it follows that $\varphi_i(f-\frac{1}{2}p)=\|f-\frac{1}{2}p\|$ and that $\varphi_i\in A_{(f-\frac{1}{2}p)}$. Since φ is an extreme point of the latter set, $\varphi_1=\varphi_2$. Hence φ is an extreme point of A_{f-p}.

(2) $\Rightarrow$ (3). If $\varphi \in p^{\perp} \cap \operatorname{ext} A_{f-p}$, then

$$N_{f-p}(f) \leqq N_f(f) = \|f\| \leqq \|f-p\| = N_{f-p}(f-p) = \varphi(f-p) = \varphi(f) \leqq N_{f-p}(f) .$$

(3) $\Rightarrow$ (4). If $\varphi \in A_{f-p} \cap P^{\perp}$, then $\varphi \in A_f$ because $\|f\| \geqq |\varphi(f)| = \varphi(f-p) = \|f-p\| \geqq \|f\|$. If $\varphi \in A_f \cap P^{\perp}$, then $\varphi \in A_{f-p}$ because $\|f-p\| \geqq \varphi(f-p) = \varphi(f) = \|f\| \geqq N_{f-p}(f) = N_{f-p}(f-p) = \|f-p\|$.

(4) $\Rightarrow$ (1). Since 0 is a best approximation to f, $A_f \cap P^{\perp}$ is nonempty by Lemma 2. By (4) $A_{f-p} \cap P^{\perp}$ is nonempty. By Lemma 2, 0 is a best approximation to $f-p$, and hence p is a best approximation to f.

5. PROPOSITION. *Let E be a normed linear space and let P and Q be subspaces of E such that $P \subset Q$. Then the following are true*:

(1) *If q is a best approximation to f from Q, then $q+P$ is a best approximation to $f+P$ from Q/P.*

(2) *If $q+P$ is a best approximation to $f+P$ from Q/P and p is a best approximation to $f-q$ from P, then $q+p$ is a best approximation to f from Q.*

(3) *If P is an $\mathscr{E}$-space in E and Q/P is an $\mathscr{E}$-space in E/P, then Q is an $\mathscr{E}$-space in E.*

(4) *If P is a $\mathscr{U}$-space in E and Q/P is a $\mathscr{U}$-space in E/P, then Q is a $\mathscr{U}$-space in E.*

(5) *If Q is an $\mathscr{E}$-space in E, then Q/P is an $\mathscr{E}$-space in E/P.*

(6) *If P is an $\mathscr{E}\mathscr{U}$-space in E and Q/P is an $\mathscr{E}\mathscr{U}$-space in E/P, then Q is an $\mathscr{E}\mathscr{U}$-space in E.*

(7) *If P is an $\mathscr{E}$-space in E and Q is a $\mathscr{U}$-space in E, then Q/P is a $\mathscr{U}$-space in E/P.*

(8) *If P is an $\mathscr{E}$-space in E and Q/P is a dual space, then Q is an $\mathscr{E}$-space in E.*

PROOF. (1) If $q+P$ is not a best approximation to $f+P$ then, for some $q' \in Q$, we have $\operatorname{dist}(f-q',P) < \operatorname{dist}(f-q,P) \leqq \|f-q\|$. Hence, for some $p' \in P$, we have $\|f-q'-p'\| < \|f-q\|$. But $q'+p' \in Q$ because $P \subset Q$. Hence this last inequality shows that q is not a best approximation to f.

(2) For any $q' \in Q$, we have $\|f-q-p\| = \operatorname{dist}(f-q,P) \leqq \operatorname{dist}(f-q',P) \leqq \|f-q'\|$. Hence $q+p$ is a best approximation to f from Q.

(3) This follows at once from (2).

(4) If the conclusion is false, then some $f \in E$ has two distinct best approximations q and q' from Q. By part (1), $q+P$ and $q'+P$ are best approximations to $f+P$ from Q/P. Since Q/P is a $\mathscr{U}$-space, we have $q+P = q'+P$. Hence $q' = q+p$ for some $p \in P \setminus \{0\}$. Thus

$$\|f-q-p\| = \|f-q'\| = \|f-q\| \\ = \operatorname{dist}(f,Q) = \operatorname{dist}(f-q,Q) \leqq \operatorname{dist}(f-q,P) .$$

This shows that both p and 0 are best approximations to $f-q$ from P. Hence P is not a $\mathscr{U}$-space.

(5) This follows immediately from (1).

(6) This follows immediately from (3) and (5).

(7) If the conclusion is false, then for some $f \in E$, $f+P$ has two distinct best approximations in Q/P, say $q+P$ and $q'+P$. Then $q-q' \notin P$. Since P is an $\mathscr{E}$-space, there exist best approximations p and p' for $f-q$ and $f-q'$ respectively. By part (2), $q+p$ and $q'+p'$ are best approximations to f from Q. Since Q is a $\mathscr{U}$-space, $q+p=q'+p'$. Thus $q-q' \in P$, a contradiction.

(8) This will follow from (3) after establishing that any dual space M in a normal linear space is an $\mathscr{E}$-space. Indeed, for any f the closed spheres S_n with center f and radius $\operatorname{dist}(f,M)+1/n$ intersect M in weak* compact sets, because M is a dual. Hence there is a point of M common to all the spheres S_n, and this point is a best approximation to f from M.

Given a (not necessarily closed) linear subspace P in a normed linear space E, we define

$$P^\circ = \{f \in E : \|f\| = \operatorname{dist}(f,P)\} .$$

The notation is suggestive because the elements of P° have 0 as a best approximation in P. The next proposition records some of the elementary properties of P°. The proof is straightforward.

6. Proposition. (1) *P° is always closed.*

(2) *P is an $\mathscr{E}$-space if and only if $P+P^\circ = E$.*

(3) *P is an $\mathscr{E}\mathscr{U}$-space if and only if $P \oplus P^\circ = E$.*

Let P be an $\mathscr{E}\mathscr{U}$-subspace of a normed linear space E. Thus each $f \in E$ has in P a unique best approximation, which we denote by Tf. The operator T is called the *metric projection* of E onto P, or the *Tchebycheff map.* Define for $\lambda \geqq 0$,

$$P_\lambda(f) = \{g \in E : \|T(f-g) - Tf + Tg\| \leqq \lambda \|f-g\|\} .$$

The elementary properties of T and P_λ are summarized in the next result.

7. Proposition. *Let P be an $\mathscr{E}\mathscr{U}$-subspace of a normed linear space E.*

(1) *If, for some fixed λ, $f_n \in P_\lambda(f)$ and $f_n \to f$, then $Tf_n \to Tf$.*

(2) *If, for some λ, $P_\lambda(f)$ contains a neighborhood of f, then T is continuous at f.*

(3) *If* $P^\circ + P^\circ \subset P^\circ$, *then* T *is continuous.*

(4) *If* P *is a hyperplane, then* T *is continuous.*

(5) *If* P° *is boundedly compact, then* T *is continuous.*

(6) *If* T *is continuous at the points of* P°, *then* T *is continuous.*

PROOF. (1) $\|Tf-Tf_n\| \leqq \|Tf-Tf_n-T(f-f_n)\| + \|T(f-f_n)-(f-f_n)\| + \|f-f_n\| \leqq \lambda\|f-f_n\| + 2\|f-f_n\| \to 0$.

(2) This follows immediately from (1).

(3) If $P^\circ + P^\circ \subset P^\circ$, then for any f and g, $(f-Tf)+(Tg-g) \in P^\circ$. Hence $0 = T(f-Tf+Tg-g) = T(f-g)-Tf+Tg$. Thus $P_0(f) = E$, and (2) applies.

(4) In this case, elementary arguments show that T is linear. Hence $P_0(f) = E$, and (2) applies.

(5) If T is discontinuous, then for some sequence $\{f_n\}$ we have $f_n \to f$ yet the elements $h_n \equiv f_n - Tf_n$ remain outside a neighborhood of $f - T_f$. If P° is boundedly compact, then we may assume (passing to a subsequence if necessary) that $h_n \to f-g$ for some g. Since

$$g = \lim(f-h_n) = \lim(f_n - h_n) = \lim Tf_n$$

and since P is closed, it follows that $g \in P$. Since

$$\|f-g\| = \lim\|h_n\| = \lim \operatorname{dist}(f_n, P) = \operatorname{dist}(f,P)\ ,$$

we have $g = Tf$. But this is not possible, because $f-g = \lim h_n \neq f - Tf$.

(6) If T is discontinuous, let $f_n \to f$ and $Tf_n \nrightarrow Tf$. Put $g_n = f_n - Tf$. Then $g_n \to f - Tf \in P^\circ$ but $Tg_n = Tf_n - Tf \nrightarrow 0$.

8. THEOREM. *For an* $\mathscr{E}\mathscr{U}$*-subspace* P *of finite codimension in a normed linear space* E, *the following properties are equivalent*:

(1) *The metric projection onto* P *is continuous.*

(2) P° *is boundedly compact.*

PROOF. The implication (2) $\Rightarrow$ (1) is contained in Proposition 7. For the converse, assume that (2) is false, and let $\{f_n\}$ be a bounded sequence in P° which has no convergent subsequence. The cosets $f_n + P$ are bounded in E/P because $|||f_n + P||| = \operatorname{dist}(f_n, P) = \|f_n\|$. Since E/P is finite-dimensional, we can assume (passing to a subsequence if necessary) that $f_n + P$ converges, say to $f + P$, where $f \in P^\circ$. Let p_n be the best approximation of $f_n - f$. Then

$$\|f_n - f - p_n\| = \operatorname{dist}(f_n - f, P) \equiv |||(f_n + P) - (f + P)||| \to 0\ .$$

It follows that $f_n - p_n \to f$. But $T(f_n - p_n) \nrightarrow Tf$ because

$$T(f_n - p_n) - Tf = Tf_n - p_n - Tf = -p_n = (f_n - f - p_n) + (f - f_n) \nrightarrow 0 .$$

An $\mathscr{E}\mathscr{U}$-subspace P in a normed linear space E will be said to be an $\mathscr{E}\mathscr{U}^*$-subspace if the metric projection T_P of E onto P is continuous.

9. Theorem. *Let P be an $\mathscr{E}\mathscr{U}^*$-subspace in a normal linear space E, and let Q be a subspace of E which contains P. Then the following are equivalent:*

(1) *Q is an $\mathscr{E}\mathscr{U}^*$-subspace in E.*
(2) *Q/P is an EU^*-subspace in E/P.*

Proof. Assume (1). By Proposition 5, Q/P is an $\mathscr{E}\mathscr{U}$-subspace, and we must show that its metric projection $T_{Q/P}$ is continuous. Let $f_n + P \to f + P$. Put $g_n = f_n - T_Q f$ and $g = f - T_Q f$. Then $g_n + P \to g + P$ and $g \in Q^\circ$. It follows that $\operatorname{dist}(g_n - g, P) \to 0$, that $\|g_n - g - T_P(g_n - g)\| \to 0$, and that $g_n - T_P(g_n - g) \to g$. By the continuity of T_P,

$$T_P g_n - T_P(g_n - g) \to T_P(g) = 0 .$$

Thus, $g_n - T_P g_n \to g$. By the continuity of T_Q,

$$T_Q g_n - T_P g_n \to T_Q g = 0 .$$

Thus $\operatorname{dist}(T_Q g_n, P) \to 0$, $\operatorname{dist}(T_Q f_n - T_Q f, P) \to 0$, and $T_Q f_n + P \to T_Q f + P$. By Proposition 5, this implies that $T_{Q/P}(f_n + P) \to T_{Q/P}(f + P)$.

Now assume (2). By Proposition 5, Q is an $\mathscr{E}\mathscr{U}$-space, and we must show that its metric projection T_Q is continuous. By Proposition 6 it suffices to prove the continuity of T_Q at points of Q°. Let $f_n \to f \in Q^\circ$. Then $f_n + P \to f + P$, and by the continuity of $T_{Q/P}$ we have $T_{Q/P}(f_n + P) \to T_{Q/P}(f + P)$. By Proposition 5, $T_Q f_n + P \to T_Q f + P = P$. Hence $f - f_n + T_Q f_n + P \to P$. It follows that

$$\operatorname{dist}(f - f_n + T_Q f_n, P) \to 0 ,$$
$$\|f - f_n + T_Q f_n - T_P(f - f_n + T_Q f_n)\| \to 0 ,$$
$$f_n - T_Q f_n + T_P(f - f_n + T_Q f_n) \to f .$$

By the continuity of T_P,

$$T_P(f_n - T_Q f_n) + T_P(f - f_n + T_Q f_n) \to T_P f = 0 .$$

Since $T_P(f_n - T_Q f_n) = 0$ we have $T_P(f - f_n + T_Q f_n) \to 0$. From a previous equation it follows that $-T_Q f_n + f_n \to f$, whence $T_Q f_n \to 0$.

The converse of Theorem 9 is not true. That is, there exists an example of spaces $P \subset Q \subset E$ such that P is an $\mathscr{E}\mathscr{U}$-space in E, Q is an $\mathscr{E}\mathscr{U}^*$-space in E, Q/P is an $\mathscr{E}\mathscr{U}^*$-space in E/P, and P is not an $\mathscr{E}\mathscr{U}^*$-space in E. Such an example is as follows. Let P be an $\mathscr{E}\mathscr{U}$-subspace of codimension 2 which is not an $\mathscr{E}\mathscr{U}^*$-subspace. (See Theorem 36 below.) Let Q be a hyperplane in E which contains P and is an $\mathscr{E}\mathscr{U}$-space. The $\mathscr{E}\mathscr{U}^*$-property of Q and Q/P follows then from the general results in Propositions 5 and 8.

III. Approximation in *C*.

In this section X denotes a compact Hausdorff space, and C the space of continuous real-valued functions on X with supremum norm. Given a subspace P of C, we define as an *α-set* any set of the form $\{x\colon |f(x)| = \|f\|\}$ for some $f \in P^\circ$. Thus Y is an α-set if $Y \subset X$ and if Y is the critical point set for some function f which has 0 for a best approximation in P. With the aid of this concept, Haar's Theorem on approximation in C can be stripped of dimensionality restrictions and given the following form.

10. THEOREM. *The following properties of a subspace P in C are equivalent:*

(1) *P is a $\mathscr{U}$-space.*
(2) *0 is the only element of P which vanishes on an α-set.*

PROOF. If (1) is denied, then there exists an f which has two distinct best approximations in P, say p_1 and p_2. Then $p \equiv \frac{1}{2}(p_1 + p_2)$ is another best approximation to f, and 0 is a best approximation to $f - p$. The set $Y = \operatorname{crit}(f - p)$ is then an α-set. For $y \in Y$ we have

$$\pm\|f-p\| = (f-p)(y) = \tfrac{1}{2}(f-p_1)(y) + \tfrac{1}{2}(f-p_2)(y)\,.$$

Since $|(f-p_i)(y)| \leqq \|f-p_i\| = \|f-p\|$, we see that $(f-p_1)(y) = (f-p_2)(y)$, whence $(p_1 - p_2)(y) = 0$. Thus $p_1 - p_2$ vanishes on Y.

For the converse, suppose that some nonzero element q in P vanishes on an α-set Y. Then $Y = \operatorname{crit}(f)$ for some $f \in P^0$. Without loosing generality we assume that $\|f\| = 1$ and $\|q\| < 1$. Following Haar's original proof, define $h = (1 - |q|)f$. If $0 \leqq \theta \leqq 1$, then

$$|h - \theta q| \leqq |h| + \theta|q| \leqq 1 - |q| + \theta|q| \leqq 1\,.$$

Hence $\|h - \theta q\| \leqq 1$. Since $f \in P^\circ$, Lemma 1 asserts the existence of an element φ in $P^\perp \cap \overline{\operatorname{co}}\, A_f$, where the closure is taken in the $\sigma(C^*, C)$-topology and the set A_f is defined as $\{\hat{y}\ \mathrm{sgn} f(y): y \in Y\}$. Here $\hat{y}$ denotes the evaluation functional corresponding to the point y. It follows that $\|\varphi\| \leqq 1$ and that $\varphi = \lim_\alpha \sum_i \lambda_i^\alpha \hat{y}_i^\alpha$, with $y_i^\alpha \in Y$, $\sum_i |\lambda_i^\alpha| = 1$, and $\mathrm{sgn}\,\lambda_i^\alpha = \mathrm{sgn} f(y_i^\alpha)$. For any $p \in P$ we have

$$\begin{aligned}\|h-p\| \geqq \varphi(h-p) &= \varphi(h) = \lim_\alpha \textstyle\sum_i \lambda_i^\alpha h(y_i^\alpha) \\ &= \lim_\alpha \textstyle\sum_i \lambda_i^\alpha f(y_i^\alpha) = \lim_\alpha \|f\| = 1 .\end{aligned}$$

Thus $\mathrm{dist}(h,P) \geqq 1$, and all the elements θq are best approximations to h. It is possible to prove that $\mathrm{dist}(h,P) \geqq 1$ by assuming on the contrary that $\|h-p\| < 1$ for some $p \in P$. Then for $x \in \mathrm{crit}(f)$ we have $q(x) = 0$, $h(x) = \pm 1$, and $h(x)p(x) > 0$. It is then possible to select $\theta > 0$ in such a way that $\|f - \theta p\| < 1$, a contradiction. This alternative proof avoids use of the Hahn–Banach Theorem and the Krein–Milman Theorem, but involves more calculation.

In the preceding result the hypothesis of compactness on X can be relaxed to pseudocompactness (every continuous real-valued function on X is bounded).

Another characterization of the $\mathscr{U}$-subspaces in $C[X]$ has been given by Phelps [12, p. 649]. His theorem involves the annihilator of the subspace and is also proved by an appropriate modification of Haar's original argument. In order to clarify the relationship between Phelps' Theorem and Theorem 10 two technical terms are convenient. If φ is an element of C^*, then the *support* of φ is the smallest closed set A such that $I_A \subset \varphi_\perp$. Here I_A is the ideal $\{f \in C: f(x) = 0 \text{ for all } x \in A\}$ and $\varphi_\perp = \{f \in C: \varphi(f) = 0\}$. A subset of X is termed a *θ-set* if it is the support of an element of $P^\perp$ which achieves its supremum on the unit sphere of C. In these terms, Phelps' Theorem asserts that a subspace P of C is a $\mathscr{U}$-space if and only if 0 is the only element of P which vanishes on a θ-set in X. The connection between α-sets and θ-sets is as follows.

11. Proposition. *Every α-set contains a θ-set. The converse is generally not true.*

Proof. If Y is an α-set, then for some $f \in P^\circ$, $Y = \mathrm{crit}(f)$. By Proposition 3 there is an element φ in $P^\perp$ such that $\|\varphi\| = 1$ and $\varphi(f) = \|f\|$. By the Krein–Milman Theorem, φ is the limit in the $\sigma(C^*, C)$-topology of a net $\sum_i \theta_i^\alpha \hat{y}_i^\alpha$, where $y_i^\alpha \in Y$ and $\sum_i |\theta_i^\alpha| = 1$. Here $\hat{y}$ is the evaluation at y. If $g \in C$ and g vanishes on Y, then $\varphi(g) = 0$. Thus Y contains the support of φ. Since $\varphi \in P^\perp$ and φ attains its supremum on the unit sphere at $f/\|f\|$, it follows that its support is a θ-set.

In order to see that a θ-set need not contain an α-set, let X be the set of ordinals not exceeding the first uncountable ordinal Ω. See, for example, [7, pp. 24 and 59]. Give X the order topology, and define $P=\{f\in C: f(\Omega)=0\}$. The evaluation functional $\hat{\Omega}$ annihilates P and achieves its supremum on the unit sphere at 1. Hence the support of $\hat{\Omega}$ is a θ-set. But the support of $\hat{\Omega}$ is $\{\Omega\}$, and this contains no α-set.

In order to illuminate somewhat the nature of α-sets, and in order to recover Haar's Unicity Theorem, two elementary results are helpful.

12. LEMMA. *If P is a finite-dimensional subspace of C and if Y is an α-set, then there exist points $y_1,\ldots,y_n$ in Y such that the corresponding set of evaluation functionals is linearly dependent on P.*

PROOF. If Y is an α-set, then $Y=\operatorname{crit}(f)$ for some $f\in P^\circ$. In Lemma 2, take B to be the set of all evaluation functionals and their negatives. The conclusion is then that there exist points $y_i\in Y$ such that $\sum_i\lambda_i\hat{y}_i\in P^\perp$ for some λ_i with $\sum_i|\lambda_i|=1$.

13. LEMMA. *Let P be any subspace of C, and let $y_1,\ldots,y_k$ be points of X such that the corresponding set of evaluation functionals is dependent on P. If $\{A_1,\ldots,A_k\}$ is a disjoint family of closed G_δ-sets such that $y_i\in A_i$ for $i=1,\ldots,k$, then $\bigcup A_i$ is an α-set.*

PROOF. If $\sum_i\lambda_i\hat{y}_i\in P^\perp$ and $\sum_i|\lambda_i|=1$, then by Tietze's Lemma there exists an $f\in C$ such that $f(x)=\operatorname{sgn}\lambda_i$ for $x\in A_i$, $i=1,\ldots,k$, and $|f(x)|<1$ for $x\notin\bigcup A_i$. By Lemma 1, $f\in P^\circ$. Hence $\bigcup A_i$ is an α-set.

14. COROLLARY. (Haar). *Let P be an n-dimensional subspace of C. The following properties of P are equivalent:*

(1) *P is a $\mathscr{U}$-space.*
(2) *0 is the only element of P vanishing on n points.*

PROOF. If (2) is true, then every set of n evaluation functionals is linearly independent on P. By Lemma 12, every α-set must contain $n+1$ points. By (2), 0 is the only element of P vanishing on an α-set. Theorem 10 yields (1).

Conversely, if (2) is false, then some nonzero element $p\in P$ vanishes on a set of n points, say $y_1,\ldots,y_n$. Thus $\{\hat{y}_1,\ldots,\hat{y}_n\}$ is dependent on P. By Tietze's Lemma, there exist closed G_δ-sets A_i such that $y_i\in A_i$. By Lemma 13, $Z(p)\cap\bigcup A_i$ is an α-set. By Theorem 10, P is not a $\mathscr{U}$-space.

The following elementary lemma, given without proof, will be useful below.

15. LEMMA. *Let A be a closed subset of a Hausdorff space X (not necessarily compact). Let I be the ideal of all continuous functions on X which vanish on A. If $X \setminus A$ contains n or more isolated points, then* $\dim(I) \geqq n$. *If $X \setminus A$ consists of precisely n points, then* $\dim(I) = n$. *If X is completely regular and $X \setminus A$ contains n or more points, then* $\dim(I) \geqq n$.

16. PROPOSITION. *If P is a $\mathcal{U}$-space of codimension n in C then* $\operatorname{card}[X \setminus \operatorname{crit}(f)] \leqq n$ *for each $f \in P^\circ$.*

PROOF. If $\operatorname{card}[X \setminus \operatorname{crit}(f)] > n$ for some $f \in P^\circ$, and if I is the ideal of functions vanishing on $\operatorname{crit}(f)$, then by Lemma 15, $\dim(I) > n$. Since P is of codimension n, it contains a nonzero element p in common with I. Thus p vanishes on the α-set $\operatorname{crit}(f)$, and by Theorem 10, P is not a $\mathcal{U}$-space.

IV. Approximation in L_1.

In this section, a set X, a σ-ring Σ of subsets of X, and a measure μ on Σ are prescribed. We denote by L_1 the Banach space of (equivalence classes of) measurable real-valued functions f on X for which

$$\|f\| \equiv \int |f| < \infty .$$

For $f \in L_1$, put $Z(f) = \{x \in X : f(x) = 0\}$. Since f stands for an equivalence class of functions, $Z(f)$ stands for an equivalence class of sets, which differ from each other only by sets of measure zero.

The following theorem characterizes best approximations in subspaces of L_1. The equivalence of (1) and (2) here is due to R. C. James [6, p. 291]. Kripke and Rivlin have recently proved this equivalence in the complex case [9]. The implication (1) $\Rightarrow$ (3) provides a quantitative form for certain unicity theorems. The condition (4), which is more complicated than (2), has the advantage that a *strict* inequality in (4) for all $p \in P \setminus \{0\}$ can be shown to be equivalent to the assertion that 0 is the *unique* best approximation to f. The proof of this is not included, however.

17. PROPOSITION. *Let P be a linear subspace of L_1. For $f \in L_1$ the following properties are equivalent:*

(1) $f \in P^\circ$(*that is,* 0 *is a best approximation to* f *from* P).

(2) $\int p \operatorname{sgn} f \leqq \int_{Z(f)} |p|$ *for all* $p \in P$.

(3) $\int |f+p| \geqq \int |f| + \int_A |f| + \int_A |f+p|$ *for all* $p \in P$. *Here*

$$A = A(f,p) = \{x\colon |f(x)| < |p(x)| \text{ and } f(x)p(x) < 0\}\,.$$

(4) $\int p \operatorname{sgn}(f+p) \geqq \int_{Z(f+p)} |f|$ *for all* $p \in P$.

PROOF. For the implication (1) $\Rightarrow$ (2) see the remarks above.

For the implication (2) $\Rightarrow$ (3), assume (2), and let p be any element of P. Define $\sigma(x)$ to be $\operatorname{sgn} f(x)$ on $X \setminus Z(f)$ and to be $\operatorname{sgn} p(x)$ on $Z(f)$. Then

$$(*) \qquad \int |f+p| \geqq \int_{X \setminus A} (f+p)\sigma + \int_A |f+p|$$

$$= \int (f+p)\sigma + \int_A |f+p| - \int_A (f+p)\sigma = \int |f| + \int_A |f+p| + \int p\sigma - \int_A (f+p)\sigma\,.$$

From (2) it follows that

$$\int p\sigma = \int_{Z(f)} |p| + \int_{X \setminus Z(f)} p\sigma \geqq \left|\int p \operatorname{sgn} f\right| + \int_{X \setminus Z(f)} p\sigma$$

$$= \left|\int_{X \setminus Z(f)} p\sigma\right| + \int_{X \setminus Z(f)} p\sigma \geqq 0\,.$$

Furthermore, on the set A we have $fp < 0$ and $|p| > |f|$, so that $\operatorname{sgn}(f+p) = \operatorname{sgn} p = -\operatorname{sgn} f = -\sigma$. Hence $\int_A (f+p)\sigma \leqq 0$. Thus inequality (*) remains true when the last two terms are dropped. This establishes (3).

For the implication (3) $\Rightarrow$ (4) we have at once

$$\int p \operatorname{sgn}(f+p) = \int |f+p| - \int f \operatorname{sgn}(f+p)$$

$$\geqq \int |f| - \int_{X \setminus Z(f+p)} |f| = \int_{Z(f+p)} |f|\,.$$

For the implication (4) $\Rightarrow$ (1), assume that (1) is false. Then for some $p \in P$ we have $\int |f+2p| < \int |f+p| < \int |f|$. Define a linear functional φ on L_1 by the equation

$$\varphi(g) = \int g \operatorname{sgn}(f+p) + \int_{Z(f+p)} g \operatorname{sgn} f\,.$$

Then $\|\varphi\| \leqq 1$. From the inequality

$$\|f+p\| = \varphi(f+p) = \tfrac{1}{2}\varphi(f) + \tfrac{1}{2}\varphi(f+2p)$$

$$\leqq \tfrac{1}{2}\varphi(f) + \tfrac{1}{2}\|f+2p\| < \tfrac{1}{2}\varphi(f) + \tfrac{1}{2}\|f+p\|$$

we conclude that $\varphi(f) > \|f+p\|$. Hence

$$\int f \operatorname{sgn}(f+p) + \int_{Z(f+p)} |f| > \int |f+p| = \int (f+p) \operatorname{sgn}(f+p)$$

in denial of (4).

18. Corollary. *Let P be a subspace of L_1. Then for any $f \in P^\circ$ and for any $p \in P$ we have*

$$\| |f+p| - |f| \| \leqq \|p\| \leqq 3 \| |f+p| - |f| \| .$$

Proof. The inequality on the left follows immediately from the triangle inequality. Now let A be the set of points x such that $|f(x)| < |p(x)|$ and $p(x)f(x) < 0$. By Proposition 17

$$\begin{aligned} \int_A |p| &= 2\int_A |f+p| - \int_A (|f+p| - |f|) \\ &\leqq 2\int (|f+p| - |f|) - \int_A (|f+p| - |f|) \\ &= \int (|f+p| - |f|) + \int_{X\setminus A} (|f+p| - |f|) \leqq 2\int \big||f+p| - |f|\big| . \end{aligned}$$

On the set $X \setminus A$ we have $|f| \geqq |p|$ or $fp \geqq 0$. Hence

$$\int_{X\setminus A} |p| = \int_{X\setminus A} \big||f+p| - |f|\big| \leqq \int \big||f+p| - |f|\big| .$$

Thus we have

$$\int |p| = \int_A |p| + \int_{X\setminus A} |p| \leqq 3\int \big||f+p| - |f|\big| .$$

In some of the proofs below it is necessary to know that the conjugate space of L_1 can be identified with the space L_∞ of essentially bounded measurable functions, via the correspondence

$$\varphi(f) = \int fg, \qquad \varphi \in L_1^*,\ g \in L_\infty,\ f \in L_1 .$$

This is the case, for example, when the underlying measure space is σ-finite. We prefer, however, to assume outright (when necessary) that "$L_1^* = L_\infty$".

The next theorem has been given by I. Singer [17, p. 183]. It is also a consequence of Proposition 3.

19. PROPOSITION. (Singer). *Assume that* $L_1^* = L_\infty$, *and let* P *be a subspace of* L_1. *Then* $f \in P^\circ$ *if and only if some* $g \in L_\infty \cap P^\perp$ *has the properties* $|g| \leqq 1$ *and* $\int fg = \int |f|$.

It is possible to show by an example that this theorem cannot be improved by dropping the hypothesis "$L_1^* = L_\infty$". This example is due to T. Botts and is described by J. Schwartz in [16]. Let $X = [0,1]$, let Σ consist of all subsets of X which are countable or have countable complement, and let $\mu(A)$ be the number of elements in A. Define $\varphi(f) = \int x f(x) d\mu$. Then φ is a continuous linear functional on L_1 which is not of the form $\varphi(f) = \int fg$ for any measurable g. The hyperplane $P = \varphi^{-1}[0]$ is an $\mathscr{E}\mathscr{U}$-space in L_1, but best approximations cannot be characterized as in Singer's Theorem.

20. PROPOSITION. *Let* X *be a nonatomic measure space such that* $L_1^* = L_\infty$. *If* P *is a finite-dimensional subspace of* L_1, *then the following conditions on* $f \in L_1$ *are equivalent*:

(1) $f \in P^\circ$.

(2) *for some* $g \in L_\infty$, $g^2 = 1$, $g \perp P$, *and* $\int gf = \int |f|$.

PROOF. By Theorem 8 of Singer's paper [18], property (1) is equivalent to the existence of an $h \in L_\infty$ such that $h \perp P$, $|h| \leqq 1$, and $\int hf = \int |f|$. By Lemma 2 of Phelps' paper [13], this assertion in turn is equivalent to property 2.

Corresponding to any subspace P in L_1 we define a class of sets called "β-sets". They are the sets $Z(f)$ for some $f \in P^\circ$. The next proposition characterizes the $\mathscr{U}$-subspaces of L_1 in the manner of Theorem 10.

21. THEOREM. *For a subspace* P *in* L_1, *the following properties are equivalent*:

(1) P *is a* $\mathscr{U}$-*space.*

(2) 0 *is the only element of* P *vanishing on a* β-*set.*

PROOF. If (1) is false, then some $f \in L_1$ has two best approximations, p_1 and p_2 in P. Let $p = \frac{1}{2}(p_1 + p_2)$. Then

$$\int (|f-p| - \tfrac{1}{2}|f-p_1| - \tfrac{1}{2}|f-p_2|) = 0 .$$

Since the integrand is nonpositive, it must vanish. Hence $|f-p| = \frac{1}{2}|f-p_1| + \frac{1}{2}|f-p_2|$. This shows that for every x in the β-set $Z(f-p)$ we must have $(f-p_1)(x) = (f-p_2)(x) = 0$ and $(p_1-p_2)(x) = 0$.

If (2) is false, then the three conditions $f \in P^\circ$, $p \in P \setminus \{0\}$, and $Z(f) \subset Z(p)$ can be met simultaneously. Define $h = |p| \operatorname{sgn} f$. If q is arbitrary in P, then by Proposition 17 we have

$$\begin{aligned}\int q \operatorname{sgn} h = \int_{X\setminus Z(h)} q \operatorname{sgn} f &= \int q \operatorname{sgn} f - \int_{Z(h)} q \operatorname{sgn} f \\ &\leqq \int_{Z(f)} |q| - \int_{Z(h)\setminus Z(f)} q \operatorname{sgn} f \\ &= \int_{Z(h)} |q| - \int_{Z(h)\setminus Z(f)} \{|q| + q \operatorname{sgn} f\} \leqq \int_{Z(h)} |q| .\end{aligned}$$

By Proposition 17, this inequality (for all q in P) implies that $h \in P^\circ$. Now let $0 \leqq \theta < 1$. If $(h-\theta p)(x) \neq 0$, then $p(x) \neq 0, f(x) \neq 0$, $|h(x)| = |p(x)| \neq 0$, and $\operatorname{sgn}(h-\theta p)(x) = \operatorname{sgn} h(x) = \operatorname{sgn} f(x)$. Hence

$$\int |h-\theta p| = \int (h-\theta p) \operatorname{sgn} f \leqq \int h \operatorname{sgn} f + \theta \int_{Z(f)} |p| = \int h \operatorname{sgn} f = \int |h| .$$

Here we have used the fact that $\int_{Z(f)} |p| = 0$, which follows from the inclusion $Z(f) \subset Z(p)$. Also, $-\theta \int p \operatorname{sgn} f \leqq \theta \int_{Z(f)} |p|$ because $f \in P^\circ$ and Proposition 17 applies. Thus the elements θp are all best approximations to h.

V. Approximation in C_1.

In this section, a set X, a σ-algebra $\mathscr{A}$ of subsets of X, a measure μ on $\mathscr{A}$, and a topology $\mathscr{T}$ on X are all prescribed. We make the following explicit assumptions:

(1) The topological space $(X, \mathscr{T})$ is Hausdorff and completely regular.

(2) Every open set is measurable (that is, $\mathscr{T} \subset \mathscr{A}$), and every nonempty open set has positive measure.

(3) Every singleton (a set with only one element) is of finite measure.

We denote then by C_1 the subspace of L_1 consisting of continuous functions. Alternatively, C_1 is the normed linear space of integrable continuous functions with norm $\|f\| = \int |f|$. Corresponding to a subspace P in C_1 we define a class of sets in X called "γ-sets". They are the sets $Z(f)$ for $f \in P^\circ$. The next result is proved just as Theorem 21 except that one must observe that the function h constructed there belongs to C_1 since $Z(f) \subset Z(p)$.

22. **Theorem.** *For a subspace P of C_1 the following properties are equivalent*:

(1) *P is a $\mathscr{U}$-space.*
(2) *0 is the only element of P which vanishes on a γ-set.*

The following result is the analogue for C_1 of a result given by Phelps [13] for L_1. We use the notations $\|g\|_\infty = \operatorname{ess\,sup}|g(x)|$, $S_g = \{x\colon |g(x)| < \|g\|_\infty\}$ and $D_g = \{x\colon g \text{ is discontinuous at } x\}$. A subset of X is termed a "δ-set" (with respect to a subspace P) if it is of the form $S_g \cup D_g$ for some measurable, essentially bounded, nonzero g satisfying $g \perp P$.

23. **Theorem.** *Assume that $L_1{}^* = L_\infty$. For a subspace P of C_1 the following properties are equivalent*:

(1) *P is a $\mathscr{U}$-space.*
(2) *0 is the only element of P vanishing on a δ-set.*

Proof. If (2) is false, then the conditions $p \in P \setminus \{0\}$, $Y \subset Z(p)$, $Y = S_g \cup D_g$, $g \in L_\infty$, $g \perp P$ can be simultaneously fulfilled. Without loss of generality, we assume that $\|g\|_\infty = 1$. Define $f = g|p|$. Then $f \in C_1$ because $D_g \subset Z(p)$. If $f(x) \neq 0$, then $p(x) \neq 0$, $|g(x)| = 1$, and $\operatorname{sgn} f(x) = g(x)$. Hence for arbitrary $p' \in P$,

$$\int |f-p'| \geqq \int (f-p')g = \int fg = \int |f| \,.$$

Thus $f \in P^\circ$. Now let $0 \leqq \theta < 1$. If $(f-\theta p)(x) \neq 0$, then $p(x) \neq 0$, $g(x) = \operatorname{sgn} f(x)$, $|\theta p(x)| < |f(x)|$ and $\operatorname{sgn}(f-\theta p)(x) = g(x)$. Hence

$$\int |f-\theta p| = \int (f-\theta p)g = \int fg = \int |f| \,.$$

Thus P is not a $\mathscr{U}$-space.

For the implication (2) $\Rightarrow$ (1), it suffices (in view of Theorem 22) to prove that every γ-set contains a δ-set. Let Y be a γ-set. Then $Y = Z(f)$ for some $f \in P^\circ$. By Proposition 19, there exists a function $g \in L_\infty$ such that $\|g\|_\infty \leqq 1$, $g \perp P$, and $\int gf = \int |f|$. If $f(x) \neq 0$, then $g(x) = \operatorname{sgn} f(x)$ and $x \notin S_g \cup D_g$. This proves that the δ-set $S_g \cup D_g$ is contained in the γ-set $Z(f)$.

A subset Y of X is termed an "ε-set" (with respect to a subspace P of C_1) if Y is the boundary of a measurable set A satisfying $\int_A p = \int_{X \setminus A} p$ for all $p \in P$.

24. **Theorem.** *Let X be a nonatomic measure space such that $L_1{}^* = L_\infty$,*

and let P be a finite-dimensional subspace of C_1. The following are equivalent:

(1) *P is a $\mathscr{U}$-space.*

(2) *0 is the only element of P vanishing on an ε-set.*

PROOF. In order to establish that (1) implies (2) it suffices (in view of Theorem 23) to prove that each ε-set is a δ-set. If A is a measurable set such that $\int_A p = \int_{X \setminus A} p$ for all $p \in P$, then define $g(x)$ to be 1 on A and -1 on $X \setminus A$. Then $g \in L_\infty$, and $g \perp P$. Furthermore, $D_g \cup S_g = D_g =$ boundary of A. Hence the boundary of A is a δ-set.

If (1) is false, then by Theorem 23, there is an element $g \in L_\infty$ and an element $p \in P \setminus \{0\}$ such that $g \perp P$ and $S_g \cup D_g \subset Z(p)$. By Lemma 2 of Phelps [13], there is no loss of generality in assuming that $g^2 = 1$. Thus D_g is the boundary of the set $A = \{x : g(x) = 1\}$. Since $g \perp P$, we have $\int_A p = \int_{X \setminus A} p$ for all $p \in P$. Hence D_g is an ε-set, and (2) is false.

An n-dimensional subspace P of $C[X]$ is said to be *interpolating* if for any set of n distinct points $x_i \in X$ and for any n real numbers c_i there exists an element $p \in P$ such that $p(x_i) = c_i$ for $i = 1, \ldots, n$.

25. COROLLARY. (Jackson–Krein). *Every interpolating subspace of $C_1[a,b]$ is a $\mathscr{U}$-space.*

PROOF. If P is an n-dimensional interpolating subspace of $C[a,b]$ then each function g such that $g^2 = 1$ and $g \perp P$ must have at least n discontinuities, because otherwise an element $p \in P$ would exist such that $\operatorname{sgn} p = g$, and then $\int gp > 0$. In this connection see, for example, [4, p. 62]. It follows that each ε-set must contain at least n points. By the interpolating property, 0 is the only element of P which vanishes on n or more points. Hence by Theorem 24 P is a $\mathscr{U}$-space.

If x is an isolated point of X, then let $\bar{x}$ denote the characteristic function of the set $\{x\}$ divided by its measure. Since x is isolated, $\{x\}$ is open and of positive finite measure. Thus $\bar{x} \in C_1$ and $\int \bar{x} = 1$.

26. LEMMA. *If x is an isolated point of X, then $\bar{x}$ and $-\bar{x}$ are extreme points of the unit sphere S of C_1. Conversely, every extreme point of S is an $\bar{x}$ or $-\bar{x}$ for some isolated point x.*

PROOF. Let x be an isolated point and suppose that $\bar{x} = \frac{1}{2}(f+g)$, where $\int |f| = \int |g| = 1$. Then $\bar{x}(x) = \frac{1}{2} f(x) + \frac{1}{2} g(x)$. If $f(x) > \bar{x}(x)$, then $\int |f| > \bar{x}(x) \mu\{x\} = 1$. If $f(x) < \bar{x}(x)$, then $g(x) > \bar{x}(x)$ and $\int |g| > 1$. Thus $f(x) =$

$g(x)=\bar{x}(x)$. It follows that $f(y)=g(y)=0$ for all y different from x. Thus $f=g=\bar{x}$.

For the converse, let f be an extreme point of S. If the open set $\{x:f(x)\neq 0\}$ contains just one point, x, then x is isolated and $f=c\bar{x}$. Since $\int|f|=1$, $|c|=1$ and $f=\pm\bar{x}$. In the other case there exist two points x_i and an $\varepsilon>0$ such that $|f(x_i)|>\varepsilon$. Select open sets V_i such that $x_i\in V_i$, $V_1\cap V_2=\emptyset$, $|f|>\varepsilon$ on V_i, and f is of constant sign on V_i. Select $h_i\in C$ such that $h_i(x_i)f(x_i)>0$, $h_i=0$ off V_i, and either $0\leqq h_i\leqq 1$ or $-1\leqq h_i\leqq 0$. Since $|h_i|\leqq 1$ and $|f|>\varepsilon$ on V_i it is permissible to assume (multiplying each h_i by a small positive constant if necessary) that $|h_i|\leqq|f|$ and $\int|h_1|=\int|h_2|$. Put $g=h_1-h_2$. Since $f=\frac{1}{2}(f+g)+\frac{1}{2}(f-g)$. we shall have a contradiction if $\int|f\pm g|=1$. In fact,

$$\int|f\pm g| = \int_{V_1}|f\pm h_1| + \int_{V_2}|f\mp h_2| + \int_{X\setminus(V_1\cup V_2)}|f|$$

$$= \int_{V_1}(|f|\pm|h_1|) + \int_{V_2}(|f|\mp|h_2|) + \int_{X\setminus(V_1\cup V_2)}|f| = 1\,.$$

27. Lemma. *Let P be a subspace of C_1 and let f be an element of P°. If x is an isolated point of X such that $f(x)\neq 0$, then $\bar{x}\in P^\circ$.*

Proof. Since $f\in P^\circ$, Proposition 3 implies the existence of $\varphi\in P^\perp$ such that $\|\varphi\|=1$ and $\varphi(f)=\|f\|$. By multiplying f by an appropriate scalar, we can assume that $f(x)=\bar{x}(x)$. It follows that $\|f\|=\|f-\bar{x}\|+\|\bar{x}\|$. Consequently, $\varphi(f-\bar{x})+\varphi(\bar{x})=\|f-\bar{x}\|+\|\bar{x}\|$. Since $|\varphi(f-\bar{x})|\leqq\|f-\bar{x}\|$ and $|\varphi(\bar{x})|\leqq\|\bar{x}\|$, we conclude that $\varphi(\bar{x})=\|\bar{x}\|$. By Proposition 3, $\bar{x}\in P^\circ$.

For a set of functions, A, we define $Z(A)=\bigcap\{Z(f):f\in A\}$.

28. Proposition. *If P is a $\mathscr{U}$-space of dimension n in C_1, then* $\operatorname{card}[Z(f)\setminus Z(P)]\geqq n$ *for every $f\in P^\circ$.*

Proof. If not, the conditions $f\in P^\circ$ and $\{x_1,\ldots,x_{n-1}\}\supset Z(f)\setminus Z(P)$ can be fulfilled. Let $\{p_1,\ldots,p_n\}$ be a basis for P. Since the equations

$$\sum_{i=1}^{n} c_i p_i(x_j) = 0, \qquad j=1,\ldots,n-1\,,$$

have a nontrivial solution, there exists an element p in $P\setminus\{0\}$ such that $Z(f)\setminus Z(P)\subset Z(p)$. Since $Z(P)\subset Z(p)$, we have $Z(f)\subset Z(p)$. By Theorem 22, P is not a $\mathscr{U}$-space.

29. Proposition. *If P is a $\mathscr{U}$-space of codimension n in C_1, then* $\operatorname{card}[X \setminus Z(f)] \leqq n$ *for every* $f \in P^\circ$.

Proof. Suppose that $f \in P^\circ$ and that $\operatorname{card}[X \setminus Z(f)] > n$. By Lemma 15, the ideal $I = \{g : Z(f) \subset Z(g)\}$ has dimension at least $n+1$. Every element of I vanishes on $Z(f)$, but 0 is the only element of P which vanishes on $Z(f)$, by Theorem 22. Hence $I \cap P = \{0\}$, and P has codimension at least $n+1$.

30. Proposition. *If P is an $\mathscr{E}$-space of codimension n in C or C_1, then* $\operatorname{card}[X \setminus Z(P^\circ)] \geqq n$.

Proof. Let I be the ideal of functions vanishing on $Z(P^\circ)$. Then $P^\circ \subset I$. If $\operatorname{card}[X \setminus Z(P^\circ)] < n$, then $\dim(I) < n$ by Lemma 15. In that case, we could not have $P + P^\circ = C$ as required by Proposition 6.

31. Theorem. *Let P be a subspace of codimension n in C_1 such that the set $A = \bigcup \{X \setminus Z(f) : f \in P^\circ\}$ is finite and such that each $\varphi \in P^\perp$ attains its supremum on the unit sphere of C_1. A necessary and sufficient condition that P be a $\mathscr{U}$-space is that $X \setminus Z(f)$ contain at most n elements, for each $f \in P^\circ$.*

Proof. The necessity is established by Proposition 29. For the sufficiency, assume all the hypotheses, yet P is not a $\mathscr{U}$-space. By Proposition 19, there exist $f \in P^\circ$ and $p \in P \setminus \{0\}$ such that $Z(f) \subset Z(p)$ and $\|f\| = 1$. Let I denote the ideal of all functions $g \in C_1$ for which $Z(p) \subset Z(g)$. The number $r = n + 1 - \operatorname{card}[X \setminus Z(p)]$ is positive because

$$\operatorname{card}[X \setminus Z(p)] \leqq \operatorname{card}[X \setminus Z(f)] \leqq n .$$

Assertion (1): $\dim(P^\perp \cap I^\perp) \geqq r$. Since $0 \neq p \in P \cap I$, we have $\dim(P \cap I) \geqq 1$. Also $\dim(I) \leqq \operatorname{card}[X \setminus Z(p)] = n + 1 - r$. Hence

$$\begin{aligned} \dim(P^\perp \cap I^\perp) &= \dim(P+I)^\perp = \operatorname{codim}(P+I) \\ &= \operatorname{codim}(P) - \dim(I) + \dim(P \cap I) \\ &\geqq n - (n+1-r) + 1 = r . \end{aligned}$$

This proves Assertion (1).

Select $\varphi_0 \in P^\perp$ such that $\|\varphi_0\| = 1 = \varphi_0(f)$. For each isolated point x in X, define $\bar{x}$ as was done in the remarks preceding Lemma 26. Observe that if $x \notin Z(P)$, then $x \notin Z(f)$ and $\varphi_0(\bar{x}) = \operatorname{sgn} f(x)$.

Assertion (2): To each $\varphi \in P^\perp$ there corresponds an $x \in A$ such that $\bar{x} \in C_1$ and $|\varphi(\bar{x})| = \|\varphi\|$. Indeed, if $\varphi \in P^\perp$, then by hypothesis there exists an element $g \in C_1$ such that $\|g\|=1$ and $\varphi(g)=\|g\|$. By Proposition 3, $g \in P^\circ$. By hypothesis, $\operatorname{card}[X \setminus Z(g)] \leqq n$. Thus g is of the form $g=\sum_i c_i \bar{x}_i$ with $x_i \in X \setminus Z(g) \subset A$. Then $1=\|g\|=\sum_i |c_i|$ and $\|\varphi\|=|\varphi(g)| \leqq \sum_i |c_i|\,|\varphi(\bar{x}_i)| \leqq \|\varphi\|$. It follows that $|\varphi(\bar{x}_i)|=\|\varphi\|$. This proves Assertion (2).

Now we define inductively (for $i=1,2,\ldots,r$) functionals $\varphi_i \in P^\perp \cap I^\perp$, reals c_i, and points $x_i \in A$ in such a way that

(i) $\varphi_i(\bar{x}_j)=0$ if $1 \leqq j < i$ (this being vacuous when $i=1$);
(ii) $\{\varphi_1,\ldots,\varphi_i\}$ is linearly independent;
(iii) $\|\varphi_0+c_1\varphi_1+\ldots+c_i\varphi_i\|=1=|(\varphi_0+c_1\varphi_1+\ldots+c_i\varphi_i)(\bar{x}_i)|$;
(iv) $\varphi_i(\bar{x}_i) \neq 0$.

To this end, suppose that φ_j, c_j, and x_j have been defined, subject to (i) ... (iv) for all j less than i. If $i \leqq r$, then by Assertion (1), there exists $\psi \in P^\perp \cap I^\perp$ such that $\{\varphi_1,\ldots,\varphi_{i-1},\psi\}$, is independent. Put $\varphi_i=\psi-\sum_{j=1}^{i-1} a_j\varphi_j$, where a_j are determined in such a way that $\varphi_i(\bar{x}_k)=0$ for $k<i$. This is possible because the system of equations $\sum_j a_j\varphi_j(\bar{x}_k)=\psi(\bar{x}_k)$, which determines a_j, is triangular with nonzero coefficients $\varphi_j(\bar{x}_j)$ on the diagonal. This construction produces $\varphi_i \in P^\perp \cap I^\perp$ such that $\varphi_i(\bar{x}_j)=0$ for $j<i$. Also $\{\varphi_1,\ldots,\varphi_i\}$ is independent. For convenience put $B=\varphi_0+c_1\varphi_1+\ldots+c_{i-1}\varphi_{i-1}$. Define $c_i=\sup\{c : \|B+c\varphi_i\| \leqq 1\}$. Note that $c_i \geqq 0$ because of property (iii).

Assertion (3): There exists an $x_i \in A$ such that $|(B+c_i\varphi_i)(\bar{x}_i)|=1$ and $\varphi_i(\bar{x}_i) \neq 0$. If this is false, then

$$\max\{|(B+c_i\varphi_i)(\bar{x})| : x \in A \text{ and } \varphi_i(x) \neq 0\} < 1$$

because A is a finite set. Hence there exists $t>c_i$ such that

$$(*) \qquad \max\{|(B+t\varphi_i)(\bar{x})| : x \in A \text{ and } \varphi_i(\bar{x}) \neq 0\} \leqq 1\,.$$

By Assertion (2), there exists an $x \in A$ for which $|(B+t\varphi_i)(\bar{x})|=\|B+t\varphi_i\|$. If $\varphi_i(\bar{x})=0$, then $\|B+t\varphi_i\|=|B(\bar{x})| \leqq \|B\|=1$. If $\varphi_i(\bar{x}) \neq 0$, then by $(*)$, $\|B+t\varphi_i\| \leqq 1$. In either case, the existence of such a t contradicts the definition of c_i. This proves Assertion (3). The induction is now fully established.

At the end of the process we have a functional $\varphi=\varphi_0+c_1\varphi_1+\ldots+c_r\varphi_r$ such that $\|\varphi\|=1$, $\varphi \in P^\perp$, and $|\varphi(\bar{x})|=1$ for all x in the set

$$D = [X \setminus Z(p)] \cup \{x_1,\ldots,x_r\}\,.$$

Define $g=\sum\{\bar{x}\operatorname{sgn}\varphi(\bar{x}) : x \in D\}$. Then $\varphi(g)=\sum\{|\varphi(\bar{x})| : x \in D\}=\|g\|$. Hence $g \in P^\circ$ by Proposition 3. But

$$\operatorname{card}[X \setminus Z(g)] = \operatorname{card}(D) = n+1-r+r > n\,,$$

a contradiction.

32. PROPOSITION. *The hypothesis that each* $\varphi \in P^{\perp}$ *attain its supremum on the unit sphere of* C_1 *cannot be omitted from Theorem* 31.

PROOF. Let $X = [0,1] \cup \{2\} \cup \{3\}$, with Lebesgue measure on $[0,1]$ and unit mass on each isolated point. Let h_1, h_2, and h_3 be respectively the characteristic functions of $[0,1]$, $\{2\}$, and $\{3\}$. Put $g_1(x) = xh_1(x)$ and $g_2(x) = h_2 + h_3$. Define $\varphi_i(f) = \int fg_i$. Then φ_i annihilates the subspace $P = \{f\colon \varphi_1(f) = \varphi_2(f) = 0\}$ but φ_1 does not achieve its supremum on the unit sphere. If $f \in P^{\circ}$, then for some α_i we must have $\int |f| = \int f(\alpha_1 g_1 + \alpha_2 g_2)$ and $\|\alpha_1 g_1 + \alpha_2 g_2\| = 1$. This implies that $f(x) = 0$ for $x \in [0,1]$. Hence $X \setminus Z(f) \subset \{2,3\}$. The function g_2 has many best approximations in P, for example, 0 and $h_2 - h_3$. Indeed, one can verify that $\int |g_2| = 2 = \int |g_2 - h_2 + h_3| = \int g_2^2$.

Phelps [11, p. 249] has given an example which establishes the following proposition.

33. PROPOSITION. *The hypothesis that* $\bigcup \{X \setminus Z(f) : f \in P^{\circ}\}$ *be finite cannot be omitted from Theorem* 31.

34. THEOREM. *The following properties of* X *are equivalent*:

(1) $C_1[X]$ *contains an* $\mathscr{E}\mathscr{U}$*-subspace of codimension* n.
(2) X *contains at least* n *isolated points*.

PROOF. Assume (2), and let Y be a set of n isolated points in X. The subspace $P = \{f \in C_1\colon Y \subset Z(f)\}$ has codimension n because $P \oplus g_1 \oplus \cdots \oplus g_n = C_1$ for an appropriate set $\{g_1, \ldots, g_n\}$ of which no nontrivial linear combination belongs to P. Indeed, we can take g_i to be the characteristic function of the ith point in Y. It is easy to see that P is an $\mathscr{E}\mathscr{U}$-subspace. In fact, the best approximation in P of an arbitrary $f \in C_1$ is fh, where h is the characteristic function of $X \setminus Y$.

Conversely, suppose that P is an $\mathscr{E}\mathscr{U}$-subspace of codimension n. By induction we select $f_1, \ldots, f_n$ in $C_1 \setminus P$ such that

$$\|f_i\| = 1 = \operatorname{dist}(f_i, P \oplus f_1 \oplus \cdots \oplus f_{i-1}) .$$

This is possible because $P \oplus f_1 \oplus \cdots \oplus f_{i-1}$ is an $\mathscr{E}$-space, by Proposition 5. Clearly $\|f_i\| = \operatorname{dist}(f_i, P)$, and by Proposition 29 it follows that $X \setminus Z(f_i)$ contains at most n elements. Hence these elements are isolated points. If $\bigcup_{i=1}^{n} [X \setminus Z(f_i)]$ contains only k elements, with $k < n$, then the set of functions $\{f_1, \ldots, f_n\}$ is linearly independent, since each f_i can be identified with the k-tuple of its nonzero values. But, by the construction, $\{f_1, \ldots, f_n\}$ is independent.

35. Theorem. *For an $\mathscr{E}$-space P of codimension n in C_1 the following properties are equivalent*:

(1) *P is a $\mathscr{U}$-space with a continuous metric projection.*

(2) $\operatorname{card}[X \setminus Z(f)] \leqq n$ *for each* $f \in P^\circ$, *and* $Z(P^\circ)$ *has a finite complement.*

Proof. Since P is an $\mathscr{E}$-space of finite codimension, each φ in $P^\perp$ achieves its supremum on the unit sphere of C_1, by a result of Phelps [12, p. 649].

Now assume (2). Then P is a $\mathscr{U}$-space by Theorem 31. The continuity of the metric projection will follow from Proposition 7 if P° is boundedly compact. By (2) every $f \in P^\circ$ may be identified with an element of the finite-dimensional space $C_1[X \setminus Z(P^\circ)]$. Hence P° is boundedly compact.

Now assume (1). Then $\operatorname{card}[X \setminus Z(f)] \leqq n$ for $f \in P^\circ$ by Proposition 29. If $X \setminus Z(P^\circ)$ is infinite, then it contains a sequence of distinct isolated points, $x_1, x_2, \ldots$. By Lemma 27, $\bar{x}_i \in P^\circ$. Also $\|\bar{x}_i - \bar{x}_j\| = 2$ when $i \neq j$. Hence P° is not boundedly compact, and by Theorem 8 the metric projection is discontinuous.

36. Theorem. *In the Banach space l_1 there exists an $\mathscr{E}\mathscr{U}$-space of codimension 2 whose metric projection is discontinuous under the (strong, weak*)-pair of topologies, and hence a fortiori discontinuous under the (strong, strong)- and the (strong, weak)-pairs of topologies.*

Proof. Define the bounded sequences

$$g = \left(1, \frac{3}{4}, \frac{3}{4}, \frac{7}{8}, \frac{7}{8}, \ldots, \frac{2^n - 1}{2^n}, \frac{2^n - 1}{2^n}, \ldots\right),$$

$$h = \left(0, \frac{2}{4}, \frac{-2}{4}, \frac{3}{8}, \frac{-3}{8}, \ldots, \frac{n}{2^n}, \frac{-n}{2^n}, \ldots\right),$$

and let $P = \{f \in l_1 : \sum_i f_i g_i = \sum_i f_i h_i = 0\}$. Clearly P is a closed subspace of codimension 2 in l_1. In the rest of the proof, the following facts are needed:

Assertion (1). An element f belongs to P° if and only if there exists an element $q = \alpha g + \beta h$ such that $\|q\|_\infty = 1$ and $\sum_i q_i f_i = \|f\|_1$. This is immediate from Proposition 4.

Assertion (2). For scalars α and β the set $\operatorname{crit}(\alpha g + \beta h)$ is given by the following table, the proof of which is straightforward. (Here n is a positive integer.)

α *and* β	$\operatorname{crit}(\alpha g+\beta h)$
$\beta = 0$	$\{1\}$
$\alpha = 0$	$\{2,3\}$
$n-1 < \alpha/\beta < n$	$\{2n\}$
$n-1 < -\alpha/\beta < n$	$\{2n+1\}$
$\alpha/\beta = n$	$\{2n,\ 2n+2\}$
$\alpha/\beta = -n$	$\{2n+1,\ 2n+3\}$

It is easy to see that in order to prove the $\mathscr{E}$-property of P it is enough to prove that the vectors $(0,1,0,0,\ldots)$ and $b'=(1,b,0,0,\ldots)$ possess best approximations in P, for all b. Using the two preceding assertions, one can prove that if $b\leqq -\frac{2}{3}$, then a best approximation to b' is $(1,-\frac{2}{3},-\frac{2}{3},0,0,\ldots)$. If $b\geqq\frac{12}{5}$, then a best approximation to b' is $(1,\frac{12}{5},0,-\frac{16}{5},0,0,\ldots)$. If

$$\frac{4n}{2(1-2^n)-3n} \geqq b \geqq \frac{4(n-1)}{2(1-2^{n-1})-3(n-1)}, \qquad n\geqq 3\,,$$

then a best approximation of b' is $p=(1,b,0,\ldots,0,p_{2n-3},0,p_{2n-1},0,0,\ldots)$, with

$$p_{2n-3} = \frac{2^{n-2}[b(2-2^{n+1}-3n)-4n]}{2^n(2-n)-2}, \qquad p_{2n-1} = \frac{1}{n}\,[2^{n-1}b-(2n-2)p_{2n-3}]\,.$$

Finally, if

$$\frac{4(n-1)}{2^n-2-3(n-1)} \geqq b \geqq \frac{4n}{2^{n+1}-2-3n}, \qquad n \geqq 4\,,$$

then a best approximation of b' is $p=(1,b,0,\ldots,0,p_{2n-4},0,p_{2n-2},0,0,\ldots)$ with

$$p_{2n-4} = \frac{2^{n-2}[b(2^{n+1}-2-3n)-4n]}{2^n(2-n)-2}, \qquad p_{2n-2} = -\frac{1}{n}[2^{n-1}b+(2n-2)p_{2n-4}].$$

In order to prove that P is a $\mathscr{U}$-space, assume the contrary. By Theorem 23, some nonzero element p in P vanishes on the set S_q for some $q\perp P$. Thus $X\setminus Z(p)\subset\operatorname{crit}(q)$. By one of the previous assertions, it follows that $X\setminus Z(p)$ contains at most 2 points. Since $\sum_i g_i p_i=0$, $X\setminus Z(p)$ contains at least 2 points. Hence if $X\setminus Z(p)=\{i,j\}$ then $g_ip_i+g_jp_j=0$, and similarly for h. This implies that $g_ih_j-g_jh_i=0$ which is impossible. Thus P is a $\mathscr{U}$-space.

Now consider the points, for $n>4$,

$$f^n = \left(1,\frac{4n}{2^{n+1}-2-3n},0,0,0,\ldots\right).$$

Then f^n converge to $(1,0,0,..)$, which belongs to P°. However, Tf^n is of the form $(1,a,0,...,0,b,0,...)$ with $a=f_2{}^n$ and $b=2^{n+1}/(2^{n+1}-2-3n)$, occupying the $2n-2$ coordinate position. Thus Tf^n does not converge pointwise to zero.

37. PROPOSITION. *For an $\mathscr{E}\mathscr{U}$-space P of codimension n in C_1 the following properties are equivalent*:

(1) *P° is a linear space.*

(2) $\mathrm{card}(X \setminus A)=n$, *where* $A=\bigcap\{Z(f): f \in P^\circ\}$.

PROOF. Suppose that (2) is false. By Proposition 30, $\mathrm{card}(X\setminus A)>n$. Select $x_0,\ldots,x_n$ in $X\setminus A$. By Proposition 29, each x_i is an isolated point of X. By Lemma 27, $\bar{x}_i \in P^\circ$ for each i. Hence P° contains a linearly independent set of $n+1$ elements. If P° is a subspace, this contradicts the equation $P \oplus P^\circ = C_1$ (Proposition 6).

Now suppose that (2) is true. Let I denote the ideal of functions which vanish on A. We shall prove (1) by establishing that $P^\circ = I$. The inclusion $P^\circ \subset I$ is obvious. By induction we select $f_1,\ldots,f_n$ such that

$$1 = \|f_i\| = \mathrm{dist}(f_i, P\oplus f_1 \oplus \ldots \oplus f_{i-1})\,.$$

Then $f_i \in P^\circ$. It is easy to see that no non-trivial linear combination of $f_1,\ldots,f_n$ can belong to P. Thus, in particular, $\{f_1,\ldots,f_n\}$ is independent. Since $f_i \in P^\circ \subset I$, and since $\dim(I)=n$ by Lemma 15, it follows that $\{f_1,\ldots,f_n\}$ is a basis for I. We have thus proved that $P \cap I = \{0\}$. Now let f be an arbitrary element of I and let p be its best approximation in P. Then $f-p \in P^\circ \subset I$. Hence $p \in I \cap P$. It follows that $p=0$ and that $f \in P^\circ$. This proves that $I \subset P^\circ$.

The following seem to be among the principal questions that remain to be answered in this subject:

(1) What can be said about the existence or the structure of $\mathscr{E}\mathscr{U}$-subspaces in C_1 or C which have infinite dimension and infinite codimension?

(2) What are the exact conditions on an $\mathscr{E}\mathscr{U}$-subspace in C or C_1 in order that its metric projection be continuous?

(3) What are the exact conditions on X in order that C_1 should contain an $\mathscr{E}\mathscr{U}$-subspace of dimension n, for each n?

(4) If the space C is given an arbitrary norm, can the $\mathscr{U}$-spaces be characterized by the zero sets of their elements, as has been done here in Theorems 10 and 22?

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Since the submission of this manusfript, several related works have appeared, as follows:

20. J. Blatter, P. D. Morris, and D. E. Wulbert, *Continuity of the set-valued metric projection*, Math. Ann. 178 (1968), 12–24.
21. W. W. Breckner and I. Kolumban, *Über die Characterisierung von Minimallösungen in linearen normierten Räumen*, Mathematica (Cluj) 10 (1968), 33–46.
22. R. Holmes and B. Kripke, *Smoothness of approximation*, Mich. Math. J. 15 (1968), 225–248.
23. Y. Ikebe, *A characterization of best Tchebycheff approximations in function spaces*, Proc. Japan Acad. 44 (1968), 484–488.

24. A. J. Lazar, P. D. Morris, and D. E. Wulbert, *Continuous selections for metric projections*, to appear.
25. P. D. Morris, *Metric projections onto subspaces of finite codimension*, Duke Math, J., to appear.
26. I. Singer, *Cea mai buna aproximare in spatii vectoriale normate prin elemente din subspatii vectoriale*, Bucharest, 1967.
27. D. E. Wulbert, *Convergence of operators and Korovkin's theorem*, J. Approximation Theory, to appear.

The following contributions to problems (1) and (2) should be noted. Paper [27] gives an example in $C(X)$ of an $\mathscr{E}\mathscr{U}$-subpsace of infinite dimension and infinite codimension. In paper [25] it is proved that the metric projection of $C(X)$ onto an $\mathscr{E}\mathscr{U}$-subspace of finite codimension greater than 1 is necessarily discontinuous (X being an infinite compact Hausdorff space).

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