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ON STEINER AND SIMILAR TRIPLE SYSTEMS

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1. Introduction.

Given a finite set S of s elements, a Steiner triple system over the set S is a system of triples of elements of S such that each pair of elements is contained in one and only one of the triples. It is well known [3] that a necessary and sufficient condition for the existence of a Steiner triple system is that $s=6n+1$ or $6n+3$ for some positive integer n. In [5] Th. Skolem gave a method for constructing a Steiner triple system in the case $s=6n+1$, $n\equiv 0$ or 1 modulo 4, and in [2] H. Hanani extended this to the case $s=6n+1$, $n\equiv 2$ or 3 modulo 4. In this paper a similar method for constructing a Steiner triple system in all cases is given. In [1] Fort and Hedlund gave an inductive method for constructing a minimal system of triples such that each pair occurs in at least one triple, and here a non-inductive method is given. Similarly in [4] J. Schönheim gave a construction for a maximal system of triples such that no pair is contained in more than one triple and, again, another construction is given here.

2. Construction of a Steiner triple system if $s=6n+1$.

In [5] Th. Skolem showed that the numbers $1,2,\ldots,2n$ can be distributed in n pairs with differences $1,\ldots,n$ if and only if $n\equiv 0$ or 1 modulo 4. In [6] he used this to construct a Steiner triple system when $s=6n+1$ and $n\equiv 0$ or 1 modulo 4. He also remarked that a necessary condition for the distribution of the integers $1,\ldots,2n-1,2n+1$ into n pairs with differences $1,\ldots,n$ is $n\equiv 2$ or 3 modulo 4 and noted that if such a distribution could be found, then a Steiner triple system in the case $n\equiv 2$ or 3 modulo 4 could be constructed. Later H. Hanani [2] showed that if $n\equiv 3$ modulo 4, then the integers $1,\ldots,n,n+2,\ldots,2n+1$ can be distributed into n pairs with differences $1,\ldots,n$ and used this to construct a Steiner triple system in the case $s=6n+1$, $n\equiv 2$ or 3 modulo 4.

Firstly, for $n\equiv 0$ or 1 modulo 4, different distributions of the numbers $1,\ldots,2n$ into n pairs (a_r,b_r) with $b_r-a_r=r$, $r=1,\ldots,n$, from those of

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Skolem are given. For $n=4m$ and $m \geqq 1$ the distribution is given in the following chart.

r	a_r	b_r	
2α	$2m+1-\alpha$	$2m+1+\alpha$	$\alpha=1,2,\ldots,2m;$
1	$7m$	$7m+1$	
$1+2\alpha$	$6m-\alpha$	$6m+1+\alpha$	$\alpha=1,2,\ldots,m-2;$
$2m-1$	$4m+2$	$6m+1$	$m \geqq 2;$
$2m-1+2\alpha$	$5m+2-\alpha$	$7m+1+\alpha$	$\alpha=1,2,\ldots,m-1;$
$3m-1$	$2m+1$	$6m.$	

Chart 1.

For $n=4m+1$ and $m \geqq 2$ the distribution is

r	a_r	b_r	
2α	$2m+1-\alpha$	$2m+1+\alpha$	$\alpha=1,2,\ldots,2m;$
1	$5m+2$	$5m+3$	
$1+2\alpha$	$6m+2-\alpha$	$6m+3+\alpha$	$\alpha=1,2,\ldots,m-2;$
$2m-1$	$6m+3$	$8m+2$	
$2m-1+2\alpha$	$5m+2-\alpha$	$7m+1+\alpha$	$\alpha=1,2,\ldots,m;$
$4m+1$	$2m+1$	$6m+2.$	

Chart 2.

Next, for $n \equiv 2$ or 3 modulo 4, distributions of the integers $1,\ldots,2n-1,2n+1$ into n pairs (a_r,b_r) with $b_r-a_r=r$, $r=1,\ldots,n$, are given, which answers Skolem's query [6]. The distributions of the integers $1,\ldots,2n$ given above are very similar to each other and further, they are very similar to the distributions of the integers $1,\ldots,2n-1,2n+1$ given immediately below. For $n=4m+2$ and $m \geqq 2$ the distribution is

r	a_r	b_r	
2α	$2m+2-\alpha$	$2m+2+\alpha$	$\alpha=1,2,\ldots,2m+1;$
1	$7m+4$	$7m+5$	
$1+2\alpha$	$6m+2-\alpha$	$6m+3+\alpha$	$\alpha=1,2,\ldots,m;$
$2m+3$	$6m+2$	$8m+5$	
$2m+3+2\alpha$	$5m+2-\alpha$	$7m+5+\alpha$	$\alpha=1,2,\ldots,m-2;$
$4m+1$	$2m+2$	$6m+3\ .$	

For $n=4m+3$ and $m\geqq 1$ the distribution is

r	a_r	b_r	
2α	$2m+2-\alpha$	$2m+2+\alpha$	$\alpha=1,2,\ldots,2m+1;$
1	$5m+4$	$5m+5$	
$1+2\alpha$	$6m+5-\alpha$	$6m+6+\alpha$	$\alpha=1,2,\ldots,m-1;$
$2m+1$	$6m+6$	$8m+7$	
$2m+1+2\alpha$	$5m+4-\alpha$	$7m+5+\alpha$	$\alpha=1,2,\ldots,m;$
$4m+3$	$2m+2$	$6m+5$.	

Now let A be the set of all triples

$$\{x,x+r,x+n+b_r\},\qquad r=1,\ldots,n;\ x=1,\ldots,6n+1\ ,$$

where each number is taken modulo $6n+1$. Then A is a Steiner triple system. The number of distinct triples of A is at most $n(6n+1)=s(s+1)/6$ which is well-known [3] to be the number of triples in a Steiner triple system. It is therefore only necessary to show that each pair $\{y,z\}$ of unequal numbers between 1 and $6n+1$ occurs at least once in a triple of A.

It is clear that

$$(1,2,\ldots,n,n+a_1,n+a_2,\ldots,n+a_n,n+b_1,n+b_2,\ldots,n+b_n)$$

is a permutation of $(1,\ldots,3n)$ if $n\equiv 0$ or 1 modulo 4, or of $(1,\ldots,3n-1, 3n+1)$ if $n\equiv 2$ or 3 modulo 4. Exactly one of the differences of the pair $\{y,z\}$ is congruent modulo $6n+1$ to one of $1,\ldots,3n$ and similarly exactly one is congruent modulo $6n+1$ to one of $1,\ldots,3n-1,3n+1$. We may therefore assume that $z-y$ is congruent modulo $6n+1$ to one of $1,\ldots,3n$ if $n\equiv 0$ or 1 modulo 4, and to one of $1,\ldots,3n-1,3n+1$ if $n\equiv 2$ or 3 modulo 4. For some r, $1\leqq r\leqq n$, if $z-y=r$ or $n+b_r$, then

$$\{y,z\}\subset\{y,y+r,y+n+b_r\}\in A\ ,$$

and if $z-y=n+a_r$, then

$$\{y,z\}\subset\{z-n-b_r,z-n-a_r,z\}\in A\ .$$

It follows therefore that A is a Steiner triple system.

3. Construction of a Steiner triple system if $s=6n+3$.

A very similar method exists in this case to that already described in the case $s=6n+1$. Firstly, if $n\equiv 0$ or 3 modulo 4, the integers $1,\ldots,n$, $n+2,\ldots,2n+1$ are distributed into n pairs (c_r,d_r) with $d_r-c_r=r$, $r=1,\ldots,n$. The distribution when $n=4m$ and $m\geqq 1$ is as follows:

r	c_r	d_r	
2α	$2m-\alpha$	$2m+\alpha$	$\alpha=1,2,\ldots,2m-1;$
$4m$	$2m$	$6m$	
1	$7m+1$	$7m+2$	
$1+2\alpha$	$6m-\alpha$	$6m+1+\alpha$	$\alpha=1,2,\ldots,m-1;$
$2m+1$	$4m$	$6m+1$	
$2m+1+2\alpha$	$5m+1-\alpha$	$7m+2+\alpha$	$\alpha=1,2,\ldots,m-1.$

The distribution when $n=4m+3$ and $m\geqq 0$ is given in the chart immediately below, and was given by H. Hanani in [2].

r	c_r	d_r	
2α	$2m+2-\alpha$	$2m+2+\alpha$	$\alpha=1,2,\ldots,2m+1;$
1	$7m+6$	$7m+7$	
$1+2\alpha$	$6m+4-\alpha$	$6m+5+\alpha$	$\alpha=1,2,\ldots,m;$
$2m+3$	$6m+4$	$8m+7$	$m\geqq 1;$
$2m+3+2\alpha$	$5m+4-\alpha$	$7m+7+\alpha$	$\alpha=1,2,\ldots,m-1;$
$4m+3$	$2m+2$	$6m+5.$	

Next, for $n\equiv 1$ or 2 modulo 4 the integers $1,\ldots,n,n+2,\ldots,2n,2n+2$ are distributed into n pairs (c_r,d_r) with $d_r-c_r=r$, $r=1,2,\ldots,n$. For $n=4m+1$ and $m\geqq 2$ the distribution is

r	c_r	d_r	
2α	$2m+1-\alpha$	$2m+1+\alpha$	$\alpha=1,2,\ldots,2m;$
1	$7m+3$	$7m+4$	
$1+2\alpha$	$6m+1-\alpha$	$6m+2+\alpha$	$\alpha=1,2,\ldots,m;$
$2m+3$	$6m+1$	$8m+4$	
$2m+3+2\alpha$	$5m+1-\alpha$	$7m+4+\alpha$	$\alpha=1,2,\ldots,m-2;$
$4m+1$	$2m+1$	$6m+2.$	

The distribution for $n=4m+2$ and $m\geqq 2$ is

r	c_r	d_r	
2α	$2m+1-\alpha$	$2m+1+\alpha$	$\alpha=1,2,\ldots,2m;$
$4m+2$	$4m+2$	$8m+4$	
1	$7m+4$	$7m+5$	
$1+2\alpha$	$6m+2-\alpha$	$6m+3+\alpha$	$\alpha=1,2,\ldots,m;$
$2m+3$	$6m+3$	$8m+6$	
$2m+3+2\alpha$	$5m+2-\alpha$	$7m+5+\alpha$	$\alpha=1,2,\ldots,m-2;$
$4m+1$	$2m+1$	$6m+2.$	

Now let B be the set of all triples

$$\{x, x+r, x+n+d_r\}, \qquad r=1,\ldots,n;\ x=1,\ldots,6n+3\ ,$$

where each number is taken modulo $6n+3$, and let C be the set of all triples

$$\{x, x+2n+1, x+4n+2\}, \qquad x=1,\ldots,2n+1\ .$$

Then the union D of B and C is a Steiner triple system. The number of distinct triples of D is at most $n(6n+3)+2n+1=(6n+3)(6n+2)/6=s(s+1)/6$, which is the correct number of triples for a Steiner triple system. It is therefore only necessary to verify that each pair $\{y,z\}$ of unequal numbers between 1 and $6n+3$ occurs at least once in a triple of D. We omit this verification as it is very similar to the one at the end of section 2.

4. Further distributions and a lemma.

If $n \equiv 0$ or 1 modulo 4 then, as shown in charts 1 and 2 of section 2, it is possible to distribute the integers $1,\ldots,2n$ into n pairs (e_r,f_r) with $f_r-e_r=r$, $r=1,\ldots,n$ (putting $a_r=e_r$ and $b_r=f_r$ for $r=1,\ldots,n$). For $n\equiv 2$ or 3 modulo 4 distributions are now given of the integers $1,\ldots,2n-1, 2n+3$ into n pairs (e_r,f_r) with $f_r-e_r=r$, $r=1,\ldots,n$. For $n=4m+2$ and $m \geqq 3$ the distribution is

r	e_r	f_r	
2α	$2m+2-\alpha$	$2m+2+\alpha$	$\alpha=1,2,\ldots,2m+1;$
1	$7m+5$	$7m+6$	
$1+2\alpha$	$6m+2-\alpha$	$6m+3+\alpha$	$\alpha=1,2,\ldots,m+1;$
$2m+5$	$6m+2$	$8m+7$	
$2m+5+2\alpha$	$5m+1-\alpha$	$7m+6+\alpha$	$\alpha=1,2,\ldots,m-3;$
$4m+1$	$2m+2$	$6m+3\ .$	

For $n=4m+3$ and $m \geqq 1$ the distribution is

r	e_r	f_r	
2α	$2m+2-\alpha$	$2m+2+\alpha$	$\alpha=1,2,\ldots,2m+1;$
1	$5m+3$	$5m+4$	
$1+2\alpha$	$6m+5-\alpha$	$6m+6+\alpha$	$\alpha=1,2,\ldots,m;$
$2m+3$	$6m+6$	$8m+9$	
$2m+3+2\alpha$	$5m+3-\alpha$	$7m+6+\alpha$	$\alpha=1,2,\ldots,m-1;$
$4m+3$	$2m+2$	$6m+5\ .$	

Let E be the set of all triples

$$\{x, x+r, x+n+f_r\}, \qquad x=1,\ldots,6n+3;\; r=1,\ldots,n\,,$$

where each number is taken modulo $6n+3$. The set E is the foundation of several triple systems in the next two sections.

LEMMA. *Let $\{y,z\}$ be a pair of unequal numbers between 1 and $6n+3$ and let neither of the differences of $\{y,z\}$ be congruent to $3n+1$ modulo $6n+3$. Then $\{y,z\}$ is a subset of exactly one of the triples of E.*

PROOF. It is clear that $(1,\ldots,n,n+e_1,\ldots,n+e_n,n+f_1,\ldots,n+f_n)$ is a permutation of $(1,\ldots,3n)$ if $n\equiv 0$ or 1 modulo 4 or of $(1,\ldots,3n-1,3n+3)$ if $n\equiv 2$ or 3 modulo 4. Since neither of the differences of $\{y,z\}$ are congruent to $3n+1$ modulo $6n+3$, one of the differences of $\{y,z\}$ is congruent modulo $6n+3$ to one of $1,\ldots,3n$ and similarly one is congruent modulo $6n+3$ to one of $1,\ldots,3n-1,3n+3$. We may therefore suppose that $z-y$ is congruent modulo $6n+3$ to one of $1,\ldots,3n$ if $n\equiv 0$ or 1 modulo 4 and to one of $1,\ldots,3n-1,3n+3$ if $n\equiv 2$ or 3 modulo 4. For some r, $1\leqq r\leqq n$, if $z-y=r$ or $n+f_r$ then

$$\{y,z\} \subset \{y,y+r,y+n+f_r\} \in E$$

whilst if $z-y=n+e_r$ then

$$\{y,z\} \subset \{z-n-f_r, z-n-e_r, z\} \in E\,.$$

Therefore $\{y,z\}$ is a subset of at least one triple of E.

The number of triples in the set E is clearly at most $n(6n+3)$ and so the number of distinct pairs contained by the triples of E is at most $3n(6n+3)$; equality here implies that no pair occurs in more than one triple of E. But each pair $\{y,z\}$ satisfying the conditions of the lemma is a subset of a triple of E, and the number of such pairs is clearly

$$\binom{6n+3}{2} - (6n+3) \;=\; 3n(6n+3)\,.$$

Therefore no pair $\{y,z\}$ occurs in more than one triple of E.

5. Each pair in at least one triple.

Suppose that a set T of triples of elements of S has the following property P: *each pair of elements is in at least one triple.* An element $a\in S$ occurs at least once with every other element of S in a triple of T, and so the number of triples in which a occurs is at least $\lceil\frac{1}{2}(s-1)\rceil$, where

$\lceil x \rceil$ denotes the smallest integer which is not less than x. Therefore the number $|T|$ of triples of T is at least $\frac{1}{3}s\lceil\frac{1}{2}(s-1)\rceil$ and since the number of triples must be an integer we obtain

$$|T| \geqq \left\lceil \tfrac{1}{3}s\lceil\tfrac{1}{2}(s-1)\rceil\right\rceil = \nu .$$

Fort and Hedlund [1] constructed inductively a triple system T of ν triples which had property P. We now give such a minimal triple system simply and explicitly, for all but a few small values of s. For completeness, we give all cases although cases 1 and 2 are given by Fort and Hedlund. We first observe that

$$\nu = \begin{cases} 6n^2+n & \text{if } s=6n+1 , \\ 6n^2+4n+1 & \text{if } s=6n+2 , \\ 6n^2+5n+1 & \text{if } s=6n+3 , \\ 6n^2+8n+3 & \text{if } s=6n+4 , \\ 6n^2+9n+4 & \text{if } s=6n+5 , \\ 6n^2+12n+6 & \text{if } s=6n+6 . \end{cases}$$

CASE 1. $s=6n+1$ or $6n+3$. Then a Steiner triple system on s elements has the property P and has ν elements.

CASE 2. $s=6n+2$ or $6n+4$. To a Steiner triple system on the numbers $1,\ldots,s-1$ add the triples

$$\{s,x,x+\tfrac{1}{2}s\}, \quad x=1,2,\ldots,\tfrac{1}{2}s-1, \quad \textit{and} \quad \{s,\tfrac{1}{2}s,1\} .$$

The triple system obtained has the property P and contains ν triples.

CASE 3. $s=6n+5$. Let F be the set of all triples

$$\{6n+4,1+2\alpha(3n+1),1+(2\alpha+1)(3n+1)\}, \qquad \alpha=1,2,\ldots,3n+1 ,$$
$$\{6n+5,1+(2\alpha+1)(3n+1),1+(2\alpha+2)(3n+1)\}, \qquad \alpha=0,1,\ldots,3n+1 ,$$

where the last two numbers in each triple are taken modulo $6n+3$. The union G of E (defined in section 4), F and the set containing just the triple $\{6n+5,6n+4,3n+2\}$ contains $(6n+3)n+(3n+1)+(3n+2)+1=\nu$ triples and has property P.

To see that G has property P let $\{y,z\}$ be a pair of unequal numbers between 1 and $6n+5$. If $\{y,z\}\subset\{6n+5,6n+4,3n+2\}$ then $\{y,z\}$ is a subset of a triple of G. Suppose therefore that $\{y,z\}\not\subset\{6n+5,6n+4,3n+2\}$. Since $(6n+3,3n+1)=1$, each residue class modulo $6n+3$ contains one of the numbers $1+\beta(3n+1)$, $\beta=2,3,\ldots,6n+3$, except for the residue class which contains $3n+2$. Therefore if $(6n+4)$ or $(6n+5) \in \{y,z\}$ then $\{y,z\}$ is a subset of a triple of F. If $\{6n+4, 6n+5\} \cap \{y,z\} = \emptyset$ and one of the differences of $\{y,z\}$ is congruent modulo $6n+3$ to $3n+1$ then $\{y,z\}$ is a subset of a triple of F, whilst if neither of the differences of $\{y,z\}$ is congruent to $3n+1$ modulo $6n+3$ then, by the lemma, $\{y,z\}$ is a subset of a triple of E. Since G is the union of E, F and $\{\{6n+5, 6n+4, 3n+2\}\}$ it follows that G has property P.

CASE 4. $s = 6n+6$. Let H be the set of all triples

$$\{6n+6, x, x+3n+2\}, \qquad x = 1, 2, \ldots, 3n+2 .$$

Then the union J of E (defined in section 4), F (defined in case 3 above), H and the set containing just the triple $\{6n+6, 6n+5, 6n+4\}$ contains $(6n+3)n + (3n+1) + 2(3n+2) + 1 = \nu$ triples and has property P. The verification of this is omitted since it is similar that of case 3.

6. Each pair in at most one triple.

Now suppose that a set T of triples of elements of S has the following property Q: *no pair of elements is in more than one triple.* An element $a \in S$ occurs at most once with every other element of S in a triple of T, and so the number of triples in which a occurs is at most $[\frac{1}{2}(s-1)]$, where $[x]$ denotes the largest integer not larger than x. Therefore the number $|T|$ of triples of T is at most $\frac{1}{3}s[\frac{1}{2}(s-1)]$ and since the number of triples must be an integer we obtain

$$|T| \leqq \left[\tfrac{1}{3}s[\tfrac{1}{2}(s-1)]\right] .$$

J. Schönheim [4] has shown that, if

$$\mu = \begin{cases} \left[\frac{1}{3}s[\frac{1}{2}(s-1)]\right] & \textit{for} \quad s \not\equiv 5 \pmod 6 , \\ \left[\frac{1}{3}s[\frac{1}{2}(s-1)]\right] - 1 & \textit{for} \quad s \equiv 5 \pmod 6 , \end{cases}$$

then $|T| \leqq \mu$, and further, in all cases he constructed a triple system T of μ triples which had the property Q. We now give such a maximal triple system simply and explicitly for all but a few small values of s. For completeness again, we give the construction in all cases, although cases 1 and 2 are given by Schönheim. We first observe that

$$\mu = \begin{cases} 6n^2 - 2n & \textit{if } s = 6n , \\ 6n^2 + n & \textit{if } s = 6n+1 , \\ 6n^2 + 2n & \textit{if } s = 6n+2 , \\ 6n^2 + 5n + 1 & \textit{if } s = 6n+3 , \\ 6n^2 + 6n + 1 & \textit{if } s = 6n+4 , \\ 6n^2 + 9n + 2 & \textit{if } s = 6n+5 . \end{cases}$$

CASE 1. $s = 6n+1$ or $6n+3$. Then a Steiner triple system on s elements has the property Q and has μ elements.

CASE 2. $s = 6n$ or $6n+2$. From a Steiner triple system on $s+1$ elements delete all triples containing a particular element. The triple system remaining has the property Q and has μ elements.

CASE 3. $s = 6n+4$. Let K be the set of all triples

$$\{6n+4, x, x+3n+1\}, \qquad x = 1, \ldots, 3n+1 .$$

Then the union L of E (defined in section 4) and K contains $(6n+3)n+3n+1 = \mu$ triples and also has property Q. This is easily verified using the lemma.

CASE 4. $s = 6n+5$. Let M be the set of all triples

$$\begin{matrix} \{6n+4, 1+2\alpha(3n+1), 1+(2\alpha+1)(3n+1)\}, \\ \{6n+5, 1+(2\alpha+1)(3n+1), 1+(2\alpha+2)(3n+1)\}, \end{matrix} \qquad \alpha = 1, \ldots, 3n+1 ,$$

where the last two numbers of each triple are taken modulo $6n+3$. The union N of E (defined in section 4) and M contains $(6n+3)n+2(3n+1) = \mu$ triples and also has property Q. Again, this is easily verified using the lemma.

Added in proof.

Let E' be the set of all triples $\{x, x+3i+1, x+6i+3\}$, $x = 1, \ldots, 6n+3$, $i = 0, \ldots, n-1$, where each number is taken modulo $6n+3$. Then E' can be used instead of E, whenever E was used, since E' also satisfies the lemma. The advantage of E' over E is that E' has a simpler construction.

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