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ON L^p ESTIMATES FOR QUASI-ELLIPTIC BOUNDARY PROBLEMS

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The aim of this note is to indicate how the methods of [1] can be applied to derive a priori L^p-inequalities for quasi-elliptic boundary problems in the half-space $R_+{}^n$ when $1<p<\infty$. However, we have comparatively stronger assumptions on the coefficients, as the norms are no longer rotation invariant.

1. Preliminaries.

Whenever convenient, we use the notations of [1]. If α is a multi-index, i.e. a sequence $\alpha_1,\ldots,\alpha_n$ of non-negative integers, we write

$$|\alpha| = \sum \alpha_j, \quad D^\alpha = D_1{}^{\alpha_1}\ldots D_n{}^{\alpha_n} \text{ with } D_j = -i\,\partial/\partial x_j\,.$$

If m is a sequence of positive integers $m_1,\ldots,m_n$, set

$$|\alpha:m| = \sum_{j=1}^{m} \alpha_j/m_j\,.$$

The constant coefficient differential operator

$$A = A(D) = \sum_{|\alpha:m|\leq 1} a_\alpha D^\alpha = A_0(D) + \sum_{|\alpha:m|<1} a_\alpha D^\alpha$$

is "properly" quasi-elliptic, if the corresponding polynomial

$$A(\xi) = A(\xi_1,\xi'), \quad \xi' = (\xi_2,\ldots,\xi_n)\,,$$

has the following properties:

(1) $$A_0(\xi) \neq 0 \quad \text{for every } \xi\in R^n,\ \xi\neq 0;$$

(2) for all real $\xi'\neq 0$, $A_0(\xi_1,\xi')$ has exactly m_+ zeros $\xi_1=\varrho(\xi')$ with positive and $m_-=m_1-m_+$ zeros with negative imaginary part.

Because of the structure of the quasi-elliptic operator A, it is natural to introduce function spaces defined by use of the weight functions

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$$\Lambda(\xi) = \delta\xi_1 + i\left(1+\delta^{2m_1}\sum_{j=2}^{n}\xi_j^{2m_j}\right)^{1/2m_1}, \quad \delta>0 .$$

Thus for $1<p<\infty$ we take

$$H_{s;p} = \{f\,;\ f\in S',\ \bar{F}\Lambda^s Ff\in L^p\},$$

where F is the Fourier transform, normed by

$$\|f\|_{s;p} = \|\bar{F}\Lambda^s Ff\|_p .$$

Let $(H_{s;p})_-$ consist of those elements of $H_{s;p}$, whose supports are contained in the half-space $x_1 \leqq 0$, and set

$$H^+_{s;p} = H_{s;p}/(H_{s;p})_- ,$$

with the quotient norm $\|\cdot\|^+_{s;p}$.

The traces of the $H_{s;p}$-functions on $x_1=0$ belong to the spaces $W_{s;p}$ (for exact statements see [2, p. 95]). For

$$sm_i/m_1 = s_i > [s_i], \quad i=2,\ldots,n ,$$

let $W_{s;p}$ be the space of functions

$$u = u(x'), \quad x'\in R^{n-1} ,$$

with for all j

$$D_j^k u\in L^p, \quad k\leqq[s_j] \quad \text{and} \quad |D_j^{[s_j]}u|_{s-[s_j],\,j} < \infty ,$$

where

$$|v|_{\theta,j} = \left(\int_0^\infty t^{-p\theta}\|\Delta_{j,t}v\|_p{}^p\frac{dt}{t}\right)^{1/p}$$

and

$$\Delta_{j,t}v = v(x_2,\ldots,x_j+t,\ldots,x_n) - v(x_2,\ldots,x_n) .$$

$W_{s;p}$ is a Banach space, normed by

$$\langle u\rangle_{s;p} = \sum_j\left\{\sum_{k=0}^{[s_j]}\delta^{km_1/m_j}\|D_j^k u\|_p + \delta^s|D_j^{[s_j]}u|_{s_j-[s],\,j}\right\}.$$

For $s<0$, we set $W_{s;p}=(W_{-s;p'})'$.

The next class of functions takes $W_{s;p}$ into $W_{s;p}$ by multiplication. For $s\geqq 0$ denote by $K_{s;p}$ the set of functions φ, for which

$$D_j^k\varphi\in L_\infty, \quad k\leqq[s_j] ,$$

s_j as above, and such that

$$\sum_j\sum_{k=0}^{[s_j]}\left(\int_0^\infty t^{-p(s_j-[s_j])}\|\Delta_{j,t}D_j^k\varphi\|_\infty{}^p\frac{dt}{t}\right)^{1/p} < \infty .$$

For $s<0$, set $K_{s;p}=K_{-s;p'}$. As in [1, Theorem 1.5], we have

LEMMA 1. *If $f \in W_{s;p}$ and $\varphi \in K_{s;p}$, then $\varphi f \in W_{s;p}$ and*

$$\langle \varphi f \rangle_{s;p} \leqq (K\|\varphi\|_\infty + o(1))\langle f \rangle_{s;p} \quad as\ \delta \to +0 .$$

Here K is independent of φ, but $o(1)$ depends on φ.

2. A priori estimates.

Let

$$B_j = B_j(D) = \sum_{|\alpha:m| \leqq n_j/m_1} b_{j\alpha} D^\alpha = B_{0j}(D) + \sum_{|\alpha:m| < n_j/m_1} b_{j\alpha} D^\alpha$$

be differential operators with constant coefficients, and let the order of D_1 in B_j equal the order ν_j of D_1 in B_{0j}. Let $F_{ij}(\xi')$ be the characteristic matrix corresponding to the boundary problem

$$A_0 u = f \quad \text{in} \quad R_+{}^n ,$$
$$B_{0j} u = g_j \quad \text{on} \quad x_1 = 0,\ j = 1,\ldots,m_+ ,$$

so that

$$\sum_{j=0}^{m_+-1} F_{ij}(\xi')\, D_1{}^j u(0,\xi') = B_i(D_1,\xi')\, u(0,\xi'), \quad i = 1,\ldots,m_+ ,$$

if $Au = 0$ in $R_+{}^n$. We require that

(3) $$\det F_{ij}(\xi') \neq 0 \quad \text{for} \quad \xi' \neq 0 .$$

Then using Krée's version of Mihlin's theorem (cf. [3, Th. 8, p. 74]) and reasoning as in [1] we obtain

THEOREM 1. *Let $A(D)$ and $B_1(D),\ldots,B_{m_+}(D)$ satisfy* (1), (2) *and* (3). *For*

(4) $$sm_j/m_1 \neq 1/p \bmod 1, \quad j = 1,\ldots,n ,$$
$$s > s_0 = \max(m_+ - 1, \nu_1,\ldots,\nu_{m_+}) + 1/p ,$$

the following a priori estimate holds:

(5)
$$\|u\|_{s;p}^+ \leqq C\left(\delta^{m_1}\|Au\|_{s-m_1;p}^+ + \sum_{j=1}^{m_+} \delta^{n_j+1/p} \langle B_j u(0,\cdot')\rangle_{s-n_j-1/p;p} + \|u\|_{s-1;p}^+\right) .$$

Here C is independent of $u \in H_{s;p}$ and $\delta \in (0,\delta_0)$ for some $\delta_0 > 0$.

Let finally

$$A(x,D) = \sum_{|\alpha:m| \leqq 1} a_\alpha(x) D^\alpha$$

be a differential operator with C^∞-coefficients in $R_+{}^n$, and let

$$B_j(x,D) = \sum_{|\alpha:m| \leqq n_j/m_1} b_{j\alpha}(x) D^\alpha, \quad j=1,\ldots,m_+ ,$$

be differential operators with $K_{s-n_j-1/p;p}$-coefficients on $x_1=0$, such that, for every $x^0 \in R_+{}^n$, $A(x^0,D)$ is properly quasi-elliptic and of order m_j in D_j and such that if $x_1{}^0=0$, the conditions of Theorem 1 are satisfied for the corresponding "frozen" operators $A(x^0,D)$ and $B_j(x^0,D)$. Then using Theorem 1 and Lemma 1, we immediately obtain

THEOREM 2. *Under the assumption of* (4) *the following a priori estimate holds*:

$$\|u\|_{s;p}^+ \leqq C\left(\delta^{m_1}\|Au\|_{s-m_1;p}^+ + \sum_{j=1}^{m_+} \delta^{n_j+1/p}\,\langle B_j u(0,\cdot')\rangle_{s-n_j-1/p;p} + \|u\|_{s-1;p}^+\right)$$

for $u \in H_{s;p}^+$ *with compact support, the constant* C *depending only on the support of* u.

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