MATH. SCAND. 24 (1969), 145–166

MARKOV SETS

J. HOFFMANN-JØRGENSEN

1. Introduction.

Kingman has introduced the notion of a regenerative event (see [2]). A regenerative event is a family $\{E(t) \mid t>0\}$ of measurable sets in the probability space, $(\Omega,\mathscr{M},P)$, such that:

$$(1) \qquad P\left(\bigcap_{j=1}^{n} E(t_j)\right) = P(E(t_1))\, P\left(\bigcap_{j=2}^{n} E(t_j-t_1)\right)$$

for all $t_1, \ldots, t_n$ with $0<t_1<\ldots<t_n$.

In his papers [2] and [3] Kingman gives a detailed study of the analytic properties of the function $p(t)=P(E(t))$, under the assumption:

$$(2) \qquad p(t) \to 1 \quad \text{as } t \to 0+ \ .$$

These results have important applications to Markov processes with discrete state space, and one of the most important examples of a regenerative event is

$$(3) \qquad E(t) = \{X(t)=0\} ,$$

where X is a Markov process with discrete state space and where 0 is a fixed state, such that $X(0)=0$ a.s. In this case $p(t)=p_{00}(t)$.

If X is a Markov process with a general state space, and 0 is a fixed state such that $X(0)=0$ a.s., then (3) of course still defines a regenerative event, but in many cases we have $p(t) \equiv 0$ (for example if $X(t)$ has a continuous distribution, $\forall t>0$). In this case (1) gives no information, and (2) is not satisfied.

We shall here give a definition of a strong Markov set, which is more restrictive than (1). We shall then construct a canonical strong Markov process X, with $X(0)=0$ a.s., associated to the strong Markov set, such that the strong Markov set itself is given by (3). In [5] Krylov and Yuškevič have worked along the same line, but in their definition of a "Markov random set" they start with the canonical process, and the set itself is not mentioned, which seems rather unnatural.

The problem was proposed to me by Professor P.-A. Meyer, and I am indebted to him for his extended help and encouragement in connection with this work.

Received May 1, 1968.

2. Definitions and basic lemmas.

Let $(\Omega,\mathscr{M}^\circ,P)$ be a probability space, $\mathscr{M}$ the completion under P of $\mathscr{M}^\circ$, and $(\mathscr{M}_t)_{t\geqq 0}$ an increasing family of σ-algebras all contained in $\mathscr{M}$, such that P restricted to $\mathscr{M}_t$ is complete, $\forall t\geqq 0$, and $(\mathscr{M}_t)$ is right-continuous, that is,

$$\mathscr{M}_t = \bigcap_{s>t}\mathscr{M}_s = \mathscr{M}_{t+} \quad \forall t\geqq 0 .$$

Let $\{\theta_t \mid t\geqq 0\}$ be a family of measurable translation operators in Ω, that is, θ_t is a map of Ω into itself, such that

(i) $\theta_t\circ\theta_s = \theta_{t+s}\,\forall t,s \geqq 0;\ \theta_0(\omega)=\omega\ \forall\,\omega\in\Omega$,
(ii) $(t,\omega)\rightsquigarrow\theta_t(\omega)$ is $(\mathscr{B}_+\times\mathscr{M},\mathscr{M}^\circ)$-measurable.

These data are fixed in the sequel.

A map T of Ω into $[0,\infty)$ is a stopping time if and only if $\{T\leqq t\}\in\mathscr{M}_t$, $\forall t\geqq 0$.

If T is a stopping time, then we define:

$$\mathscr{M}_T = \{A\in\mathscr{M} \mid A\cap\{T\leqq t\}\in\mathscr{M}_t\,\forall t\geqq 0\} .$$
$$[T] = \big\{(T(\omega),\omega) \mid \omega\in\{T<\infty\}\big\} .$$

(See, for example, [6; IV, § 3]).

If K is a subset of $\mathsf{R}_+\times\Omega$, where $\mathsf{R}_+=[0,\infty)$, and R is a map of Ω into $[0,\infty]$, then we define:

(a) $K^\omega = \{t \mid (t,\omega)\in K\}$.
(b) $K_t = \{\omega \mid (t,\omega)\in K\}$.
(c) $\bar K = \{(t,\omega) \mid t\in\overline{K^\omega}\}$, where $\overline{K^\omega}$ is the closure of K^ω in R_+ .
(d) $K(R) = \{(t,\omega) \mid R(\omega)<\infty,\ (t+R(\omega),\omega)\in K\}$.
(e) $K_R = \{\omega \mid (R(\omega),\omega)\in K,\ R(\omega)<\infty\}$.
(f) $K_A = \bigcup_{s\in A}K_s$ for $A\subseteqq\mathsf{R}_+$.
(g) $\theta_R(\omega) = \theta_{R(\omega)}(\omega)$ if $R(\omega)<\infty$ and undefined if $R(\omega) = \infty$.

DEFINITION. A subset M, of $\mathsf{R}_+\times\Omega$ is said to be a *strong Markov set* if and only if

(i) M is progressively measurable with respect to $(\mathscr{M}_t)$ (see, for example, [6; IV, D. 45]).
(ii) $P(M_0)=1$.
(iii) $\theta_s^{-1}(M_t)=M_{t+s}\ \forall t,s\geqq 0$.
(iv) M^ω is right-closed $\forall\,\omega\in\Omega$.*)

* A subset B of R_+ is right-closed, if it is closed in the topology generated by the right closed intervals: $[a,b)$, $0\leqq a<b<\infty$.

(v) $P(\overline{M}_t \setminus M_t) = 0 \ \forall t \geqq 0$.

(vi) If T is a stopping time with $[T] \subseteq M$, then

$$P(A \cap \theta_T^{-1}(B)) = P(A)\,P(B) \quad \forall A \in \mathscr{M}_T\,, \quad A \subseteq \{T < \infty\} \quad \forall B \in \mathscr{M}\,.$$

Let X be a strong homogeneous Markov process w.r.t. $(\mathscr{M}_t)$ whose paths are right-continuous, have left limits, and admit no fixed discontinuities, and such that $\mathscr{M}_t$ is equal to the completion of the σ-algebra generated by $\{X(s) \mid 0 \leqq s \leqq t\}$, under P. If E is the state space and $x_0 \in E$, such that $X(0) = x_0$ a.s., then

$$M = \{(t,\omega) \mid X(t,\omega) = x_0\}$$

is a strong Markov set, since:

(i) follows from the fact that X is progressively measurable. (See, for example, [6; IV, T 47]).

(ii) follows from $X(0) = x_0$ a.s.

(iii) follows from $X_{t+s} = X_t \circ \theta_s$.

(iv) follows from the right-continuity of X.

(v) follows from the absense of fixed discontinuities of X.

(vi) follows from the strong Markov property:

$$P(A \cap \theta_T^{-1}(B)) = E\{1_A\, P_{X_T}(B)\} = E\{1_A\, P_{x_0}(B)\} = P(A)\, P(B)$$

since $[T] \subseteq M$ implies that $X_T \equiv x_0$ on $\{T < \infty\}$ and $P_{x_0} = P$.

The main theorem of this paper states, that all strong Markov sets are generated in this way.

In all that follows, M will denote a strong Markov set.

LEMMA 1. *Let T be a stopping time such that $[T] \subseteq M$ and let F be a $\mathscr{M}_T \times \mathscr{M}$-measurable, bounded map of $\Omega \times \Omega$ into* R. *Then*

$$\int_{\{T<\infty\}} F(\omega, \theta_T(\omega))\, P(d\omega) = \int_{\{T<\infty\}} P(d\omega') \int_{\Omega} F(\omega', \omega'')\, P(d\omega'')\,.$$

PROOF. Let H denote the set of all bounded $\mathscr{M}_T \times \mathscr{M}$ measurable functions F satisfying the hypothesis of the lemma. Then H is clearly a vector space, and H is closed under monotone, uniformly bounded, pointwise limits.

By (vi) we have that $1_{A \times B} \in H$, $\forall A \in \mathscr{M}_T$, $\forall B \in \mathscr{M}$. Hence the lemma follows from [6; I, T 20].

LEMMA 2. *If $B \in \mathscr{M}$ and $A \in \mathscr{M}_s$ such that $A \subseteq M_s$, then*

$$P(A \cap \theta_s^{-1}(B)) = P(A)\, P(B)\,. \tag{2.1}$$

In particular we see that $\{M_t \mid t>0\}$ *is a regenerative event in the sense of* [2], *and we have*

$$P(A) = 0 \text{ or } 1 \quad \forall A \in \mathcal{M}_{0+}\,. \tag{2.2}$$

PROOF. Let Σ_s be defined by:

$$\Sigma_s(\omega) = \begin{cases} s & \text{if} \quad \omega \in M_s\,, \\ \infty & \text{if} \quad \omega \notin M_s\,. \end{cases}$$

Then Σ_s is a stopping time, such that $[\Sigma_s] \subseteqq M$ and

$$\mathcal{M}_{\Sigma_s} = \{A \in \mathcal{M} \mid A \cap M_s \in \mathcal{M}_s\}\,.$$

Since

$$\theta_{\Sigma_s}(\omega) = \theta_s(\omega) \quad \forall\, \omega \in M_s\,,$$

the lemma follows immediately from (vi), and from the fact that $\mathcal{M}_{0+} = \mathcal{M}_0$ and $P(M_0)=1$.

Now let T^* be the first exit time from M,

$$T^*(\omega) = \inf\{t>0 \mid t \notin M^\omega\}$$

($\inf\{\emptyset\}=\infty$), and let S be defined by

$$S = \limsup_{t\to 0+} M_t = \bigcap\nolimits_{\varepsilon>0} M_{(0,\varepsilon)} = \{\omega \mid 0 \text{ is an accumulation point for } M^\omega\}\,.$$

By [6; IV, T 52] we see that T^* is a stopping time, and it is easily seen that

$$\{T^* > t+s\} = \{T^* > s\} \cap \{T^* \circ \theta_s > t\}\,.$$

Since $\{T^*>s\} \in \mathcal{M}_s$ and is contained in M_s we have from Lemma 2:

$$P(T^*>t+s) = P(T^*>s)\,P(T^*>t) \quad \forall t,s \geqq 0\,.$$

By [6; III, T9, T12 and No. 24] we have that $M_{(0,\varepsilon)} \in \mathcal{M}_\varepsilon$ $\forall \varepsilon>0$, hence $S \in \mathcal{M}_{0+} = \mathcal{M}_0$, and we get

LEMMA 3.

(3.1) $P(S) = 0$ *or* 1 .

(3.2) $\exists\, q \in [0,\infty]$ *such that* $P(T^*>t) = e^{-qt}$ $\forall t>0$.

If $q=0$, then $P(T^*=\infty)=1$, and hence $M^\omega = \mathsf{R}$ for a.a. $\omega \in \mathcal{M}$. This case is the most trivial example of a strong Markov set and will be excluded, that is:

In the following we assume that $0<q\leqq\infty$.

We shall now introduce some basic processes associated with the strong Markov set M. Let

$$S_t(\omega) = \inf\{s \mid s \geqq t, (s,\omega) \in M\}, \qquad \omega \in \Omega,$$

$$U_t(\omega) = \begin{cases} \sup\{s \mid 0 \leqq s \leqq t, (s,\omega) \in M\} & \text{if} \quad t \geqq S_0(\omega), \\ \text{undefined} & \text{if} \quad t < S_0(\omega), \end{cases}$$

$$V_t(\omega) = \begin{cases} S_t(\omega) - U_t(\omega) & \text{if} \quad t \geqq S_0(\omega), \\ \text{undefined} & \text{if} \quad t < S_0(\omega), \end{cases}$$

$$\begin{aligned} T(t,a,\omega) &= \inf\{s \mid s \geqq S_t(\omega), s - U_s(\omega) = a\}, && \omega \in \Omega, \\ S^*(t,a,\omega) &= S_{T(t,a,\omega)}(\omega), && \omega \in \Omega, \\ U^*(t,a,\omega) &= U_{T(t,a,\omega)}(\omega), && \omega \in \Omega, \\ V^*(t,a,\omega) &= V_{T(t,a,\omega)}(\omega), && \omega \in \Omega. \end{aligned}$$

Here $\inf\{\emptyset\} = \infty$ and $S_\infty \equiv V_\infty \equiv U_\infty \equiv \infty$. If $t = 0$ we shall drop t in T, S^*, U^* and V^*.

If $T(t,a,\omega) < \infty$, then $T(t,a,\omega) \geqq S_t(\omega) \geqq S_0(\omega)$, hence U^* and V^* are defined everywhere.

LEMMA 4.

(4.1) $(t,\omega) \rightsquigarrow S_t(\omega)$ *is* $\mathscr{B}_+ \times \mathscr{M}$*-universally measurable.*

(4.2) S_t *is a stopping time* $\forall t \geqq 0$ *and* $[S_t] \subseteqq M$.

(4.3) $S_t(\theta_r(\omega)) = S_{t+r}(\omega) - r \quad \forall \omega \in \Omega \quad \forall t, r \geqq 0$.

PROOF. (4.1) Let $a > 0$, then

$$\{(t,\omega) \mid S_t(\omega) > a\} = \mathrm{proj}_{\mathbf{R}_+ \times \Omega}\{(s,t,\omega) \mid (s,\omega) \in M, \ t \leqq s \leqq a\}$$

and (4.1) follows from [6; III, T 9, T 12 and No. 24].

(4.2) follows from [6; IV; T 52] and the right-closedness of M^ω.

(4.3) follows from: $\theta_r^{-1}(M_t) = M_{t+r} \ \forall t, r \geqq 0$.

LEMMA 5.

(5.1) $(t,\omega) \rightsquigarrow U_t(\omega)$ *is progressively measurable.*

(5.2) $U_t(\theta_r(\omega)) = U_{t+r}(\omega) - r$ *if* $S_r(\omega) \leqq t + r$.

(5.3) $t \rightsquigarrow U_t(\omega)$ *is right continuous on* $[S_0(\omega), \infty)$ *for all* $\omega \in \Omega$.

PROOF. (5.3) follows immediately from the right-closedness of M^ω.

(5.2) follows from $\theta_r^{-1}(M_s) = M_{s+r} \ \forall s \geqq 0$ and from the fact, that $S_0(\theta_r(\omega)) = S_r(\omega) - r$.

(5.1) follows from (5.3) and the easy fact that $\omega \rightsquigarrow U_t(\omega)$ is $\mathscr{M}_t$-measurable.

LEMMA 6.

(6.1) $(t,\omega) \rightsquigarrow V_t(\omega)$ *is* $\mathscr{B}_+ \times \mathscr{M}$*-universially measurable.*

(6.2) $V_t(\theta_r(\omega)) = V_{t+r}(\omega)$, *if* $S_r(\omega) \leqq t+r$.

PROOF. Immediate consequence of Lemmas 4 and 5.

LEMMA 7. *Let a, t and r be non-negative. Then*

(7.1) $T(t,a)$ *is a stopping time*;

(7.2) $T(t,a) = S_t + T(a) \circ \theta_{S_t}$;

(7.3) $T(t,a,\theta_r(\omega)) = T(t+r,a,\omega) - r \;\; \forall\, \omega \in \Omega$;

(7.4) *if* $a > 0$ *and* $a_n \downarrow a$, *then* $T(t,a_n) \downarrow T(t,a)$.

PROOF. Let

$$T(\omega) \;=\; \inf\,\{t \mid t \geqq 0,\, t - U_t(\omega) = a\}\;.$$

Then T is a stopping time (see, for example, [6; T 52]), and since

$$T(t,a) \;=\; S_t + T \circ \theta_{S_t}$$

we have that $T(t,a)$ is stopping time (see, for example, [7; XV, (3.1)] and [7; XIII, T 19]).

If $S_0(\omega) = 0$, then $T(\omega) = T(a,\omega)$. If $r = S_t(\omega) < \infty$, then $r \in M^\omega$ (M^ω is right closed) and $S_r(\omega) = r$. Hence

$$S_0(\theta_r(\omega)) \;=\; S_r(\omega) - r \;=\; 0\,,$$

and $T(\theta_{S_t}(\omega)) = T(a,\, \theta_{S_t}(\omega))$, that is,

$$T \circ \theta_{S_t} \;=\; T(a) \circ \theta_{S_t}\,,$$

and (7.1) and (7.2) are proved.

(7.3) follows easily from Lemma 4 and 5.

(7.4). It is obvious that $T(t,a_n,\omega) \geqq T(t,a,\omega) = r$. If $r = \infty$ there is nothing to prove. Let $r < \infty$, then by the right continuity of U

$$r - U_r(\omega) \;=\; a \;>\; 0$$

that is, $r \notin M^\omega$. Since M^ω is right closed, there exists an $e > 0$, so that

$$[r, r+e) \cap M^\omega \;=\; \emptyset\,.$$

Let $n_0 \geqq 1$, such that $a_n - a < e \;\forall\, n \geqq n_0$, then

$$(r+s)-U_{r+s}(\omega) = a+s$$

for $a \leqq s < e+a$ and

$$r = T(t,a,\omega) \leqq T(t,a_n,\omega) \leqq r+(a_n-a) \quad \forall n \geqq n_0;$$

hence $T(t,a_n,\omega) \downarrow r = T(t,a,\omega)$.

Lemma 8. *Let a, t and r be non-negative. Then*

(8.1) *$S^*(t,a)$ is a stopping time such that $[S^*(t,a)] \subseteq M$;*

(8.2) $S^*(t,a,\theta_r(\omega)) = S^*(t+r,a,\omega)-r \quad \forall \omega \in \Omega;$

(8.3) $U^*(t,a,\theta_r(\omega)) = U^*(t+r,a,\omega)-r \quad \forall \omega \in \Omega;$

(8.4) $V^*(t,a,\theta_r(\omega)) = V^*(t+r,a,\omega) \quad \forall \omega \in \Omega.$

Proof. (8.1). Since $S^*(t,a) = S_0 \circ \theta_{T(t,a)} + T(t,a)$ the result follows from [7; XIII, T 19].

(8.2–4) are trivial consequences of the preceeding lemmas.

Lemma 9. *Let t and a be non-negative. Then*

(9.1) $S^*(t,a) = S^*(a) \circ \theta_{S_t} + S_t;$

(9.2) $U^*(t,a) = U^*(a) \circ \theta_{S_t} + S_t;$

(9.3) $V^*(t,a) = V^*(a) \circ \theta_{S_t}.$

Proof. Let $r = S_t(\omega) < \infty$. Then $r \in M^\omega$ and $S_r(\omega) = r$, and

$$T(r,a,\omega) = \inf\{s \mid s \geqq r, s-U_s(\omega) = a\} = T(t,a,\omega).$$

From Lemma 8 we then have:

$$\begin{aligned} S^*(t,a,\omega) &= S^*(r,a,\omega) = S^*(a,\theta_r(\omega))+r, \\ U^*(t,a,\omega) &= U^*(r,a,\omega) = U^*(a,\theta_r(\omega))+r, \\ V^*(t,a,\omega) &= V^*(r,a,\omega) = V^*(a,\theta_r(\omega)), \end{aligned}$$

which proves the lemma.

Lemma 10. *Let $a_0 = \sup\{a \mid a \geqq 0, P(T(a) < \infty) = 1\}$. Then $a_0 > 0$ and*

$$P(T(a) < \infty) = \begin{cases} 1, & 0 \leqq a < a_0, \\ 0, & a_0 \leqq a < \infty. \end{cases}$$

Proof. Since $T(a) \leqq T(b)$, if $0 \leqq a \leqq b < \infty$, we see that $P(T(a) < \infty) = 1$ $\forall a \in [0,a_0)$. By an easy argument, it follows that

$$\{T(a) = \infty\} \cap M_0 = \{T(a) \geqq S_t\} \cap \{T(a) \circ \theta_{S_t} = \infty\} \cap M_0.$$

Since $T(a)$ and S_t are both stopping times, we have

$$\{T(a) \geqq S_t\} \in \mathscr{M}_{S_t} .$$

Since $T(a,\omega)=\infty$, implies $S_t(\omega)<\infty$, we have

$$\{T(a) \geqq S_t\} \subseteqq \{S_t < \infty\} .$$

Then (vi) gives us

$$P(T(a)=\infty) = P(T(a) \geqq S_t)\, P(T(a)=\infty) .$$

Since $S_t \geqq t$ we get (by letting $t \to \infty$)

$$P(T(a)=\infty) = P(T(a)=\infty)^2 ,$$

that is, $P(T(a)<\infty)=0$ or 1 $\forall a \geqq 0$, hence $P(T(a)<\infty)=0$ $\forall a \in (a_0,\infty)$.

Since $q>0$, we have that

$$P(T^*=\infty) = P(t-U_t=0 \ \forall t) = 0 .$$

We can therefore find $a>0$ such that

$$P(t-U_t \leqq a \ \forall t) < 1 ,$$

hence $P(T(a)=\infty)<1$, and we get $a_0 \geqq a > 0$.

If $a_0<\infty$, then there exists $a_n \downarrow a_0$ such that $a_n > a_0$. Further $T(a_n)=\infty$ a.s., and by Lemma 7 we have $T(a_n) \downarrow T(a_0)$, that is, $T(a_0)=\infty$ a.s., and the lemma is proved.

LEMMA 11. *Let R be a $\mathscr{M}$-measurable map of Ω into* $[0,\infty)$, *and* $t \geqq 0$. *Then*

$$P(S_t > t = U_R) = 0 .$$

PROOF. Since $\{S_t>t=U_R\} \subseteqq \bar{M}_t \setminus M_t$, the lemma follows from (v).

LEMMA 12. *Let* $0 \leqq u, s \leqq t$; $0<v<a_0$; $B \in \bar{\mathscr{B}}_+$; $K \in \mathscr{M}_u$; *and* $A \in \mathscr{B}_+$ *such that* $A \subseteqq (v+t,\infty)$. *Then*

$$P(T(s,v) \in A;\ V^*(s,v) \in B;\ K) = P(T(s,v) \in A;\ K)\, P(V^*(v) \in B) .$$

Here $\mathscr{B}_+$ is the Borel subsets of $[0,\infty)$, *and $\bar{\mathscr{B}}_+$ is the Borel subsets of* $[0,\infty]$.

PROOF. Clearly it is enough to prove the lemma when $A=[v+p,\infty)$ with $p>t$.

Let $\omega \in \{v+p \leqq T(s,v)<\infty\} \setminus \{S_p>p=U^*(s,v)\}$. Then $T(s,v,\omega)>p$.

If $U^*(s,v,\omega)=p$, then $S_p(\omega)=p \leqq T(s,v,\omega)$.

If $U^*(s,v,\omega)>p$, then $S_p(\omega) \leqq U^*(s,v,\omega) \leqq T(s,v,\omega)$, since $p>s$.

So in any case $S_p(\omega) \leqq T(s,v,\omega)$, and $S_p(\omega)<\infty$.

Let $\omega \in \{S_p \leqq T(s,v) < \infty\}$. Since M^ω is disjoint from

$$(U^*(s,v,\omega),\ T(s,v,\omega)]\ ,$$

we have $S_p(\omega) \leqq U^*(s,v,\omega)$, hence

$$\infty > T(s,v,\omega) = v + U^*(s,v,\omega) \geqq S_p(\omega)+v \geqq p+v\ .$$

From (vi) and Lemmas 7 and 10 we get

$$\begin{aligned} P(T(s,v)=\infty,\ S_s<\infty) &= P(T(v)\circ\theta_{S_s}=\infty,\ S_s<\infty) \\ &= P(T(v)=\infty)\,P(S_s<\infty) = 0\ . \end{aligned}$$

Since $S_p \geqq S_s$ we have

$$P(T(s,v)=\infty,\ S_p<\infty) = 0\ .$$

By Lemma 11 we have

$$P(p=U^*(s,v)<S_p) = 0\ ,$$

hence

$$(*)\qquad \begin{aligned} \{T(s,v)\in[v+p,\infty)\} &= \{S_p<\infty;\ S_p\leqq T(s,v)\} \quad \text{a.s. } [P] \\ &= \{S_p<\infty;\ S_p\leqq U^*(s,v)\}. \end{aligned}$$

If $u \leqq p \leqq S_p$, then $K \in \mathscr{M}_{S_p}$, and from (*) we see that

$$\{v+p \leqq T(s,v) < \infty\}$$

belongs to $\mathscr{M}_{S_p}$ and is contained in $\{S_p<\infty\}$ a.s. Since for obvious reasons

$$V^*(s,v,\omega) = V^*(v,\theta_{S_p}(\omega)) \qquad \forall\,\omega\in\{S_p<\infty;\ S_p\leqq U^*(s,v)\}\ ,$$

we get from (vi)

$$\begin{aligned} &P(v+p\leqq T(s,v)<\infty;\ K;\ V^*(s,v)\in B) \\ &\qquad\qquad = P(v+p\leqq T(s,v)<\infty;\ K)\,P(V^*(v)\in B), \end{aligned}$$

and the lemma is proved.

Lemma 13. *Let $\mathscr{F}_t$ be the σ-algebra generated by the family $\{U_s \mid 0\leqq s\leqq t\}$, and let*

$$H(v,B) = P(V^*(v)\in B), \qquad B\in\bar{\mathscr{B}}_+,\ v\geqq 0\ .$$

If $t\geqq 0$ and $B\in\bar{\mathscr{B}}_+$, then a.s. we have:

$$P(V_t\in B \mid \mathscr{F}_t) = P(V_t\in B \mid U_t) = H(t-U_t, B)\ .$$

Proof. The proof runs along the same lines as the proof of Lemma 9 in [5]. First we note, that $V^*(\cdot,\omega)$ is right-continuous on $(0,\infty)$ (see, for example, the proof of point (7.4) in Lemma 7). This implies that $H(\cdot, B)$ is right-continuous on $(0,\infty)$ for $B=[c,d)$. Clearly it is enough to prove that

$$(*) \qquad P(V_t \in B; K) = \int_K H(t-U_t, B)\, dP ,$$

where $K=\bigcap_{j=1}^k \{U_{t_j} \in (a_j, b_j]\}$. For all $0 \leqq t_1 < t_2 < \ldots < t_k = t$, $B=[c,d)$ and $a_1 < b_1, a_2 < b_2, \ldots, a_k < b_k$.

Since $U_s \leqq s$ we may assume, that $b_j \leqq t_j$. Let $t_0 = 0$, then

$$(a_k, b_k] = \bigcup_{j=1}^k (a_k, b_k] \cap (t_{j-1}, t_j] .$$

Hence it is enough to prove (*) in the case where

$$(a_k, b_k] \subseteqq (t_{j-1}, t_j] \quad \text{for some } j=1,\ldots,k .$$

Let $s=t_j$ and $u=t_{j-1}$, then $t \geqq s > u$.

Case I. $(a_k, b_k] \subseteqq (t_{j-1}, t_j]$ with $1 \leqq j \leqq k$ and $b_k < t$.

Since $U_{t_k}(\omega) < t_j$ implies $U_{t_k}(\omega) = U_{t_{k-1}}(\omega) = \ldots = U_{t_j}(\omega)$, we see that K takes the form

$$K = \{U_t = U_s \in (a,b]\} \cap K_0 = \{U_t \in (a,b]\} \cap K_0 ,$$

where $K_0 \in \mathscr{M}_u$, $u \leqq a < b \leqq s$; $b < t$. Now let $a_1^n = a < a_2^n < \ldots < a_{n+1}^n = b$ be a partition of $[a,b]$, such that $a_j^n - a_{j-1}^n = (b-a)/n = d_n$, and put

$$\begin{aligned}
l_j^n &= t - a_j^n , & j &= 1,\ldots,n+1 , \\
T_j^n &= T(a_j^n, l_j^n) , & j &= 1,\ldots,n+1 , \\
\Delta_j^n &= (a_j^n, a_{j+1}^n] , & j &= 1,\ldots,n , \\
A_j^n &= \{U_t \in \Delta_j^n\} , & j &= 1,\ldots,n , \\
B_j^n &= \{T_j^n \in \Delta_j^n + l_j^n\} , & j &= 1,\ldots,n , \\
D_n &= \{U_t = t;\ U_r = r \text{ for some } r \in (t, t+d_n]\} .
\end{aligned}$$

Let $\omega \in B_j^n$. Then $a_{j+1}^n + l_j^n \geqq T_j^n(\omega) \geqq t$. If $r = T_j^n(\omega)$, then $r - U_r(\omega) = l_j^n$, hence $M^\omega \cap (r - l_j^n, r] = \emptyset$. Now, since $r - l_j^n \leqq a_{j+1}^n \leqq t < r$, we find that

$$U_t(\omega) = U_r(\omega) = r - l_j^n \in \Delta_j^n ,$$

that is,

$$(4) \qquad B_j^n \subseteqq A_j^n .$$

Let $\omega \in A_j^n \setminus B_j^n$. Then $r = S_{a_j^n}(\omega) \leqq a_{j+1}^n < t$ and $(r,t] \cap M^\omega = \emptyset$. Since $t + d_n - r \geqq l_j^n > 0$ and $\omega \notin B_j^n$, we see that $(t, t+d_n] \cap M^\omega \neq \emptyset$, that is,

$$(5) \qquad \bigcup_{j=1}^n A_j^n \setminus B_j^n \subseteqq D_n .$$

From the right continuity of U it follows that

$$(6) \qquad D_n \downarrow \emptyset .$$

From (4), (5), and (6) we get

$$(7)\quad P(V_t\in B;\ U_t\in(a,b];\ K_0) = \sum_{j=1}^{n} P(V_t\in B;\ U_t\in \Delta_j^n;\ K_0) = \lim_{n\to\infty}\sum_{j=1}^{n} P(V_t\in B;\ B_j^n;\ K_0)\ .$$

If $\omega\in B_j^n$, then $S_{a_j^n}(\omega)\leqq a_{j+1}^n<t<T_j^n(\omega)=r$, hence $S_r(\omega)=S_t(\omega)$, $U_t(\omega)=U_r(\omega)$ and

$$V^*(a_j^n,\ l_j^n,\ \omega) = V_r(\omega) = V_t(\omega)\ .$$

By Lemma 12 we get, using $u\leqq a_j^n$, $(t,t+d_n]\subseteq(a_j^n+l_j^n,\ \infty)$ and $l_j^n>0$,

$$(8)\quad P(V_t\in B;\ B_j^n;\ K_0) = P(V^*(a_j^n,\ l_j^n)\in B,\ T(a_j^n,\ l_j^n)\in(t,t+d_n];\ K_0) = P(B_j^n\cap K_0)\,H(l_j^n,\ B)\ .$$

Substituting (8) in (7) and using (4), (5) and (6) we get

$$\begin{aligned} P(V_t\in B;\ K) &= \lim_{n\to\infty}\textstyle\sum_{j=1}^{n} P(B_t^n\cap K_0)\,H(l_j^n,\ B)\\ &= \lim_{n\to\infty}\textstyle\sum_{j=1}^{n} P(A_j^n\cap K_0)\,H(l_j^n,\ B)\\ &= \lim_{n\to\infty}\textstyle\sum_{j=1}^{n} P(l_{j-1}^n\leqq t-U_t<l_j^n;\ K)\,H(l_j^n,\ B)\\ &= \int_K H(t-U_t,B)\,dP\ ,\end{aligned}$$

where the last equality follows from right-continuity of $H(\cdot,B)$ in the interval $(0,\infty)$ (note, that $t-U_t>0$ inside K).

Case II. $(a_k,b_k]\subseteq(t_{k-1},t_k];\ b_k=t_k$.
In this case we can write K on the form

$$K = K_1\cup K_2\ ,$$

where

$$\begin{aligned} K_1 &= \textstyle\bigcap_{j=1}^{k-1}\{U_{t_j}\in(a_j,b_j]\}\cap\{U_{t_k}\in(a_j,t_k)\}\ ,\\ K_2 &= \textstyle\bigcap_{j=1}^{k-1}\{U_{t_j}\in(a_j,b_j]\}\cap\{U_{t_k}=t_k\}\ .\end{aligned}$$

From Case I we find that

$$P(V_{t_k}\in B;\ K_1) = \int_{K_1} H(t-U_t,\ B)\,dP\ .$$

Let $\omega\in K_2$; if $\omega\in\bar{M}_{t_k}\setminus M_{t_k}$. Then $V_{t_k}(\omega)=0$, and since $P(\bar{M}_{t_k}\setminus M_{t_k})=0$, we have

$$\{V_{t_k}\in B;\ K_2\} = \begin{cases} K_2 & \text{if } 0\in B\\ \emptyset & \text{if } 0\notin B\end{cases}\quad \text{a.s. } [P]\,.$$

Hence we get (using the fact that $H(0,B)=1_B(0)$)

$$P(V_{t_k}\in B;\, K_2) \;=\; 1_B(0)\, P(K_2) \;=\; \int\limits_{K_2} H(t_k-U_{t_k},\, B)\, dP\ ,$$

and since K_1 and K_2 are disjoint, the lemma is proved.

3. The canonical process.

We shall now divide the discussion into two parts, Case 1 with $q=\infty$, and Case 2 with $0<q<\infty$.

Case 1: $q=\infty$.

In this case the canonical process X for M is defined by

$$X_t(\omega) \;=\; t-U_t(\omega) \qquad \text{whenever } t\geqq S_0(\omega)\ .$$

Note that, since $P(M_0)=P(S_0=0)=1$, the process $X_t(\omega)$ is defined for all $t\geqq 0$ and for a.a. $\omega\in\Omega$.

Let E denote $[0,a_0)$ (see Lemma 10), equipped with the right-topology, that is, the topology generated by the right closed intervals $[a,b)$, $0\leqq a<b\leqq a_0$.

We shall now prove that X is a strong Markov process with state space E; actually we will prove that X is a right-continuous Feller process on E, with transition probabilities given by

$$P(t,x,A) \;=\; P(X_{t+T(x)}\in A), \qquad t\geqq 0,\, x\in E,\, A\in\mathscr{B}(E)\ .$$

LEMMA 14. *The map* $t \rightsquigarrow X_t(\omega)$ *is a right-continuous map of* $[S_0(\omega),\infty)$ *into* E *for a.a.* $\omega\in\Omega$.

PROOF. Let $t_0\geqq S_0(\omega)$ and $t_n \downarrow t_0$. Then by the right-continuity of U we have $X_{t_n}(\omega)\to X_{t_0}(\omega)$ (in the usual topology of R).

If $X_{t_0}(\omega)=0$, then $X_{t_n}(\omega)\geqq X_{t_0}(\omega)\ \forall_n$, hence $X_{t_n}(\omega)\to X_{t_0}(\omega)$ in the right-topology of R.

If $X_{t_0}(\omega)>0$, then $t_0\notin M^\omega$. By the right-closedness of M^ω, we can find an $e>0$, such that $[t_0,t_0+e]$ is disjoint from M^ω, and hence

$$X_t(\omega) \;=\; X_{t_0}(\omega)+(t-t_0), \qquad t_0\leqq t\leqq t_0+e\ .$$

Let n_0 be chosen such that $t_0\leqq t_n\leqq t_0+e\ \ \forall n\geqq n_0$. Then

$$X_{t_n}(\omega) \;=\; X_{t_0}(\omega)+(t_n-t_0) \;\geqq\; X_{t_0}(\omega) \qquad \forall n\geqq n_0;$$

hence $X_{t_n}(\omega)\to X_{t_0}(\omega)$ in the right-topology. Since

$$P(X_t \geqq a_0 \text{ for some } t) = P(T(a_0) < \infty) = 0,$$

we see that $X_t(\omega) \in E$ for all $t \geqq 0$ for a.a. ω.

LEMMA 15. *Let $f \in C(E)$. Then*

(15.1) $t \rightsquigarrow P(t,x,f) = \int_0^{a_0} f(y)\, P(t,x,dy)$ *is a right-continuous map of* $[0,\infty)$ *into* R,

(15.2) $x \rightsquigarrow P(t,x,f)$ *is a continuous map of E into* R.

PROOF. First we note that a function g from E into R is continuous, if and only if g is a right-continuous function from $[0,a_0)$ into R (in the usual topologies).

Let $f \in C(E)$. Then

$$P(t,x,f) = Ef(X_{t+T(x)}).$$

From Lemma 14 we immediately see, that (15.1) is fulfilled.

If $0 < x_0 < a_0$ and $x_n \downarrow x_0$, then from Lemmas 14 and 7 we get

$$P(t,x_n,f) \to P(t,x_0,f).$$

Now let $x_n \downarrow 0$; then $\{T(x_n)\}_1^\infty$ is decreasing. Let

$$T(\omega) = \lim_{n\to\infty} T(x_n,\omega).$$

If $\omega \in M_0$, then clearly $[0,T(\omega)] \subseteqq M^\omega$, hence

$$\begin{aligned} P(T \geqq e) &\leqq P(\omega \mid [0,e] \subseteqq M^\omega), \\ &= P(T^* \geqq e) = 0 \quad \forall e > 0, \end{aligned}$$

since $q = \infty$. Hence $T = 0$ a.s., and we have $T(x_n) \downarrow 0$ a.s. As before we therefore get

$$P(t,x_n,f) \to P(t,0,f) \quad \text{as } n \to \infty,$$

and the lemma is proved.

LEMMA 16.

(16.1) *X is progressively measurable with respect to $(\mathscr{M}_t)$.*

(16.2) $X_t(\theta_r(\omega)) = X_{t+r}(\omega)$ *if* $S_r(\omega) \leqq t+r$ *(undefined if* $S_r(\omega) > t+r$*)*.

LEMMA 17. *Let us define*

$$F(t,A) = P(X_t \in A), \quad t \geqq 0, \ A \in \mathscr{B}(E);$$

then we have for $t,h \geqq 0$, $x \in E$ and $A \in \mathscr{B}(E)$:

$$(17.1)\quad P(t,x,A) = \int_x^{x+t} F(t+x-w,A)\,H(x,dw) + 1_A(x+t)H(x,\,(x+t,\infty]).$$

$$(17.2)\quad P(X_{t+h} \in A \mid X_t, S_t) = P(X_{t+h} \in A \mid \mathscr{F}_t, S_t) = \begin{cases} F(h+X_t-V_t,A) & \text{if } V_t - X_t \leqq h\,, \\ 1_A(h+X_t) & \text{if } V_t - X_t > h\,. \end{cases}$$

Proof. Let $S=S^*(x)$, $T=T(x)$, and

$$F_0(\omega',\omega'') = 1_{\{t+T-S\geqq 0\}}(\omega')\,1_A(X_{t+T(\omega')-S(\omega')}(\omega''))\,.$$

Since S and T are stopping times such that $S \geqq T$, we find that T is $\mathscr{M}_S$-measurable. Hence $t+T-S$ is $\mathscr{M}_S$-measurable, and since X is progressively measurable, F is $\mathscr{M}_S \times \mathscr{M}$-measurable. Let $\omega \in \Omega$. Then, by Lemma 6,

$$F_0(\omega,\theta_S(\omega)) = 1_{\{t+T-S\geqq 0\}}(\omega)\,1_A(X_{t+T}(\omega))\,.$$

Using the fact that $T<\infty$ a.s. we get from Lemma 1:

$$\int_\Omega F_0(\omega,\theta_S(\omega))\,P(d\omega) = P(X_{t+T} \in A,\, t+T-S \geqq 0) = \int_\Omega P(d\omega') \int_\Omega F_0(\omega',\omega'')\,P(d\omega'')\,.$$

Using the fact that $V^*(x,\omega) = x+S^*(x,\omega)-T(x,\omega)$ we get

$$P(X_{t+T} \in A,\, t+T \geqq S) = \int_{\{V^*(x)\leqq t+x\}} P(X_{t+x-V^*(x,\omega)} \in A)\,P(d\omega) = \int_x^{x+t} F(t+x-w,A)\,H(x,dw)\,.$$

If $S(\omega) > t+T(\omega)$, then clearly $X_{t+T}(\omega) = t+x$, hence

$$P(X_{t+T} \in A,\, S>t+T) = 1_A(t+x)\,P(S>t+T) = 1_A(t+x)\,H(x,\,(x+t,\infty])\,,$$

and (17.1) is proved.

Let $B \in \overline{\mathscr{B}}_+$ and $K \in \mathscr{F}_t$, and define

$$F_1(\omega',\omega'') = 1_{B_1}(S_t(\omega'))\,1_K(\omega')\,1_A(X_{t+h-S_t(\omega')}(\omega''))\,,$$

where $B_1 = B \cap [0,t+h]$. Then as in the proof of (17.1) we see that

$$P(X_{t+h} \in A; S_t \in B; S_t \leqq t+h; K) = \int_{\{S_t \in B_1\} \cap K} F(t+h-S_t(\omega), A)\, P(d\omega)\,.$$

Here we have used the fact that $\mathscr{F}_t \subseteq \mathscr{M}_t \subseteq \mathscr{M}_{S_t}$.

If $S_t(\omega) > t+h$, then $X_{t+h}(\omega) = h + X_t(\omega)$. Hence, if $B_2 = B \cap (t+h, \infty]$ we get

$$P(X_{t+h} \in A, S_t \in B; S_t > t+h; K) = \int_{\{S_t \in B_2\} \cap K} 1_A(h+X_t)\, dP\,.$$

Now, since $S_t = t - X_t + V_t$ and $B_1 \cup B_2 = B$ (disjoint union), we have

$$P(X_{t+h} \in A; S_t \in B; K)$$
$$= \int_{\{S_t \in B\} \cap K} \left(1_{\{V_t - X_t \leqq h\}} F(h+X_t - V_t, A) - 1_{\{V_t - X_t > h\}}\, 1_A(h+X_t)\right) dP$$

from which (17.2) follows.

We shall now prove a general lemma about Markov processes, which is a slight modification of theorems 4.13, 4.14, 5.10, and 5.11 of [1].

Let F be a topological space, $(\mathscr{Y}_t)_{t \geqq 0}$ an increasing sequence of σ-algebras all contained in $\mathscr{M}$, and $Q(t,x,\cdot)$ a probability measure on $(F, \mathscr{B}(F))$ $\forall t \geqq 0$ $\forall x \in F$. Then we have

LEMMA 18. *Let $(Y_t)_{t \geqq 0}$ be a right-continuous process, adapted to $(\mathscr{Y}_t)_{t \geqq 0}$, and with state space F. We assume:*

(a) *$\mathscr{B}(F)$ is generated by $C(F)$.*

(b) *$t \rightsquigarrow Q(t,x,f)$ is right-continuous, $\forall f \in C(F)$ $\forall x \in F$.*

(c) *$x \rightsquigarrow Q(t,x,f)$ is continuous, $\forall f \in C(F)$ $\forall t \geqq 0$.*

(d) *For all open non-empty $\mathscr{O} \subseteq F$, $\exists\, t \geqq 0$: $P(Y_t \in \mathscr{O}) > 0$.*

(e) *If $A \in \mathscr{B}(F)$ and $t, s \geqq 0$, then*

$$P(Y_{t+s} \in A \mid \mathscr{Y}_t) = Q(s, Y_t, A) \quad a.s.$$

Let $\mathscr{Y}_t^*$ be the completion of $\mathscr{Y}_t$ under P, and $\hat{\mathscr{Y}}_t = \mathscr{Y}^*_{t+0} = \bigcap_{s>t} \mathscr{Y}_s^*$. Then we have:

(18.1) *$\hat{\mathscr{Y}}_t$ is complete under P, and right-continuous in t.*

(18.2) *Q satisfies the Kolmogorov–Chapman equation*

$$Q(t+s,x,A) = \int_F Q(t,y,A) Q(s,x,dy) \quad \forall A \in \mathscr{B}(F)\ \forall x \in F\ \forall t,s \geqq 0\,.$$

(18.3) *If T is a stopping time for $(\hat{\mathscr{Y}}_t)$, $S \in \hat{\mathscr{Y}}_T$ such that $S \geqq T$, and $A \in \mathscr{B}(F)$, then*

$$P(Y_S \in A \mid \hat{\mathscr{Y}}_T) = Q(S-T, Y_T, A) \quad a.s. \text{ on } \{S < \infty\}\,,$$

that is, $(Y_t)_{t\geqq 0}$ is a strong Markov process with respect to $(\hat{\mathscr{Y}}_t)_{t\geqq 0}$ and with Q as transition semi-group.

Proof. (18.1) is trivial, and in the usual way we see, that

$$P(Y_{t+s}\in A \mid \mathscr{Y}_t^*) = Q(s,Y_t,A) \quad \text{a.s.};$$

hence, without loosing generality, we can assume, that $\mathscr{Y}_t=\mathscr{Y}_t^*\ \forall t\geqq 0$.
Let $t,s\geqq 0$, $f\in C(F)$. Then

$$\begin{aligned} Q(t+s,Y_u,f) &= E(f(Y_{t+s+u}) \mid \mathscr{Y}_u) \\ &= E(Q(t,Y_{s+u},f) \mid \mathscr{Y}_u) = E(Q(t,Y_{s+u},f) \mid Y_u)\,. \end{aligned}$$

Since $Q(s,\cdot,\cdot)$ is the conditional distribution of Y_{s+u} given Y_u, we have

$$Q(t+s,Y_u,f) = \int_F Q(t,y,f)\,Q(s,Y_u,dy)\,.$$

From this it follows that

$$Q(t+s,x,f) = \int_F Q(t,y,f)\,Q(s,x,dy) \quad \text{a.s. } [H_u]\,, \tag{9}$$

where H_u is the distribution of Y_u.

Let F_0 be the set of $x\in F$ such that (6) holds (here t,s and f are fixed). By (d) and the above argument we get that F_0 is dense in F.

Let $g(y)=Q(t,y,f)$, then by (c) $g\in C(F)$. Again by (c) we see that both sides of (9) are continuous in x, and since they coincide on the dense set F_0, they coincide everywhere. Hence (18.2) is proved.

The proof of (18.3) follows by an inspection of the proofs of the theorems 4.13, 4.14, 5.10, and 5.11 of [1].

By definition we see immediately that $\mathscr{F}_t$ is the σ-algebra generated by $\{X_s \mid 0\leqq s\leqq t\}$. Let $\mathscr{F}_t^*$ be the completion of $\mathscr{F}_t$ under P and

$$\hat{\mathscr{F}}_t = \mathscr{F}_{t+0}^* = \textstyle\bigcap_{s>t}\mathscr{F}_s^*, \quad t\geqq 0;$$

then we have

Theorem 1. *In the case $q=\infty$, the process $(X_t)_{t\geqq 0}$ is a strong Markov process with respect to $(\hat{\mathscr{F}}_t)$ with transition semi-group P, that is,*
1° *P satisfies the Kolmogorov–Chapman equation:*

$$P(t+s,x,A) = \int_E P(t,y,A)P(s,x,dy) \quad \forall t,s\geqq 0\ \forall x\in E\ \forall A\in\mathscr{B}(E)\,.$$

2° *If T is a stopping time for $(\hat{\mathscr{F}}_t)$, $S\in\hat{\mathscr{F}}_T$, such that $S\geqq T$, and $A\in\mathscr{B}(E)$, then*

$$P(X_S \in A \mid \hat{\mathscr{F}}_t) = P(S-T, Y_T, A) \quad a.s. \text{ on } \{S < \infty\}.$$

Furthermore we have:

3° $\hat{\mathscr{F}}_t$ *is complete under* P *and right-continuous in* t.
4° $\{(t,\omega) \mid X_t(\omega) = 0\} = \bar{M}$.
5° $X_0 = 0$ *a.s.*

Proof. 4° and 5° follow immediately from the definition of X, and in order to prove 1°, 2° and 3° we only have to check, that the conditions (a)–(e) in Lemma 18 are fulfilled.

It is clear that $\mathscr{B}(E) = \mathscr{B}([0,a_0))$, and since $C(E) \supseteqq C([0,a_0))$ we see that (a) is fulfilled.

(b) and (c) follow from Lemma 15.

(d) Let $0 \leqq a < b \leqq a_0$; if $P(X_t \in [a,b)) = 0 \ \forall t$. Then

$$P(X_r \in [a,b) \text{ for some } r \text{ rational}) = 0,$$

and by the right-continuity of X in E we get

$$P(X_t \in [a,b) \text{ for some } t \geqq 0) = 0,$$

and hence from the definition of X

$$P(X_t \geqq a \text{ for some } t \geqq 0) = P(T(a) < \infty) = 0,$$

which contradicts Lemma 10; hence there exists a $t > 0$, such that $P(X_t \in [a,b)) > 0$. Thus (d) is fulfilled.

(e) By (17.2) we have

$$P(X_{t+s} \in A \mid \mathscr{F}_t, S_t) = \begin{cases} F(s+X_t-V_t, A) & \text{if } V_t - X_t \leqq s, \\ 1_A(s+X_t) & \text{if } V_t - X_t > s. \end{cases}$$

By Lemma 13 we have

$$P(V_t \in B \mid \mathscr{F}_t) = H(X_t, B).$$

Combining these two facts we get from (17.1)

$$\begin{aligned} P(X_{t+s} \in A \mid \mathscr{F}_t) &= \int_t^{X_{t+s}} F(s+X_t-w, A)\, H(X_t, dw) + 1_A(s+X_t)\, H(X_t, (X_t+t, \infty]) \\ &= P(s, X_t, A), \end{aligned}$$

and (e) is fulfilled.

Hence Theorem 1 is proved.

Case 2: $0<q<\infty$.

In this case the process X, defined as in Case 1, is no longer strongly Markovian. To see this we look at the stopping time T^* in Lemma 3. We put

$$T^{**}(\omega) = \begin{cases} T^*(\omega) & \text{if } (T^*(\omega),\omega) \in M , \\ \infty & \text{if } (T^*(\omega),\omega) \notin M . \end{cases}$$

Then T^{**} is a stopping time, such that $[T^{**}] \subseteq M$. If

$$B = \{\omega \mid \exists\, a>0 \in [0,a] \subseteq M^\omega\} ,$$

then $B \in \mathscr{M}$ and

$$P(B) = P(\textstyle\bigcup_{a>0} \bigcap_{0\leqq t\leqq a} M_t) = \lim_{a\to 0} P(\bigcap_{0\leqq t\leqq a} M_t)$$
$$= \lim_{a\to 0} P(T^* > a) = 1 .$$

Now we have by the definition of T^*:

$$\theta_{T^*}^{-1}(B) = \{\omega \mid \exists\, a>0, \text{ such that } [T^*(\omega), T^*(\omega)+a] \subseteq M\} = \emptyset .$$

Hence by (vi)

$$P(\{T^{**}<\infty\} \cap \theta_{T^*}^{-1}(B)) = P(T^{**}<\infty)\, P(B) = P(T^{**}<\infty) = 0 ,$$

that is, $(T^*(\omega),\omega) \notin M$ for a.a. ω. By the right-closedness of M we get from this that

$$P(\textstyle\bigcup_{n=1}^\infty \bigcup_{k=n}^\infty \{X_{T^*+1/k}=0\}) = 0 .$$

And since

$$P(\textstyle\bigcup_{n=1}^\infty \bigcap_{k=n}^\infty \{X_{1/k}=0\}) = 1 ,$$

we see that X is *not* strongly Markovian.

We therefore have to define a new canonical process for this case, and this can be done in the following way:

$$T_1^*(\omega) = T^*(\omega) = \inf\{t>0 \mid t \notin M^\omega\} ,$$
$$Y_1^*(\omega) = Y^*(\omega) = \inf\{t>T_1^*(\omega) \mid t \in M^\omega\} ,$$

and by induction:

$$T_{n+1}^*(\omega) = \inf\{t>Y_n^*(\omega) \mid t \notin M^\omega\} ,$$
$$Y_{n+1}^*(\omega) = \inf\{t>T_{n+1}^*(\omega) \mid t \in M^\omega\} .$$

Then it is easily seen that

(a) $T_1^* \leqq Y_1^* \leqq T_2^* \leqq \ldots \leqq T_n^* \leqq Y_n^* \leqq T_{n+1}^* \leqq \ldots \qquad (Y_0^* \equiv 0)$,
(b) Y_n^* and T_n^* are stopping times and $[Y_n^*] \subseteq M$,
(c) $M^\omega = \bigcup_{n=1}^\infty [Y_{n-1}^*(\omega), T_n^*(\omega))$ for a.a. ω.

Now we define the canonical process X^* by

$$X_t^*(\omega) = \begin{cases} 0 & \text{if } t \in [Y_{n-1}^*(\omega), T_n^*(\omega)) , \\ 1+t-T_n^*(\omega) & \text{if } t \in [T_n^*(\omega), Y_n^*(\omega)) , \end{cases}$$

and the transition probabilities by

$$P^*(t,x,A) = \begin{cases} P(X_{t+T(x-1)}^* \in A), & 1 < x < a_0+1 , \\ P(X_{t+T^*}^* \in A), & x=1 , \\ P(X_t^* \in A), & x=0 , \end{cases}$$

for $x \in E^* = \{0\} \cup [1,a_0+1)$ and $A \in \mathscr{B}(E^*)$.

In Case 1 we only used the fact that $q=\infty$ to prove that $x_n \downarrow 0$ implies $T(x_n) \downarrow 0$, and hence that $P(t,x,f)$ is continuous at $x=0$, $\forall t \geqq 0$, $\forall f \in C(E)$. In Case 2 we have $T(x_n) \downarrow T^*$, if $x_n \downarrow 0$, and since we have isolated 0, we find in exactly the same way as in Case 1:

THEOREM 2. *In case $0<q<\infty$, the process $(X_t^*)_{t \geqq 0}$ is a strong Markov process with respect to $(\hat{\mathscr{F}}_t)_{t \geqq 0}$ with transition semi-group P^*, that is:*

1′ *P^* satisfies the Kolmogorov–Chapman equation*

$$P^*(t+s,x,A) = \int_{E^*} P^*(t,y,A)P^*(s,x,dy) \ \ \forall t,s \geqq 0,\ \forall x \in E^*,\ \forall A \in \mathscr{B}(E^*) .$$

2′ *If T is a stopping time for $(\hat{\mathscr{F}}_t)$, $S \in \hat{\mathscr{F}}_T$, such that $S \geqq T$, and $A \in \mathscr{B}(E^*)$, then*

$$P(X_S^* \in A \mid \hat{\mathscr{F}}_T) = P^*(S-T,X_T^*,A) \ \text{ a.s. } on \ \{S<\infty\} .$$

Furthermore we have:

3′ *$\hat{\mathscr{F}}_t$ is complete under P, and right-continuous in t;*
4′ *$\{t \mid X_t^*(\omega)=0\} = M^\omega$ for a.a. ω;*
5′ *$X_0^*=0$ a.s.*

4. Some examples and remarks.

In [5], Krylov and Yuškevič have taken the canonical process $X(t)$ in Section 3 as the definition of a Markov random set. They proved in [5] that if $X(t)$ is a Markov process, and if $q=\infty$, then $X(t)$ is strongly Markovian (cf. [5; Lemma 1]). They also state that in some cases with $0<q<\infty$, the process $X(t)$ is strongly Markovian (cf. [5; Lemma 1′]), which is actually wrong, as we saw in the introductory remark to Case 2 in Section 3.

Theorem 1 in Section 3 states that if M is a strong Markov set with $q=\infty$, then $\bar{M}$ is a strong Markov set and $X(t)$ is a Markov random set in the sense of [5].

In [5] one can find a very deep discussion of Markov random sets in the case $q=\infty$. It is shown there that the Markov random sets may be described completely by a certain function g on $[0,a_0)$ and a non-negative number α. The case $0<q<\infty$ corresponds to the case that Kingman has considered in [2], [3] and [4].

The main purpose of this paper has been to give an intrinsic definition of a strong Markov set, since the definition in [5] can hardly be used in concrete examples.

We shall now describe a procedure for handling the case where the translation operators θ_t do not arise in a natural way. In the case $0<q<\infty$ the translation operators are not really necessary, and one may use (1) in Section 1 to derive the properties of the Markov set. In the case $q=\infty$, we can assume that M^ω is closed for all $\omega \in \Omega$ (cf. Theorem 1 in Section 3). Suppose, thus, that $(\Omega, \mathscr{M}, P)$, $(\mathscr{M}_t)_{t\geqq 0}$ and M are given, such that

(i) P restricted to $\mathscr{M}_t$ is complete for all $t \geqq 0$,
(ii) M is a subset of $\mathsf{R}_+ \times \Omega$, such that M is progressively measurable with respect to $(\mathscr{M}_t)_{t\geqq 0}$,
(iii) M^ω is a closed subset of R_+ for all $\omega \in \Omega$,
(iv) $P(M_0)=1$.

Now let W denote the set of all closed subsets of R_+, and let

$$U_0(t,w) = \begin{cases} \sup\{s \mid s \in [0,t] \cap w\}, & \text{if } [0,t]\cap w \neq \emptyset\,, \\ \text{undefined}, & \text{if } [0,t]\cap w = \emptyset\,, \end{cases}$$

$$\begin{aligned} N &= \{(t,w) \mid t \in w\}\,, & \\ S_0(t,w) &= \inf\{s \mid s \in [t,\infty)\cap w\}, & \text{for } (t,w) \in R_+ \times W\,, \\ \mathscr{G}_t^\circ &= \sigma\{U_0(s) \mid 0 \leqq s \leqq t\}, & \text{for } t \geqq 0\,, \\ \mathscr{G}^\circ &= \sigma\{U_0(s) \mid 0 \leqq s < \infty\}\,, & \\ \theta_t(w) &= \{u \mid u+t \in w\}, & \text{for } t \geqq 0 \text{ and } w \in W\,. \end{aligned}$$

Then clearly $U_0(\cdot,w)$ is right-continuous, $S_0(\cdot,w)$ is left-continuous and $N_t=\{w \mid U_0(t,w)=t\}$. From this one easily deduces:

(a) N is progressively measurable with respect to $(\mathscr{G}_t^\circ)_{t\geqq 0}$.
(b) U_0 is progressively measurable with respect to $(\mathscr{G}_t^\circ)_{t\geqq 0}$.
(c) If Q is any probability measure on $(W, \mathscr{G}^\circ)$, and $\mathscr{G}$ is the completion

of $\mathscr{G}^\circ$ with respect to Q, then S_0 is $(\mathscr{B}_+ \times \mathscr{G}, \bar{\mathscr{B}})$-measurable, and the map $(t,w) \rightsquigarrow \theta_t(w)$ is $(\mathscr{B}_+ \times \mathscr{G}, \mathscr{G}^\circ)$-measurable.

(d) $\theta_t \circ \theta_s = \theta_{t+s}$, for all $t,s \geqq 0$.

(e) $\theta_0(w) = w$, for all $w \in W$.

(f) $\theta_t^{-1}(N_s) = N_{t+s}$ for all $t,s \geqq 0$.

Now we introduce a map β from Ω into W, defined by

$$\beta(\omega) = M^\omega .$$

Let Q be the image measure of P under β, that is, $Q = \beta \cdot P$ (it is easily verified, that β is measurable with respect to $(\mathscr{M}_t, \mathscr{G}_t^\circ)$ and $(\mathscr{M}, \mathscr{G}^\circ)$ for all $t \geqq 0$). Let $\mathscr{G}_t$ and $\mathscr{G}$ be the completions of $\mathscr{G}_t^\circ$ and $\mathscr{G}^\circ$, respectively, with respect to Q. Let (compare with the definition of X)

$$X_0(t,w) = t - U_0(t,w) \quad \text{for } (t,w) \in R_+ \times W .$$

Then

$$X_0(t,\beta(\omega)) = X(t,\omega) \quad \text{and} \quad \beta^{-1}(\mathscr{G}_t^\circ) = \mathscr{F}_t$$

for all $t \geqq 0$.

From the properties (a)–(f) it follows that $(W, \mathscr{G}, (\mathscr{G}_t)_{t \geqq 0}, Q, (\theta_t)_{t \geqq 0}, N)$ possesses all of the properties described in the definition of a strong Markov set, except for the strong Markov property (vi) in the definition on page 147. If N has this property, then $X_0(t)$ is a strong Markov process, and we deduce from the above argument that $X(t)$ is a strong Markov process.

The problem is therefore reduced to proving that N has property (vi) on page 147.

We shall briefly mention two examples, which are, as well as the above reduction, due to P.-A. Meyer (private communication).

1) Let $Z(t)$ be a right-continuous stochastic process with independent stationary increments, and assume $Z(0) = 0$. Let M denote the set of ladder points of the process $Z(t)$, that is, $(t,\omega) \in M$ if and only if $Z(s,\omega) \leqq Z(t,\omega)$ for all $0 \leqq s \leqq t$. Then $\bar{M}$ is strongly Markovian in the above sense.

2) Let $Z(t)$ be a right-continuous stochastic process with independent positive stationary increments, such that $Z(0) = 0$. Let M denote the set of values of $Z(t)$, that is,

$$M = \{(t,\omega) \mid Z(s,\omega) = t \text{ for some } s \geqq 0\} .$$

Then $\bar{M}$ is strongly Markovian in the above sense.

It is fairly easy to prove, that (vi) on page 147 is satisfied in both examples, whereas a direct verification of the fact, that they are Markov random sets in the sense of [5], seems in essence to be equivalent to proving Theorem 1 in Section 3.

REFERENCES

1. E. B. Dynkin, *Die Grundlagen der Theorie der Markoffschen Prozesse*, Springer-Verlag, Berlin · Heidelberg, 1961.
2. J. F. C. Kingman, *The stochastic theory of regenerative events*, Z. Wahrscheinlichkeitstheorie 2 (1964), 180–224.
3. J. F. C. Kingman, *Some further analytic results in the theory of regenerative events*, J. Math. Anal. Appl. 11 (1965), 422–433.
4. J. F. C. Kingman, *A continuous analogue of the theory of recurrent events*, Bull. Amer. Math. Soc. 69 (1963), 268–272.
5. N. V. Krylov and A. A. Juškevič, *Markov random sets*, Trudy Moskov. Mat. Obšč. 13 (1965), 114–135 (= Trans. Moscow Math. Soc. 13 (1965), 127–153).
6. P.-A. Meyer, *Probabilités et potentiel*, Hermann, Paris 1966.
7. P.-A. Meyer, *Processus de Markov* (Lecture Notes in Mathematics 26), Springer-Verlag, Berlin · New York, 1967.

UNIVERSITY OF AARHUS, DENMARK