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BORSUK-ULAM TYPE THEOREMS FOR PROPER Z_p-ACTIONS ON (MOD p HOMOLOGY) n-SPHERES

HANS JØRGEN MUNKHOLM

0. Introduction.

In 1933 Borsuk [1] proved the following

BORSUK-ULAM THEOREM. *For any map $f\colon S^n \to R^n$ there is an $x \in S^n$ such that $f(x)=f(-x)$.*

In 1955 Bourgin and (independently) Yang published proofs of the following generalization (see [2], [3], and [16]).

BOURGIN-YANG THEOREM. *For any map $f\colon S^n \to R^k$ the covering dimension of*

$$A(f) = \{x \in S^n \mid f(x)=f(-x)\}$$

is at least $n-k$.

In 1960 Conner and Floyd generalized the result further (see [7] and [8]). They proved what here will be called

CONNER-FLOYD THEOREM. *Let T be a fixed point free, differentiable involution on the n-sphere S^n, and let $f\colon S^n \to M^k$ be a continuous map into a differentiable k-manifold M^k. Suppose that*

$$f_* = 0 : H_n(S^n ; Z_2) \to H_n(M^k ; Z_2) . \tag{0.1}$$

Then the covering dimension of

$$A(T,f) = \{x \in S^n \mid f(x)=f(xT)\}$$

is at least $n-k$.

In [8] Conner and Floyd ask the following questions:

1. Can all differentiability hypotheses be eliminated?
2. Can S^n be replaced by a closed n-manifold which is a mod 2 homology n-sphere?
3. Can M^k be replaced by non-manifolds?

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It also seems natural to ask the following

4. QUESTION. Let G be a finite group of order $|G| > 2$ acting properly on the n-sphere S^n via the map $\mu: S^n \times G \to S^n$. For any map $f: S^n \to M^m$ into a compact, topological m-manifold M^m let

$$A(\mu\,; f) = \{x \in S^n \mid f(x) = f(xg) \text{ for all } g \in G\}.$$

Is the covering dimension of $A(\mu\,; f)$ necessarily $\geqq n - (|G| - 1)m$?

REMARKS. A condition analogous to (0.1) does not appear because for $n \geqq (|G| - 1)m$ and $|G| \geqq 3$ it would automatically be true (and for $n < (|G| - 1)m$ there is no question).

Covering dimension (abbreviated cov dim) is taken in the sense of [14], say; it does not really matter in which sense it is taken since cohomological dimension (see [6]) rather than cov dim will be used.

In this paper, questions 1, 2, and 4 are treated. The (partial) answers obtained are:

Question 1: Yes.

Question 2: For the Bourgin-Yang theorem (and certain other cases): Yes.

Question 4: For $G = Z_p$, p prime, and with Z_p-orientability of M^m: Yes (see also the Note at the end of this section).

Question 1 was already considered in [12].

More precisely, a theorem (see section 4) which has the following two corollaries will be proved.

MOD p CONNER-FLOYD THEOREM. *Let $\mu: S^n \times Z_p \to S^n$ be a proper action of the cyclic group of prime order p on the n-sphere. Consider a map $f: S^n \to M^m$ into a compact, topological m-manifold M^m. If $p = 2$, assume that*

$$f_* = 0 : H_n(S^n\,; Z_2) \to H_n(M^m\,; Z_2),$$

and if p is odd, assume that M^m is Z_p-orientable. Then the cohomological dimension (with coefficients Z_p) of

$$A(\mu\,; f) = \{x \in S^n \mid f(x) = f(xg) \text{ for all } g \in Z_p\}$$

is at least $n - (p - 1)m$.

MOD p BOURGIN-YANG THEOREM. *Let $\mathscr{S}^n$ be a closed n-manifold which is a* mod p *homology n-sphere. Let $\mu : \mathscr{S}^n \times Z_p \to \mathscr{S}^n$ be a proper action* *of* Z_p. *Then for any map* $f : \mathscr{S}^n \to R^m$ *the cohomological dimension (coefficients* Z_p*) of*

$$A(\mu\,; f) \;=\; \{x \in \mathscr{S}^n \mid f(x) = f(xg) \text{ for all } g \in Z_p\}$$

is at least $n-(p-1)m$.

The proof will be based upon ideas dating back to Yang [16]; they were also used by Conner and Floyd in [8].

Note (added just before printing). Using obstruction theory it is easy to prove the following:

Let k be an odd, non-prime number $\neq 9$ and let $\mu : S^k \times Z_k \to S^k$ be the standard action. Then there exists a map $f : S^k \to R$ such that $A(\mu\,; f) = \emptyset$.

It seems possible to obtain some positive results (for maps $S^n \to R$, n large, and Z_k-action on S^n) by using K-theory characteristic classes. I hope to return to this in a future publication.

1. Notation.

Let G be a finite group of order $|G| = k+1$. If $|G|$ is even, let $q = 2$, and if $|G|$ is odd, let q be an arbitrary prime. By H_*, H^*, $\bar{H}^*$ we denote singular homology, singular cohomology and Alexander–Spanier cohomology, respectively; if no coefficients are mentioned, Z_q is understood. For Alexander–Spanier cohomology we shall freely change between the two definitions given by Spanier (p. 289 and p. 308 of [15]). By $\iota : \bar{H}^* \to H^*$ we denote the natural transformation given on p. 289 of [15].

The word *manifold* will be taken to mean a Z_q-orientable topological manifold; *cc-manifold* will mean a closed (that is, compact and without boundary) and connected manifold. For any compact pair (A,B) in a Z_q-orientable n-manifold M, there is the Alexander–Spanier duality isomorphism

$$\bar{\gamma} : \; H_p(M-B,\, M-A) \to \bar{H}^{n-p}(A,B)\;,$$

defined via the slant product as in [15].

A *G-space* X will mean a space X together with a map $\mu \colon X \times G \to X$ (written $\mu(x,g) = xg$) such that $x(gh) = (xg)h$ and $x1 = x$. As usual μ (or X) is called *proper* if

$$(\exists\, x : \; xg = x) \;\Rightarrow\; g \;=\; 1\;.$$

For G-spaces X_1 and X_2 let $X_1 \times_G X_2$ be the quotient space $(X_1 \times X_2)/G$ where G acts diagonally on $X_1 \times X_2$. If Y is any space, let YG be the product of $|G|$ copies of Y; writing its elements as $\sum_g y(g)g$ it is a G-space under the action

$$\left(\sum y(g)g\right)h = \sum y(g)gh = \sum y(gh^{-1})g .$$

In YG there is the G-invariant (and G-trivial) subspace

$$\Delta Y = \{\sum y(g)g \mid y(g)=y(1) \text{ for all } g \in G\}$$

(the "diagonal"). If y_0 is a base point of Y, then ΔY and YG receive $\star = \sum y_0 g$ as a base point.

Fix a cc-manifold $\mathscr{S}^n$ which is a mod q homology n-sphere, and let $\mu: \mathscr{S}^n \times G \to \mathscr{S}^n$ be a proper Z_q-orientation-preserving action of G upon $\mathscr{S}^n$. For any map $f: \mathscr{S}^n \to Y$ put

$$A(\mu\,;f) = \{x \in \mathscr{S}^n \mid f(x)=f(xg) \text{ for all } g \in G\} .$$

Associated with μ there is a vector bundle ξ_μ described as follows. Take IG to be the augmentation ideal of the group algebra RG, that is, IG consists of those $\sum r(g)g \in RG$ for which $\sum r(g)=0$. Now ξ_μ is obtained by screwing in IG as fibre in the principal G-bundle $\mathscr{S}^n \to \mathscr{S}^n/G$, that is,

$$\xi_\mu = (\mathscr{S}^n \times_G IG \to \mathscr{S}^n/G) .$$

It is easily seen that ξ_μ is Z_q-orientable. Hence ξ_μ has a (mod q) Euler class

$$e_q(\xi_\mu) \in H^k(\mathscr{S}^n/G) .$$

As noticed in the introduction, cohomological dimension with coefficients A (A being some abelian group) is taken in the sense of [6]; it will be abbreviated $\mathrm{cd}(\cdot;A)$. If X is a compact, proper G-space, then

$$\mathrm{cd}(X;A) = \mathrm{cd}(X/G;A) .$$

To see this, cover X by closed subsets X_i chosen so small that the projection $X \to X/G$ gives a homeomorphism when restricted to X_i. The above equality then follows immediately from the sum-theorem and the monotonicity property of $\mathrm{cd}(\cdot;A)$ (see [6], theorem 4.1 and lemma 2.2 together with remark 2.11). The inequality

$$\mathrm{cd}(X;A) \leqq \mathrm{cov\,dim}(X)$$

(see [6]) shows that it is actually better to work with cd than with cov dim.

Finally a map $f: \mathscr{S}^n \to Y$ is called *nice* provided the following holds: There is a map $f_0: \mathscr{S}^n \to Y$ and a point $y_0 \in Y$ such that

(1.1) $$f_0 \simeq f\,,$$

where $\simeq$ means "homotopic to", and

(1.2) $$\forall x \in \mathscr{S}^n : f_0(xg) \neq y_0 \text{ for at most one } g \in G\,.$$

Notice that any map $f\colon \mathscr{S}^n \to Y$ is nice under either of the following conditions:

(1.3) $$\mathscr{S}^n = S^n\,,$$

(1.4) $$Y \text{ is contractible (especially for } Y = R^m)\,.$$

2. The main proposition.

2.1. PROPOSITION. *Let $f\colon \mathscr{S}^n \to M^m$ be a map into a compact m-manifold M^m. Suppose that*

(2.1) $$\operatorname{cd}\big(A(\mu;f)/G;Z_q\big) < n-km\,,$$

(2.2) $$f \textit{ is nice}\,,$$

(2.3) $$f_* = 0 : H_n(\mathscr{S}^n) \to H_n(M^m)\,.$$

Then

(2.4) $$e_q(\xi_\mu)^m = 0\,.$$

REMARKS. 1. If $n < km$, then $e_q(\xi_\mu)^m \in H^{mk}(\mathscr{S}^n/G) = 0$; therefore, *assume* from now on that $n \geqq km$.

2. With $n \geqq km$ condition (2.3) is trivially fulfilled unless $G = Z_2$ and $m = n$ (of course I do not want to consider $G = 1$, by the way I also tacitly assume $n > 0$, $m > 0$).

3. *One may, and we do take M^m a* cc*-manifold.* In fact, if M^m is not connected one just has to regard f as a map into the relevant component of M^m. And if M^m has a non-empty boundary one considers f as a map into DM^m. (DM^m = "the double of M^m" consists of two copies of M^m identified along their boundaries. DM^m is a manifold because the boundary of M^m is collared in M^m, see [4].)

4. The assumption (2.1) implies that

(2.1′) $$\bar{H}^{n-km}\big(A(\mu,f)/G\big) = 0\,.$$

This weaker assumption is the one that is actually used in the proof.

DIGRESSION 1. Before turning to the proof of the proposition, we consider the special case of maps $f\colon \mathscr{S}^n \to R^m$ (to get this as a *special* case notice that $f(\mathscr{S}^n)$ is contained in a sufficiently big closed disc in R^m). There is the $|G|m$-dimensional Z_q-orientable vector bundle

$$\eta = (\mathcal{S}^n \times_G R^m G \to \mathcal{S}^n/G)$$

containing the trivial $|G|$-dimensional vector bundle

$$\varepsilon = (\mathcal{S}^n \times_G \Delta R^m \to \mathcal{S}^n/G)$$

as a subbundle. The quotient η/ε is easily seen to be

$$m\xi_\mu = \xi_\mu \oplus \ldots \oplus \xi_\mu$$

(the direct sum of m copies of ξ_μ). Hence there is a short exact sequence of vector bundles

$$\begin{array}{ccccc} \mathcal{S}^n \times_G \Delta R^m & \longrightarrow & \mathcal{S}^n \times_G R^m G & \xrightarrow{\pi} & \mathcal{S}^n \times_G (IG)^m \\ & {\scriptstyle \varepsilon} \searrow & \downarrow {\scriptstyle \eta} & \swarrow {\scriptstyle m\xi_\mu} & \\ & & \mathcal{S}^n/G & & \end{array}$$

The map f induces a cross-section s of η, namely

$$s(xG) = \left(x, \sum f(xg^{-1})g\right) G \, ,$$

and

$$s^{-1}(\mathcal{S}^n \times_G \Delta R^m) = A(\mu;f)/G \, .$$

This means that πs is a cross-section of $m\xi_\mu$ which is zero only above $A(\mu;f)/G$. Hence the (mod$q$) Euler class of $m\xi_\mu$ satisfies

(2.5) $e_q(\xi_\mu)^m \in \operatorname{Im}\left[H^{km}(\mathcal{S}^n/G, \mathcal{S}^n/G - A(\mu,f)/G) \to H^{km}(\mathcal{S}^n/G)\right]$.

By definition of $\bar{H}$, (2.1′) means that

$$\varinjlim H^{n-km}(U) = 0$$

as U ranges over all open neighbourhoods of $A(\mu;f)/G$ in $\mathcal{S}^n/G$. For any $x \in H^{n-km}(\mathcal{S}^n/G)$ one can then find an open $U_x \supseteq A(\mu;f)/G$ such that

(2.6) $x \in \operatorname{Im}[H^{n-km}(\mathcal{S}^n/G, U_x) \to H^{n-km}(\mathcal{S}^n/G)]$.

From (2.5) and (2.6) follows that the cup-product-map

(2.7) $e_q(\xi_\mu)^m \cup - : H^{n-km}(\mathcal{S}^n/G) \to H^n(\mathcal{S}^n/G)$

vanishes. In a (Z_q-orientable) manifold, however, the cup-product-pairing to the top-dimension is non-singular (using field coefficients). Hence (2.7) implies that $e_q(\xi_\mu)^m = 0$.

The proof of the proposition is divided into 3 lemmas. The first one of these gives a condition for the vanishing of $e_q(\xi_\mu)^m$ in terms of $\mathcal{S}^n$ and M^m. It is motivated by the following

DIGRESSION 2. Suppose for a moment that $\mathscr{S}^n, \mu$, and M^m are differentiable. There are the imbeddings

$$\mathscr{S}^n \times_G \star \overset{i}{\subseteqq} \mathscr{S}^n \times_G \Delta M \overset{j}{\subseteqq} \mathscr{S}^n \times_G MG \,,$$

where $\star$ is the basepoint of MG corresponding to some basepoint of M. Also $\mathscr{S}^n \times_G \star$ is identified with $\mathscr{S}^n/G$ in a canonical way, and it is not hard to see that under this identification

$$i^* \nu = m\xi_\mu \,,$$

where ν is the normal bundle of the imbedding j.

Consider then the commutative diagram (for any G-space X let $\overline{X} = \mathscr{S}^n \times_G X$)

$$\begin{array}{ccccc}
H_{n+m}(\overline{MG}) & \xleftarrow{j_*} & H_{n+m}(\overline{\Delta M}) & \xrightarrow{j'_*} & H_{n+m}(\overline{MG}, \overline{MG} - \bar{\star}) \\
\cong \downarrow \iota\bar{\gamma} & & & & \\
H^{km}(\overline{MG}) & & & & \cong \downarrow \iota\bar{\gamma} \\
\downarrow j^* & & & & \\
H^{km}(\overline{\Delta M}) & & \xrightarrow{\quad i^* \quad} & & H^{km}(\bar{\star})
\end{array}$$

It is well known (see, for example, [11]) that the (mod$q$) Euler class of ν is

$$e_q(\nu) = j^*(\iota\bar{\gamma}) j_* \sigma \,,$$

where σ is the orientation class of $\overline{\Delta M}$. Hence $e_q(\xi_\mu)^m = i^* e_q(\nu)$ vanishes if and only if the inclusion

$$j' : \overline{\Delta M} \to (\overline{MG}, \overline{MG} - \bar{\star})$$

has

(2.8) $$H_{n+m}(j') = 0 \,.$$

The first lemma states that this is still true if all differentiability hypotheses are dropped.

2.2. LEMMA. (2.4) *and* (2.8) *are equivalent.*

PROOF. Let D_1 be an open disc around m_0 in M, and let j'' be the inclusion

$$j'' : \overline{\Delta M} \to (\overline{MG}, \overline{MG} - \overline{D_1 G}) \,.$$

It is an easy consequence of "compact support" (see lemma 12, p. 204 of [15]) that (with D_1 sufficiently small) (2.8) is equivalent to

(2.9) $$H_{n+m}(j'') = 0 \,.$$

Then look at the commutative diagram below. Here D_2 is a closed disc around m_0 in M with D_1 contained in the interior of D_2, D is the closure of D_1, $\dot{D}$ the boundary of D and $(DG)^{\cdot}$ the boundary of DG. The map j''' is the inclusion of $\overline{\Delta(\dot{D})}$ in $\overline{(DG)^{\cdot}}$, and a, b, c, d denote maps induced by inclusion.

$$\begin{array}{ccc}
H_{n+m}(\overline{\Delta M}) & \xrightarrow{\quad j_*'' \quad} & \text{(to } H_{n+m}(\overline{MG}, \overline{MG}-\overline{D_1G})\text{)} \\
\downarrow a & & \downarrow \\
H_{n+m}(\overline{\Delta M}, \overline{\Delta(M-D_1)}) & \xrightarrow{\ b\ } & H_{n+m}(\overline{MG}, \overline{MG}-\overline{D_1G}) \\
\cong \uparrow \text{exc} & & \cong \uparrow \text{exc} \\
H_{n+m}(\overline{\Delta D_2}, \overline{\Delta(D_2-D_1)}) & \xrightarrow{\ c\ } & H_{n+m}(\overline{D_2G}, \overline{D_2G}-\overline{D_1G}) \\
\downarrow \partial & & \downarrow \partial \\
H_{n+m-1}(\overline{\Delta(D_2-D_1)}) & \xrightarrow{\ d\ } & H_{n+m-1}(\overline{D_2G}-\overline{D_1G}) \\
\cong \downarrow \text{(def retr)} & & \cong \downarrow \text{(def retr)} \\
H_{n+m-1}(\overline{\Delta(\dot{D})}) & \xrightarrow{\ j_*'''\ } & H_{n+m-1}(\overline{(DG)^{\cdot}})
\end{array}$$

The map a is monic, because $\overline{\Delta(M-D_1)}$ is an $(n+m)$-manifold with boundary (so that $H_{n+m}(\overline{\Delta(M-D_1)})=0$). The isomorphisms labelled exc are given by exciding $\overline{\Delta M}-\overline{\Delta D_2}$ and $\overline{MG}-\overline{D_2G}$, respectively. By ∂ we denote boundaries; they are monic because

$$H_{n+m}(\overline{\Delta D_2}) = H_{n+m}(\overline{D_2G}) = 0$$

(ΔD_2 and D_2G are G-equivariantly contractible so that $\overline{\Delta D_2}$ and $\overline{D_2G}$ are homotopy-equivalent to $\overline{*}=\mathscr{S}^n/G$). The deformation retractions (def retr) referred to are obtained as follows. Take D_i to be a disc in R^m of radius i $(i=1,2)$; the formula

$$\bar{\varrho}\big(x, \textstyle\sum d(g)g, t\big) = \big(x, \textstyle\sum \min\{1, (\|d(g)\|^{-1}-1)t+1\}\, d(g)g\big)$$

defines a map

$$\bar{\varrho}:\ \mathscr{S}^n \times (D_2G-D_1G)\times I \to \mathscr{S}^n \times (D_2G-D_1G)\,.$$

Since, for each t, $\bar{\varrho}(\cdot,\cdot,t)$ is G-equivariant, there is an induced map

$$\varrho:\ (\overline{D_2G}-\overline{D_1G})\times I \to \overline{D_2G}-\overline{D_1G}\ ;$$

it is easily seen that ϱ gives deformation retractions from $\overline{D_2G}-\overline{D_1G}$ onto $\overline{(DG)^{\cdot}}$ and from $\overline{\Delta(D_2-D_1)}$ onto $\overline{\Delta(\dot{D})}$.

If (in the diagram) $j_*''=0$, then $\ker c \neq 0$; it follows that $\ker j_*''' \neq 0$.

Since $H_{n+m-1}(\overline{\Delta(\dot{D})})=Z_q$, this implies that $j_*'''=0$. Conversely, it is obvious that $j_*'''=0$ implies $j_*''=0$. This means that (2.9) is equivalent to

(2.10) $$H_{n+m-1}(j''') = 0 .$$

Of course (2.10) is equivalent to

(2.11) $$H^{n+m-1}(j''') = 0 .$$

Now recall the vector bundles ε, η, and ξ_μ. Clearly the (total spaces of the) sphere bundles associated with ε and η may be taken as

$$S(\varepsilon) = \overline{\Delta(\dot{D})} \quad (=\mathscr{S}^n \times_G \Delta\dot{D}) ,$$
$$S(\eta) = \overline{(DG)^\cdot} \quad (=\mathscr{S}^n \times_G (DG)^\cdot) ,$$

and with these identifications j''' becomes the inclusion $S(\varepsilon)\subseteq S(\eta)$. There is then the commutative diagram

$$\begin{array}{ccccc}
H^{n+m-1}(S(\varepsilon)) & \xrightarrow[\cong]{\delta} & H^{n+m}(B(\varepsilon), S(\varepsilon)) & \xleftarrow[\cong]{\Phi} & H^n(\mathscr{S}^n/G) \\
\uparrow (j''')^* & & \uparrow (\text{incl})^* & & \uparrow -\cup e_q(\xi_\mu)^m \\
H^{n+m-1}(S(\eta)) & \xrightarrow{\delta'} & H^{n+m}(B(\eta), S(\eta)) & \xleftarrow[\cong]{\Phi'} & H^{n-km}(\mathscr{S}^n/G)
\end{array}$$

Here $B(\varepsilon)$ and $B(\eta)$ are the ball bundles. The coboundaries δ and δ' are iso and epic, respectively (use the long exact sequences). By Φ and Φ' we denote Thom isomorphisms. Commutativity (at any rate up to sign) of the right hand square follows from a direct computation using three facts, namely:

1) $\varepsilon\oplus m\xi_\mu=\eta$,

2) Thom classes are multiplicative,

3) $e_q(\xi_\mu)^m$ may be described as the image of the Thom class of $m\xi_\mu$ under the composition

$$H^{km}(B(m\xi_\mu), S(m\xi_\mu)) \longrightarrow H^{km}(B(m\xi_\mu)) \xrightarrow{(\text{proj}^*)^{-1}} H^{km}(\mathscr{S}^n/G) .$$

Recalling that the cup-product pairing to the top dimension in $H^*(\mathscr{S}^n/G)$ is non-singular one reads off from the diagram that

(2.4) $$e_q(\xi_\mu)^m = 0$$

is equivalent to

(2.11) $$H^{n+m-1}(j''') = 0 .$$

This finishes the proof of lemma 2.2.

This lemma ties $e_q(\xi_\mu)^m$ up with the manifolds ($\mathscr{S}^n$ and M^m) in question. The next thing to do is to bring the given map $f\colon \mathscr{S}^n \to M^m$ into the picture. This is done by introducing the map $s\colon \mathscr{S}^n/G \to \overline{MG}$, defined by the formula

$$s(xG) = \big(x, \textstyle\sum f(xg^{-1})g\big)G\,.$$

Observe that s is a cross-section in the fiber bundle

$$(\overline{MG} = \mathscr{S}^n \times_G MG \to \mathscr{S}^n/G)\,.$$

Also notice that s is a homeomorphism onto its image and that

$$s^{-1}(\overline{\Delta M}) = A(\mu;f)/G\,.$$

LEMMA 2.3. *Let f satisfy* (2.2) *and* (2.3); *then*

$$e_q(\xi_\mu)^m = 0 \tag{2.4}$$

if and only if the composition

$$H_{n+m}(\overline{\Delta M}) \xrightarrow{j_*} H_{n+m}(\overline{MG}) \xrightarrow[\cong]{\iota\bar\gamma} H^{km}(\overline{MG}) \xrightarrow{s^*} H^{km}(\mathscr{S}^n/G) \tag{2.12}$$

vanishes.

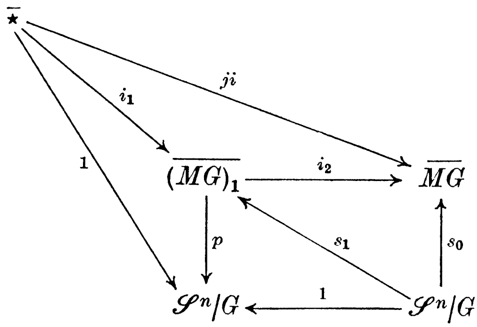
PROOF. In view of lemma 2.2 it suffices to prove that the diagram

$$\begin{array}{ccccccc}
H_{n+m}(\overline{\Delta M}) & \xrightarrow{j_*} & H_{n+m}(\overline{MG}) & \xrightarrow[\cong]{\iota\bar\gamma} & H^{km}(\overline{MG}) & \xrightarrow{s^*} & H^{km}(\mathscr{S}^n/G) \\
\downarrow j_*' & & & & \vdots\, (ji)^* & & \downarrow 1^* \cong \\
H_{n+m}(\overline{MG}, \overline{MG}-\bar\star) & & \xrightarrow[\cong]{\iota\bar\gamma} & & H^{km}(\bar\star) & \xleftarrow{\quad} &
\end{array}$$

commutes (recall that $\bar\star$ has been identified with $\mathscr{S}^n/G$). If the dotted arrow is filled in by $(ji)^*$, then the rectangle commutes, so it suffices to prove the triangle commutative, and that is where (2.2) and (2.3) come in. By (2.2) there is a point $m_0 \in M$ and a map $f_0\colon \mathscr{S}^n \to M$ such that $f \simeq f_0$ and, for each $x \in \mathscr{S}^n$, $f_0(xg) \neq m_0$ for at most one $g \in G$. It follows that s is homotopic to a cross section $s_0\colon \mathscr{S}^n/G \to \overline{MG}$ which factors through $\overline{(MG)}_1$ where

$$(MG)_1 = \{\textstyle\sum m(g)g \mid \text{at most one } m(g) \neq m_0\} \subseteq MG\,.$$

There results a commutative diagram



in which i_1, i_2 (and ji) are inclusions, s_1 is a factorization of s_0, and p is induced by the projection $\mathcal{S}^n \times (MG)_1 \to \mathcal{S}^n$.

If $x \in H^{km}(\overline{MG})$, put $x' = p^*(ji)^*x - i_2{}^*x$; then

$$(2.13) \qquad i_1{}^*x' = 0 \quad \text{and} \quad s_1{}^*x' = (ji)^*x - s^*x .$$

Hence commutativity of the triangle in question will follow from

$$(2.14) \qquad i_1{}^*x' = 0 \quad \text{implies} \quad s_1{}^*x' = 0 .$$

To prove (2.14) two cases are considered.

Case 1. $|G| > 2$. There is a relative homeomorphism

$$\alpha : \ \mathcal{S}^n \times (M, m_0) \to \big(\overline{(MG)_1}, \bar{\star}\big)$$

given by the formula

$$\alpha(x,m) = \big(x, m\cdot 1 + \sum_{g \neq 1} m_0 g\big) G .$$

From this and the Künneth formula one gets

$$H^{km}\big(\overline{(MG)_1}, \bar{\star}\big) = H^{km}(M, m_0) \oplus H^{km-n}(M, m_0) = 0 .$$

Therefore, the $i_1{}^*$ appearing in (2.14) is monic. But then (2.14) is trivially true.

Case 2. $G = Z_2$. Here $k = 1$, so $H^{km}(M, m_0) \neq 0$, and the above argument does not work. Instead one looks at the following commutative diagram

$$\begin{array}{ccccc}
H^m\big(\overline{(MZ_2)_1}, \bar{\star}\big) & \xrightarrow{j_1{}^*} & H^m\big(\overline{(MZ_2)_1}\big) & \xrightarrow{i_1{}^*} & H^m(\bar{\star}) \\
\uparrow{\scriptstyle p_2{}^*} & & \downarrow{\scriptstyle s_1{}^*} & & \\
H^m\big((MZ_2)_1/Z_2, \bar{\star}\big) & \xrightarrow{F^*} & H^m(\mathcal{S}^n/Z_2) & &
\end{array}$$

Here p_2 is induced by the projection $\mathcal{S}^n \times (MZ_2)_1 \to (MZ_2)_1$ and F is induced by $\bar{F}\colon \mathcal{S}^n \to (MZ_2)_1$, where

$$\bar{F}(x) = \sum f_0(xg^{-1})g\ .$$

Precisely as in [8, pp. 87–88] or in [12, proof of lemma 2.3] it is shown that $p_2{}^*$ is an isomorphism and that $F^* = 0$. But then $s_1{}^* j_1{}^* = 0$ and (2.14) follows by exactness of the row.

The final step in the proof of proposition 2.1 is

Lemma 2.4. *If f satisfies* (2.1′), *then* (2.12) *holds.*

Proof. Let σ be the orientation class of $\overline{\Delta M}$ and put

$$\varphi = s^*(\iota\bar{\gamma})_{j_*}\sigma\ .$$

By the so-often-referred-to non-singularity of the cup-product-pairing to $H^n(\mathscr{S}^n/G)$ it suffices to prove that

$$\varphi \cup - \ :\ H^{n-km}(\mathscr{S}^n/G) \to H^n(\mathscr{S}^n/G)$$

vanishes. Hence let $x \in H^{n-km}(\mathscr{S}^n/G)$. As in digression 1 there is an open $U_x \supseteqq A(\mu;f)/G$ such that

$$\text{(2.6)} \qquad x \in \operatorname{Im}[H^{n-km}(\mathscr{S}^n/G, U_x) \to H^{n-km}(\mathscr{S}^n/G)]$$

holds. Now choose a closed neighbourhood V of $A(\mu;f)/G$ with $V \subseteqq U_x$. There is then a neighbourhood W of $\overline{\Delta M}$ (in $\overline{MG}$) with $s^{-1}(W) \subseteqq V$. Thus s gives a map

$$(\mathscr{S}^n/G,\ \mathscr{S}^n/G - V) \to (\overline{MG},\ \overline{MG} - W)\ ,$$

and one gets a commutative diagram

$$\begin{array}{ccccccc}
 & \nearrow & H_{n+m}(W) & \xrightarrow{\iota\bar{\gamma}} & H^{km}(\overline{MG},\ \overline{MG} - W) & \xrightarrow{s^*} & H^{km}(\mathscr{S}^n/G,\ \mathscr{S}^n/G - V) \\
H_{n+m}(\overline{\Delta M}) & & & & & & \downarrow \\
 & \searrow^{j_*} & H_{n+m}(\overline{MG}) & \xrightarrow{\iota\bar{\gamma}} & H^{km}(\overline{MG}) & \xrightarrow{s^*} & H^{km}(\mathscr{S}^n/G)
\end{array}$$

from which it is seen that

$$\text{(2.15)} \qquad \varphi \in \operatorname{Im}[H^{km}(\mathscr{S}^n/G,\ \mathscr{S}^n/G - V) \to H^{km}(\mathscr{S}^n/G)]\ .$$

Since

$$(\mathscr{S}^n/G - V) \cup U_x = \mathscr{S}^n/G\ ,$$

(2.6) and (2.15) imply that $\varphi \cup x = 0$ as desired.

Hereby proposition 2.1 is proved. Borsuk–Ulam-type theorems can be derived from it by computing $e_q(\xi_\mu)^m$. This is done in the next section.

3. Computation of $e_q(\xi_\mu)^m$.

In this section $e_q(\xi_\mu)^m$ is computed for all proper actions $\mu: S^n \times G \to S^n$ of some G upon S^n and for all values of m. For actions $\mu: \mathscr{S}^n \times G \to \mathscr{S}^n$ only partial results have been obtained.

PROPOSITION 3.1. *In the following cases,* $e_q(\xi_\mu)^m \neq 0$:

$$G = Z_q \quad \text{and} \quad n \geqq (q-1)m\,, \tag{3.1}$$

$$G = Z_4, \quad m = 1, \quad \text{and} \quad n \geqq 3\,. \tag{3.2}$$

If G is periodic (hence especially if $\mathscr{S}^n = S^n$), then there are no other cases with $e_q(\xi_\mu)^m \neq 0$.

PROOF. This is divided into small steps; it is always assumed that $n \geqq (|G|-1)m$. Notice that the rings $H^*(G)$ and $H^*(\mathscr{S}^n/G)$ coincide in dimensions $<n$ (see, for example, p. 356 of [5]); this makes it possible to carry out most of the computations within $H^*(G)$.

STEP 1. If G is not q-primary, then $e_q(\xi_\mu) = 0$.

Choose a q-Sylow subgroup G_q of G. Further let $i: G_q \subseteqq G$ and $\pi: \mathscr{S}^n/G_q \to \mathscr{S}^n/G$ be inclusion and projection, respectively. There is then a commutative diagram

$$\begin{array}{ccccccccc}
\dots \to & H^j(G) & \to & H^j(\mathscr{S}^n/G) & \to & H^{j-n}(G) & \to & H^{j+1}(G) & \to \dots \\
& \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & \\
\dots \to & H^j(G_q) & \to & H^j(\mathscr{S}^n/G_q) & \to & H^{j-n}(G_q) & \to & H^{j+1}(G_q) & \to \dots
\end{array}$$

with the rows exact (see, for example, p. 358 of [5]) and i^* monic (p. 259 of [5]). From the diagram one sees that

$$\pi^*: H^{|G|-1}(\mathscr{S}^n/G) \to H^{|G|-1}(\mathscr{S}^n/G_q)$$

is monic. Hence we just have to prove that $e_q(\pi^*\xi_\mu) = 0$.

By (30.14) of [9], RG_q-modules are isomorphic precisely when they have the same characters. An easy computation of characters then shows that

$$IG \cong [G : G_q]IG_q \oplus ([G : G_q]-1)R_{\text{triv}}$$

(as RG_q-modules). Here R_{triv} is the the trivial RG_q-module. But then $\pi^*(\xi_\mu)$ splits off $([G{:}G_q]-1)$ trivial line bundles and $e_q(\pi^*\xi_\mu) = 0$.

Now we concentrate on periodic q-primary groups; these are the cyclic groups Z_{q^α} and (for $q=2$) the generalized quaternion groups $Q2^\alpha$ of order $2^\alpha \geqq 8$ (see p. 262 of [5]).

STEP 2. If $G = Q2^\alpha$ with $\alpha \geqq 3$, then $e_2(\xi_\mu) = 0$.

Recall that $Q2^\alpha$ has generators a, b and relations $a^{2^{\alpha-1}} = 1$, $b^2 = a^{2^{\alpha-2}}$, $bab^{-1} = a^{-1}$. There are three non-trivial 1-dimensional $RQ2^\alpha$-modules R_1, R_2, R_3 given by their associated representations $T_i\colon Q2^\alpha \to Z_2$ as follows:

$$T_1(a) = -1, \quad T_1(b) = 1,$$
$$T_2(a) = 1, \quad T_2(b) = -1,$$
$$T_3(a) = -1, \quad T_3(b) = -1.$$

Clearly $R_3 = R_1 \otimes R_2$, and

$$R_1 \oplus R_2 \oplus (R_1 \otimes R_2) = R_1 \oplus R_2 \oplus R_3$$

is a submodule of $IQ2^\alpha$. With λ_i denoting the real line bundle $(\mathcal{S}^n \times_{Q2^\alpha} R_i \to \mathcal{S}^n/Q2^\alpha)$ it follows that $\lambda_1 \oplus \lambda_2 \oplus (\lambda_1 \otimes \lambda_2)$ is a subbundle of ξ_μ. Hence $e_2(\xi_\mu)$ contains

$$K = e_2(\lambda_1 \oplus \lambda_2 \oplus (\lambda_1 \otimes \lambda_2)) = e_2(\lambda_1)^2 e_2(\lambda_2) + e_2(\lambda_1) e_2(\lambda_2)^2$$

as a factor.

In [13] it is shown that

$$H^*(Q2^\alpha) = Z_2[x, y, w]/I_\alpha,$$

where

$$\deg x = \deg y = 1, \quad \deg w = 4,$$
$$I_3 = \{x^2 + xy + y^2, y^3\},$$
$$I_\alpha = \{x^2 + xy, y^3\} \quad \text{for } \alpha > 3.$$

But then (no matter what is $e_2(\lambda_i)$) one gets $K = 0$. And $e_2(\xi_\mu) = 0$ as promised.

STEP 3. If $G = Z_{q^\alpha}$, then $e_q(\xi_\mu)^m = 0$ unless

$$\alpha = 1,\ n \geqq (q-1)m \quad \text{or} \quad q = 2,\ \alpha = 2,\ m = 1,\ n \geqq 3.$$

Make the complex numbers C into a CZ_{q^α}-module by the formula

$$cg = \exp(2\pi(-1)^{\frac{1}{2}} q^{-\alpha}) c$$

(g a generator of Z_{q^α}). If $q = 2$, make the real line R into an RZ_{2^α}-module using the formula $rg = -r$. If λ is the principal Z_{q^α}-bundle $(\mathcal{S}^n \to \mathcal{S}^n/Z_{q^\alpha})$, consider the complex [real] line bundle

$$\lambda_C = (\mathcal{S}^n \times_{Z_{q^\alpha}} C \to \mathcal{S}^n/Z_{q^\alpha}),$$
$$[\lambda_R = (\mathcal{S}^n \times_{Z_{2^\alpha}} R \to \mathcal{S}^n/Z_{2^\alpha})]$$

obtained by screwing in C $[R]$ as fiber in λ. Also for any complex vector bundle ζ denote its underlying real vector bundle by $\varrho(\zeta)$, and let ζ^l be the complex tensor product of l copies of ζ.

A comparison of characters reveals the following isomorphisms of RZ_{q^α}-modules

$$IZ_{q^\alpha} = C \oplus (C\otimes C) \oplus \ldots \oplus (C\otimes C\otimes \ldots \otimes C), \qquad q \text{ odd},$$
$$IZ_{2^\alpha} = R \oplus C \oplus (C\otimes C) \oplus \ldots \oplus (C\otimes \ldots \otimes C), \qquad q=2,$$

from which one gets immediately

$$(3.3) \qquad \xi_\mu = \varrho(\lambda_C) \oplus \varrho(\lambda_C{}^2) \oplus \ldots \oplus \varrho(\lambda_C{}^{\frac{1}{2}(q^\alpha-1)}), \qquad q \text{ odd},$$

$$(3.4) \qquad \xi_\mu = \lambda_R \oplus \varrho(\lambda_C) \oplus \varrho(\lambda_C{}^2) \oplus \ldots \oplus \varrho(\lambda_C{}^{2^{\alpha-1}-1}), \qquad q=2.$$

Using well-known properties of Euler classes these in turn imply

$$(3.5) \qquad e_q(\xi_\mu) = \left(\tfrac{1}{2}(q^\alpha-1)\right)!\, e_q(\lambda_C)^{\frac{1}{2}(q^\alpha-1)}, \qquad q \text{ odd},$$

$$(3.6) \qquad e_2(\xi_\mu) = (2^{\alpha-1}-1)!\, e_2(\lambda_C)^{2^{\alpha-1}-1} e_2(\lambda_R), \qquad q=2.$$

Since

$$H^{q^\alpha-1}(\mathscr{S}^n/Z_{q^\alpha}) = Z_q$$

(if $n > q^\alpha - 1$ this is a fact about $H^*(Z_{q^\alpha})$; for $n = q^\alpha - 1$ it is a fact about the (Z_q-orientable) manifold $\mathscr{S}^n/Z_{q^\alpha}$), the numerical factors in (3.5) and (3.6) will kill $e_q(\xi_\mu)$ except for q odd and $\alpha = 1$ or $q=2$ and $\alpha \leqq 2$. Hence step 3 may be finished by showing that $e_2(\xi_\mu)^m = 0$ for $m > 1$, $q=2$, and $\alpha > 1$. But this is obvious since $e_2(\lambda_R)^2 = 0$ (recall that the one-dimensional generator of $H^*(Z_{2^\alpha})$ has vanishing square when $\alpha > 1$).

The above 3 steps prove the last part of the proposition. The first part will be proved in the next three steps.

Step 4. With $G = Z_q$ and q odd, $e_q(\xi_\mu)^m \neq 0$ for all m with $n \geqq (q-1)m$.

Using Lefschetz fixed point theorem in the form given on p. 224 of [10], say, it is easily seen that n must be odd. Hence n is actually $> (q-1)m$, and all computations can take place in $H^*(Z_q)$. Since

$$e_q(\xi_\mu)^m = \left(\left(\tfrac{1}{2}(q-1)\right)!\right)^m e_q(\lambda_C)^{\frac{1}{2}(q-1)m}$$

with

$$e_q(\lambda_C) \in H^2(Z_q)$$

and

$$H^*(Z_q) = \Lambda(x) \otimes Z_q(\beta x)$$

for any non-zero $x \in H^1(Z_q)$, it is sufficient to show that $e_q(\lambda_C)$ is non-zero.

To do so identify the set of isomorphism classes of principal Z_q-bundles over $\mathscr{S}^n/Z_q$ with

$$H^1(\mathscr{S}^n/Z_q) = H^1(Z_q)$$

(for this identification, see [11]). Then λ becomes a generator of $H^1(Z_q)$ (a cross section of λ would give rise to a Z_q-equivariant map $\mathscr{S}^n \to Z_q$ which is nonsense). Therefore $\beta\lambda \neq 0$. But $\beta\lambda$ is precisely $e_q(\lambda_C)$ (this is most easily seen in the universal example $S^N \to S^N/Z_q$).

Step 5. With $G = Z_2$ one has $e_2(\xi_\mu)^m \neq 0$ for all m with $n \geqq m$.

Since

$$e_2(\xi_\mu)^m = e_2(\lambda_R)^m$$

and

$$H^*(Z_2) = Z_2[x]$$

with x one-dimensional, it suffices to show that $e_2(\lambda_R) \neq 0$ (if $n = m$ use non-singularity of the cup-product-pairing to the top-dimension). But

$$e_2(\lambda_R) = w_1(\lambda_R) = \lambda$$

under the identification of $H^1(\mathscr{S}^n/Z_2)$ with the set of isomorphism classes of principal Z_2-bundles over $\mathscr{S}^n/Z_2$. Hence it is enough to see that λ does not admit a section. This is obvious (as above in step 4).

Step 6. With $G = Z_4$ one has $e_2(\xi_\mu) \neq 0$.

Here

$$e_2(\xi_\mu) = e_2(\lambda_R) e_2(\lambda_C)$$

and

$$H^*(Z_4) = \Lambda(x) \otimes Z_2[y]$$

with x one-dimensional and y two-dimensional. Hence one just has to prove that

$$e_2(\lambda_C) \neq 0 \neq e_2(\lambda_R) .$$

This is left to the reader.

4. The main theorem.

From propositions 2.1 and 3.1 one derives the following

Theorem 4.1. *Let $\mu : \mathscr{S}^n \times Z_p \to \mathscr{S}^n$ be a proper action. Let $f : \mathscr{S}^n \to M^m$ be a nice map into an m-manifold M^m. If $p = 2$, assume that*

$$f_* = 0 : H_n(\mathscr{S}^n ; Z_2) \to H_n(M^m ; Z_2) .$$

Then

$$\mathrm{cd}(A(\mu ; f)) \geqq n - (p-1)m .$$

PROOF. Any Z_p-action on $\mathscr{S}^n$ is automatically Z_p-orientation preserving.

The two theorems in the introduction are immediate corollaries (in view of the remarks after the definition of *nice*).

There is of course also a mod 4 Borsuk–Ulam theorem for nice maps $\mathscr{S}^n \to S^1$.

5. Concluding remarks and an example.

I present first an example showing that the inequality in the mod p Borsuk–Ulam theorem cannot in general be strengthened. View S^{2n+1} as the unit sphere in C^{n+1}, and let Z_p act upon S^{2n+1} via multiplication by the powers of a p^{th} root of unity ϱ. Define

$$f = (f_1,\ldots,f_m) : S^{2n+1} \to R^m$$

by the formulae

$$f_i(z_0,\ldots,z_n) = \operatorname{Im}\left(\sum_{\nu=1}^{k} \sigma_\nu(z_{k(i-1)},\ldots,z_{ki-1})\right),$$

where $k=\frac{1}{2}(p-1)$, $\operatorname{Im} z$ denotes the imaginary part of the complex number z, and σ_ν is the ν^{th} elementary symmetric polynomial. The defining condition for $A(\mu\,;f)$ then reads

$$\operatorname{Im}\left(\sum_{\nu=1}^{k} \sigma_\nu(z_{k(i-1)},\ldots,z_{ki-1})\varrho^{j\nu} - \sum_{\nu=1}^{k} \sigma_\nu(z_{k(i-1)},\ldots,z_{ki-1})\right) = 0$$

for $j=1,2,\ldots,p$, $i=1,2,\ldots,m$. This system of equations may be solved as follows. Introduce the polynomials

$$Q_{z,i}(t) = \sum_{\nu=1}^{k} \sigma_\nu(z_{k(i-1)},\ldots,z_{ki-1})(t^\nu-1)\,.$$

The defining equations for $A(\mu\,;f)$ then read

$$Q_{z,i}(\varrho^j) \in R \quad \text{for } j=1,2,\ldots,p,\ i=1,2,\ldots,m\,.$$

It is easy to prove that if Q is a complex polynomial of degree $k=\frac{1}{2}(p-1)$ taking real values for all p^{th} roots of unity, then Q is a real constant. The solution of the defining equations for $A(\mu\,;f)$ therefore is

$$\sigma_\nu(z_{k(i-1)},\ldots,z_{ki-1}) = 0, \quad \nu=1,2,\ldots,k,\ i=1,2,\ldots,m;$$

that is,

$$A(\mu\,;f) = \{(z_0,\ldots,z_n)\in S^{2n+1} \mid z_0=\ldots=z_{km-1}=0\}\,.$$

This is a sphere of dimension $2(n-km+1)-1=2n+1-(p-1)m$, hence of cohomological dimension $2n+1-(p-1)m$.

Finally a few remarks concerning theorem 4.1:

The most obvious Z_p-action on a mod p homology n-sphere is obtained as follows. Let l be prime to p and take the usual Z_{pl}-action on S^{2n+1}. Factoring out the Z_l-action gives an

$$\mathscr{S}^{2n+1} = S^{2n+1}/Z_l = L_l^{2n+1}$$

(a lens space) and there remains a proper Z_p-action

$$\mu : L_l^{2n+1} \times Z_p \to L_l^{2n+1} .$$

It seems unlikely that every map $f: L_l^{2n+1} \to M^m$ should be *nice*. However, it is easily seen that whether f is nice or not, one has

$$\operatorname{cd}(A(\mu ; f), Z_p) \geqq (2n+1) - (p-1)m .$$

In fact, one just has to use the mod p Borsuk–Ulam theorem on the composition

$$S^{2n+1} \xrightarrow{\pi} S^{2n+1}/Z_l \xrightarrow{f} M^m$$

with Z_p-action μ' on S^{2n+1} and notice that

$$A(\mu', f\pi)/Z_l = A(\mu ; f) .$$

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UNIVERSITY OF AARHUS, DENMARK