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FINITELY ADDITIVE MEASURES ON THE NON-NEGATIVE INTEGERS

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1. Introduction.

1.1. Let S be an arbitrary semi-group and let $B(S)$ denote the Banach space of all bounded, complex-valued functions on S with the uniform norm. The conjugate space $B^*(S)$ of $B(S)$ is a convolution algebra in the sense of Definition 2.1. On page 275 of [2], Hewitt and Ross list $B^*(S)$ as one of the less tractable convolution algebras even in the case where S is the group N of integers.

Hewitt and Zuckerman [3] have obtained interesting results on a particular subalgebra $l_1(S)$ of $B^*(S)$. The subalgebra $l_1(S)$ is defined as follows. For $x \in S$, let E_x denote the linear functional on $B(S)$ whose value at f is $f(x)$; thus $E_x(f)=f(x)$. We denote by $l_1(S)$ the set of all $A \in B^*(S)$ having the form $A=\sum_{x\in S}\alpha(x)E_x$, where α is a complex-valued function on S for which $\sum_{x\in S}|\alpha(x)|<\infty$. In this sum, it is understood that the set $\{x \in S : \alpha(x) \neq 0\}$ is countable, being written, say, as $\{x_1,x_2,\ldots,x_n,\ldots\}$ and that $\sum_{n=1}^{\infty}|\alpha(n)|$ is finite. Hewitt and Zuckerman have described all the homomorphisms of $l_1(S)$ onto the complex numbers:

1.2. THEOREM. *Let S be an arbitrary semigroup and let τ be a multiplicative linear functional on $l_1(S)$ different from zero. Then τ is a bounded linear functional and there exists a semicharacter χ of S such that for all $A=\sum_{x\in S}\alpha(x)E_x \in l_1(S)$, we have*

$$\tau(A) = \sum_{x\in S} \alpha(x)\,\chi(x)\,. \tag{1.2.1}$$

Conversely, every semicharacter χ of S defines a bounded multiplicative linear functional by (1.2.1) *and two distinct semicharacters define two distinct multiplicative linear functionals.*

For commutative S, the subalgebra $l_1(S)$ is commutative and Theorem 1.2 gives all the regular maximal ideals of $l_1(S)$.

In the present paper, we let N_0 denote the semigroup, under addition,

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of all non-negative integers and consider the problem of finding all the maximal two-sided ideals of the algebra $B^*(N_0)$. We show in Theorem 3.1 that $B^*(N_0)$ admits a decomposition: $B^*(N_0)=l_1(N_0)\oplus P$, where P denotes the set of all the measures on N_0 which vanish on all finite subsets of N_0. Furthermore, P is a two-sided ideal of $B^*(N_0)$. Using this result we are able to identify some of the maximal two-sided ideals of $B^*(N_0)$. Precisely, we show (Theorem 4.2) that if J is a maximal ideal of $l_1(N_0)$, then $J\oplus P$ is a maximal two-sided ideal of $B^*(N_0)$. If χ is a semicharacter of N_0, then the mapping $L \to \int \chi(n)d\lambda(n)$ (see Section 3.1) is a homomorphism of $B^*(N_0)$ onto the complex numbers, and the kernel of the homomorphism is a maximal two-sided ideal of $B^*(N_0)$. We have so far been unable to identify other maximal two-sided ideals of $B^*(N_0)$. In Section 5, we show that P consists of all the purely finitely additive measures on N_0 (see Definition 5.4), while $l_1(N_0)$ consists of all the countably additive measures on N_0.

I like to express my gratitude to Professor Edwin Hewitt who inspired and encouraged this investigation.

2. Preliminaries.

2.1. Definition. Let S be a semi-group and let $\mathscr{F}$ be a linear space of real- or complex-valued functions on S. Suppose that, for every $x \in S$, ${}_xf \in \mathscr{F}$ whenever $f \in \mathscr{F}$, where ${}_xf$ is the function on S defined by ${}_xf(y)=f(xy)$. For a linear functional M on $\mathscr{F}$, let $\bar{M}f$ be the function on S such that $\bar{M}f(x)=M({}_xf)$ for all $x \in S$. Suppose further that M is such that $\bar{M}f \in \mathscr{F}$ whenever $f \in \mathscr{F}$. Then if L is any linear functional on $\mathscr{F}$, the functional $L*M$ whose value at f is $L(\bar{M}f)$ is called the convolution of L and M.

If $\mathscr{F}$ is as above and $\mathscr{L}$ is a linear space (real or complex according as $\mathscr{F}$ is a real or complex linear space) of linear functionals on $\mathscr{F}$ such that $L*M$ exists and is in $\mathscr{L}$ for all $L, M \in \mathscr{L}$, then $\mathscr{L}$ is called a convolution algebra with $*$ as multiplication, and addition and scalar multiplication defined in the usual way. (See [2, page 262].)

2.2 Let $B(N_0)$ denote the set of all bounded, complex-valued functions on the non-negative integers N_0. Under the uniform norm, $B(N_0)$ is a Banach space. The set of all bounded linear functionals M on $B(N_0)$ under the norm $\|M\|=\sup\{|M(f)| : f \in B(N_0), \|f\| \leqq 1\}$ will be denoted by $B^*(N_0)$.

2.3. Let l_1 denote the set of all complex-valued sequences x such that $\sum_{n=0}^{\infty}|x(n)| < \infty$. We shall regard l_1 as a semi-group under the multiplication defined by

$$(x*y)(n) = \sum_{k=0}^{n} x(k)\, y(n-k)\,.$$

If γ_n denotes the characteristic function of the point $n \in N_0$, then $\gamma_{m+n} = \gamma_m * \gamma_n$. Hence we may, and shall, regard the additive semi-group N_0 as embedded in the semi-group l_1 by the mapping $n \to \gamma_n$.

We take $\|x\| = \sum_{n=0}^{\infty} |x(n)|$ as the norm of the sequence x, and let l_1^* denote the set of all bounded linear functionals on l_1. Whenever convenient, we shall identify l_1^* with $B(N_0)$ and use the same symbol f both for the bounded linear functional on l_1 and the bounded function on N_0 corresponding to it. Furthermore, since for the characteristic function γ_n of the point $n \in N_0$, the value of $f(\gamma_n)$ is the same as $f(n)$ under this identification, we shall write $f(n)$ for $f(\gamma_n)$ whenever convenient.

2.4. We shall denote by $l_1(N_0)$ the set of all elements of $B^*(N_0)$ arising from l_1 by the mapping $x \to E_x$, where E_x is defined on l_1^* by

$$E_x(f) = f(x) = \sum_{n=0}^{\infty} x(n)\, f(n) \qquad \text{for all } f \in l_1^*\,.$$

2.5. THEOREM. *$B^*(N_0)$ is a convolution algebra with a unit element. Furthermore, under the norm $\|M\|$, $B^*(N_0)$ is a Banach algebra.*

PROOF. Taking l_1 as S and $B(N_0)$ as $\mathscr{F}$ in Definition 2.1, it is easy to verify that $B^*(N_0)$ is a convolution algebra. For $n \in N_0$, let E_{γ_n} denote the linear functional on $B(N_0)$ defined by $E_{\gamma_n}(f) = f(\gamma_n)$ for $f \in B(N_0)$. Then $E_{\gamma_n} \in B^*(N_0)$ for every $n \in N_0$ and we have $E_{\gamma_0} * L = L * E_{\gamma_0} = L$ for every $L \in B^*(N_0)$. Thus E_{γ_0} is a unit element in $B^*(N_0)$. Finally, for $n \in N_0$, $f \in B(N_0)$ and $M \in B^*(N_0)$, we have $|M_{\gamma_n}(f)| \leqq \|M\|\,\|f\|$. Hence if $L \in B^*(N_0)$,

$$|(L*M)(f)| \leqq \|L\|\,\|M\|\,\|f\| \quad \text{and} \quad \|L*M\| \leqq \|L\|\,\|M\|\,.$$

This concludes the proof.

3. A decomposition theorem.

3.1. In this section we prove the decomposition theorem mentioned in the introduction. It is well known (see, for example, [1, page 258]) that there is a one-to-one correspondence between $B^*(N_0)$ and the set of all bounded, finitely additive complex measures μ on the ring of all subsets of N_0, this correspondence being given by

$$M(f) = \int_{N_0} f(n)\, d\mu(n)\,, \quad f \in B(N_0)\,.$$

In view of this one-to-one correspondence, we shall regard $B^*(N_0)$ either as a set of linear functionals M or as a set of measures μ.

3.2. THEOREM. *Let P denote the set of all elements M of $B^*(N_0)$ such that their corresponding measures μ vanish on all finite subsets of N_0. Then*

$$B^*(N_0) = l_1(N_0) \oplus P ,$$

and P is a two-sided ideal of $B^*(N_0)$.

PROOF. First we show that $l_1(N_0) \cap P = (0)$. In fact, let M be a non-zero element of $l_1(N_0) \cap P$, where $M = \sum \beta(n) E_{\gamma n}$. We shall let ξ_n denote the characteristic function of the point n, regarded as an element of $B(N_0)$. Then, for $n_0 \in N_0$ with $\beta(n_0) \neq 0$, we would have

$$0 = \mu(n_0) = M(\xi_{n_0}) = \sum \beta(n) \xi_{n_0}(n) = \beta(n_0) ,$$

a contradiction. Hence $l_1(N_0) \cap P = (0)$. Let M be an arbitrary element of $B^*(N_0)$, and define a function α on N_0 by setting $\alpha(n) = \mu(n)$. Then $\sum_{n=0}^{\infty} |\alpha(n)| < \infty$. In fact, in the contrary case, there would exist a sequence $n_1, n_2, \ldots, n_k$ of non-negative integers such that $\sum_{m=1}^{k} |\alpha(n_m)| > \|M\|$. Then for $f_0 = \sum_{m=1}^{k} \operatorname{sgn} \overline{\alpha(n_m)} \gamma_{n_m}$, we would have

$$M(f_0) = \sum_{m=1}^{k} \operatorname{sgn} \overline{\alpha(n_m)} \alpha(n_m) = \sum_{m=1}^{k} |\alpha(n_m)| > \|M\|$$

and $\|f_0\| = 1$, which is impossible. We must therefore have $\sum_{n=0}^{\infty} |\alpha(n)| \leqq \|M\| < \infty$. Hence $\sum_{n=0}^{\infty} \alpha(n) E_{\gamma_n} \in l_1(N_0)$. Then, if ε_n is the measure corresponding to E_{γ_n},

$$(\mu - \sum_{n=0}^{\infty} \alpha(n) \varepsilon_n)(m) = 0$$

for any $m \in N_0$ and $\mu - \sum \alpha(n) \varepsilon_n \in P$. This proves the first assertion of the theorem.

To show that P is a left ideal, let $M \in P$ and $L \in B^*(N_0)$. Then if E is a finite subset of N_0 and γ_E is the characteristic function of E, we have

$$(L*M)(\gamma_E) = L(\bar{M}\gamma_E) = \int_{N_0} (\bar{M}\gamma_E)(n) \, d\lambda(n)$$

$$= \int_{N_0} \left[\int_{N_0} \gamma_E(n+m) \, d\mu(m) \right] d\lambda(n) = 0 ,$$

since the integral in the square brackets is zero. Hence P is a left ideal. To prove that P is a right ideal, we first note that $l_1(N_0)$ is in the centre of $B^*(N_0)$. In fact, for $x \in l_1, f \in B(N_0)$, we have

$$(\bar{E}_x f)(y) = E_x({}_y f) = {}_y f(x) = f(xy) = {}_x f(y) ,$$

for every $y \in l_1$. Hence $\bar{E}_x f = {}_x f$. If $L \in B^*(N_0)$, we have

$$(L*E_x)(f) = L(\bar{E}_x f) = L({}_x f) = (\bar{L}f)(x) = (E_x*L)(f) ,$$

and

$$L*E_x = E_x*L .$$

Now let L be an arbitrary element of $B^*(N_0)$ and let $M \in P$. We write $L = L_1 + L_2$, where $L_1 \in l_1(N_0)$ and $L_2 \in P$. Then

$$M*L = M*L_1 + M*L_2$$

and since $M*L_1 = L_1*M$, we have, for any finite subset E of N_0,

$$(M*L)(\gamma_E) = (L_1*M)(\gamma_E) + (M*L_2)(\gamma_E) = 0 .$$

Thus P is a right ideal of $B^*(N_0)$ and the proof is complete.

4. Maximal two-sided ideals in $B^*(N_0)$.

We now identify some of the maximal two-sided ideals of $B^*(N_0)$. We start with a lemma.

4.1. LEMMA. *Let J be a left (right) ideal of $B^*(N_0)$ containing P and let $J_1 = J \cap l_1(N_0)$. Then $J = J_1 \oplus P$.*

PROOF. It is clear that $J_1 \oplus P \subset J$. Suppose that $L \in J$; then $L = L_1 + L_2$ where $L_1 \in l_1(N_0)$ and $L_2 \in P \subset J$. Hence $L - L_2 \in J$ or $L_1 \in J$. Thus $L_1 \in J_1$, which shows that $L = L_1 + L_2 \in J_1 \oplus P$.

4.2. THEOREM. *For every maximal ideal J of $l_1(N_0)$, $J \oplus P$ is a maximal two-sided ideal of $B^*(N_0)$.*

PROOF. First we show that $J \oplus P$ is a two-sided ideal of $B^*(N_0)$. It is easy to see that $J \oplus P$ is a subspace of $B^*(N_0)$. Let L be an arbitrary element of $B^*(N_0)$ and let $M_1 + M_2 \in J \oplus P$, where $M_1 \in J$ and $M_2 \in P$. Then if

$$L = L_1 + L_2, \qquad L_1 \in l_1(N_0),\ L_2 \in P ,$$

we have

$$\begin{aligned} L*(M_1+M_2) &= (L_1+L_2)*(M_1+M_2) \\ &= L_1*M_1 + (L_1*M_2 + L_2*M_1 + L_2*M_2) \in J \oplus P , \end{aligned}$$

since P is a two-sided ideal of $B^*(N_0)$. Hence $J \oplus P$ is a left ideal of $B^*(N_0)$. The proof that $J \oplus P$ is a right ideal is similar.

Suppose now that M is a two-sided ideal of $B^*(N_0)$ such that $J\oplus P\subset \mathsf{M}$ and set $\mathsf{M}_1=\mathsf{M}\cap l_1(N_0)$. Then M_1 is an ideal of $l_1(N_0)$ and by Lemma 4.1, $\mathsf{M}=\mathsf{M}_1\oplus P$. Furthermore, $J\subseteq \mathsf{M}_1$. If $J=\mathsf{M}_1$, then $\mathsf{M}=J\oplus P$ and M does not properly contain $J\oplus P$, a contradiction. Hence $J\subsetneqq \mathsf{M}_1$ and, since M_1 is an ideal of $l_1(N_0)$ and J is maximal, $\mathsf{M}_1=l_1(N_0)$. Thus $\mathsf{M}=B^*(N_0)$ and $J\oplus P$ is a maximal two-sided ideal of $B^*(N_0)$. This concludes the proof of the theorem.

It is interesting to obtain the result in Theorem 4.2 by considering homomorphisms of $B^*(N_0)$ onto the complex numbers.

For $L\in B^*(N_0)$, we shall as before write $L=L_1+L_2$ where $L_1\in l_1(N_0)$ and $L_2\in P$, and the corresponding measure will be written $\lambda=\lambda_1+\lambda_2$.

4.3. Theorem. *Let χ be a semicharacter of N_0. Then the mapping*

$$L \to L_1(\chi) = \int_{N_0} \chi(n)\, d\lambda_1(n)$$

is a homomorphism of $B^*(N_0)$ onto the complex numbers. The kernel of this homomorphism is a maximal two-sided ideal of $B^*(N_0)$ of the form $J\oplus P$ where J is a maximal ideal of $l_1(N_0)$.

Proof. It is clear that the mapping is linear; we shall show that it is multiplicative. If $\lambda,\mu\in B^*(N_0)$, we have

$$\begin{aligned}\lambda*\mu &= (\lambda_1+\lambda_2)*(\mu_1+\mu_2)\\ &= \lambda_1*\mu_1+\lambda_2*\mu_1+\lambda_1*\mu_2+\lambda_2*\mu_2\,.\end{aligned}$$

Since P is a two-sided ideal of $B^*(N_0)$, $\lambda_2*\mu_1+\lambda_1*\mu_2+\lambda_2*\mu_2\in P$ and hence $(\lambda*\mu)_1=\lambda_1*\mu_1$. We then have

$$\begin{aligned}L_1(\chi)M_1(\chi) &= \int_{N_0}\chi(n)\,d\lambda_1(n)\int_{N_0}\chi(s)\,d\mu_1(s)\\ &= \int_{N_0}\int_{N_0}\chi(n+s)\,d\lambda_1(n)\,d\mu_1(s)\\ &= \int_{N_0}\chi(m)\,d(\lambda_1*\mu_1)(m)\\ &= \int_{N_0}\chi(m)\,d(\lambda*\mu)_1(m) = (L*M)_1(\chi)\,.\end{aligned}$$

Hence the mapping is a homomorphism. To prove the last assertion of the theorem, let J_χ be the maximal ideal of $l_1(N_0)$ determined by χ as in Theorem 1.2 and let

$$K = \left[L \in B^*(N_0) : \int_{N_0} \chi(n) d\lambda_1(n) = 0\right].$$

Then a routine argument shows that $K = J_\chi \oplus P$.

4.4. THEOREM. *Let χ be a semicharacter of N_0. Then the mapping*

$$L \to L(\chi) = \int_{N_0} \chi(n)\, d\lambda(n)$$

is a homomorphism of $B^*(N_0)$ onto the complex numbers.

The proof of this theorem is immediate and is therefore omitted.

EXAMPLE. If in Theorem 4.4, we take χ_0 to be the semicharacter defined by $\chi_0(0) = 1$, $\chi_0(n) = 0$ if $n \neq 0$, then the kernel of the homomorphism $L \to L(\chi_0)$ is of the form $J_0 \oplus P$, where J_0 is a maximal ideal of $l_1(N_0)$. In fact, let

$$K_0 = \left[\lambda \in B^*(N_0) : \int_{N_0} \chi_0(n) d\lambda(n) = 0\right].$$

If $\lambda \in K_0$ and $\lambda = \lambda_1 + \lambda_2$, then

$$\int_{N_0} \chi_0(n) d\lambda(n) = \int_{N_0} \chi_0(n) d\lambda_1(n) + \int_{N_0} \chi_0(n) d\lambda_2(n) = 0.$$

Now since $\lambda_2 \in P$, we have $\int_{N_0} \chi_0(n) d\lambda_2(n) = 0$, and therefore $\int_{N_0} \chi_0(n) d\lambda_1(n) = 0$. This implies that λ_1 is in the maximal ideal J_0 of $l_1(N_0)$ determined by χ_0. Hence for every $\lambda = \lambda_1 + \lambda_2$ in K_0, $\lambda_1 \in J_0$ and so $K_0 \subset J_0 \oplus P$. Now let $\lambda \in J_0 \oplus P$ and write $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \in J_0$, $\lambda_2 \in P$. Then $\int_{N_0} \chi_0(n) d\lambda_1(n) = 0$ and since $\int_{N_0} \chi_0(n) d\lambda_2(n) = 0$, we have $\int_{N_0} \chi_0(n) d\lambda(n) = 0$ and $\lambda \in K_0$.

5. Purely finitely additive measures.

5.1. DEFINITION. Let X be an arbitrary set and $\mathfrak{M}$ a σ-algebra of subsets of X. Let Φ denote the set of all real-valued finitely additive measures on $\mathfrak{M}$. For $\varphi \in \Phi$, we write $\varphi \geqq 0$ if $\varphi(E) \geqq 0$ for all $E \in \mathfrak{M}$. For $\varphi, \gamma \in \Phi$, we write $\varphi \leqq \gamma$ if $\gamma - \varphi \geqq 0$.

The following two results are due to Yosida and Hewitt [4].

5.2. THEOREM. *Under the partial ordering defined in* 5.1, *the set Φ is a lattice. For arbitrary φ and $\gamma \in \Phi$, the measure $\varphi \wedge \gamma$ is defined by the relation*

$$(\varphi \wedge \gamma)(E) = \inf_{T \subset E, T \in \mathfrak{M}} (\varphi(T) + \gamma(E \cap T'))$$

for all $E \in \mathfrak{M}$. *The measure* $\varphi \vee \gamma$ *is defined by the relation*

$$\varphi \vee \gamma = -((-\varphi) \wedge (-\gamma)) .$$

5.3. Theorem. *Let* φ *be an arbitrary element of* Φ. *Writing* $\varphi \vee 0$ *as* φ_+ *and* $(-\varphi) \vee 0$ *as* φ_-, *we have the relations*

$$\varphi = \varphi_+ - \varphi_- \quad \textit{and} \quad \varphi_+ \wedge \varphi_- = 0 .$$

5.4. Definition. Let φ be a measure in Φ such that $0 \leqq \varphi$. If every countably additive measure ψ such that $0 \leqq \psi \leqq \varphi$ is identically zero, then φ is said to be purely finitely additive. If $\varphi \in \Phi$ and both φ_+ and φ_- are purely finitely additive, then φ is said to be purely finitely additive.

5.5. Let X and $\mathfrak{M}$ be as in 5.1 and let Ψ denote the set of all complex-valued finitely additive measures on $\mathfrak{M}$. If $\psi \in \Psi$, and $\psi = \psi_1 + i\psi_2$, where $\psi_1, \psi_2 \in \Phi$, we shall say that ψ is purely finitely additive if both ψ_1 and ψ_2 are purely finitely additive.

5.6. Lemma. *If* $\mu \in P$ *is a real-valued measure and* $\mu = \mu_+ - \mu_-$, *then* $\mu_+ \in P$, $\mu_- \in P$.

This follows at once from 5.2.

5.7. Theorem. *A measure* $\mu \in B^*(N_0)$ *is in* P *if and only if* μ *is a purely finitely additive measure on* N_0.

Proof. Let $\mu \in P$ and let $\mu \geqq 0$. If φ is a countably additive measure such that $0 \leqq \varphi \leqq \mu$, then $\varphi(E) = 0$ for every finite subset E of N_0 and, by countable additivity, $\varphi(E) = 0$ for every subset E of N_0. Hence μ is purely finitely additive. If $\mu \in P$ and μ is real valued, we have $\mu = \mu_+ - \mu_-$, where $\mu_+ \geqq 0, \mu_- \geqq 0$. By Lemma 5.6 and what has just been proved, μ_+ and μ_- are both purely finitely additive and hence μ is purely finitely additive, by 5.4. Finally, for arbitrary $\mu \in P$, we write $\mu = \mu_1 + i\mu_2$, where μ_1 and μ_2 are real-valued; then by 5.5, μ is purely finitely additive.

Suppose now that $\mu \in B^*(N_0)$ is purely finitely additive. If $\mu \geqq 0$, then ([4, Theorem 1.16]), $\mu \wedge \varphi = 0$ for all non-negative countably additive measures φ. Then, if $n_0 \in N_0$,

$$(\mu \wedge \varphi)(n_0) = \min(\mu(n_0), \varphi(n_0)) = 0$$

for all non-negative countably additive measures φ. If φ_{n_0} is taken to be the unit mass concentrated at n_0, we have

$$\min(\mu(n_0), \varphi_{n_0}(n_0)) = \min(\mu(n_0), 1) = 0$$

and $\mu(n_0) = 0$. Hence $\mu \in P$. If μ is real and purely finitely additive, with $\mu = \mu_+ - \mu_-$, then again $\mu \in P$. Finally, if μ is an arbitrary purely finitely additive measure in $B^*(N_0)$ and $\mu = \mu_1 + i\mu_2$, μ_1, μ_2 real-valued, then $\mu \in P$. This concludes the proof of the theorem.

From Theorem 1.24 of Yosida and Hewitt [4] it follows that every measure $\mu \in B^*(N_0)$ can be uniquely written as the sum of a countably additive measure μ_c and a purely finitely additive measure μ_p. We therefore have the following:

5.8. COROLLARY. *$l_1(N_0)$ is the set of all countably additive measures on N_0.*

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators* I, Interscience, New York, 1958.
2. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Springer-Verlag, 1963.
3. E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. 83 (1956), 70–97.
4. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. 72 (1952), 46–66.

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