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THE BACKWARD AND FORWARD SUMMATION OF INFINITE SERIES OF ISOLS

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Infinite series of isols were introduced and studied in [2]. In that paper, only series of non-negative integers were considered. In this note we wish to show that in a certain sense, to be made precise below, a large class of infinite series of non-negative integers can be summed both backwards and forwards.

We employ the following notation and terminology. By a number, we mean a non-negative integer. The set of all numbers is denoted by ε. If n is a number, $\nu(n)$ denotes the initial segment of ε determined by n. The function $j(x,y)$ is the usual recursive pairing function mapping ε^2 one to one onto ε. For α a set of numbers and x any number, $j(x,\alpha)$ denotes $\{j(x,y) \mid y \in \alpha\}$ and $\mathrm{Req}\,\alpha$ denotes the recursive equivalence type of α. Lastly, $\simeq$ denotes the relation of recursive equivalence and if f is a recursive function, f_Λ denotes its extension to the isols.

One of the basic results of [5] is the fact that for T an infinite, regressive isol,

$$(1) \qquad \textstyle\sum_T:\ 1+2+3+\ldots\ =\ \sum_T:\ T+(T-1)+(T-2)+\ldots\,.$$

This result might be considered an extension of the statement that the forward and backward summations of the first n integers are equal. Without stretching one's imagination too far, it could be said that the right hand side of (1) represents the backward summation of the positive integers with respect to the infinite, regressive isol T-even though all the terms on the left hand side of (1) are finite and all those on the right hand side are infinite. In the hope of obtaining other results similar to (1), we proceed to consider a "natural" definition of the summation of a large class of infinite series of infinite isols.

Definition 1. Let f be any function ε into ε. The e-difference of f, denoted by e_f, is given by

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$$e_f(0) = f(0) \quad \text{and} \quad e_f(n+1) = f(n+1) - f(n) .$$

DEFINITION 2. Let f be an increasing, recursive function and T an infinite, regressive isol with t_n any regressive function ranging over a set in T.

$$\Sigma_T f_\Lambda(T-(n+1)) := \text{Req} \bigcup_k j\left[t_k, \bigcup_n j\left(t_{k+n}, \nu(e_f(n))\right)\right] . \tag{2}$$

In (2) and elsewhere $\bigcup_{i=0}^{\infty}$ is denoted $\bigcup_i$.

Note that since f is increasing and recursive, it is indeed the case by [1, Proposition 2, Corollary 4] that the terms of such a series are regressive.

THEOREM 1. *For every increasing, recursive function* f,

$$\Sigma_T f_\Lambda(T-(n+1))$$

is a function from $\Lambda_R - \varepsilon$ *into* Λ_R.

PROOF. Let f be increasing, recursive and let T be an infinite, regressive isol. Let t and t^* be any two regressive functions ranging over sets in T. In order to show that the sum is a function on $\Lambda_R - \varepsilon$, we must prove that

$$\bigcup_k j\left[t_k, \bigcup_n j\left(t_{k+n}, \nu(e_f(n))\right)\right] \simeq \bigcup_k j\left[t^*_k, \bigcup_n j\left(t^*_{k+n}, \nu(e_f(n))\right)\right] . \tag{3}$$

Let the left hand side and right hand side of (3) be denoted by α and β respectively. By [2, Proposition 3], there is a partial recursive, one to one function p such that for each n, $p(t_n) = t^*_n$. Let q be defined by

$$q(z) = j\left[pk(z), j(pkl(z), ll(z))\right] ,$$

where k and l are the functions such that $j(k(n), l(n)) = n$. Since k and l are recursive, it follows that q is a partial recursive function. It is clearly the case that q is one to one and moreover q maps α onto β. Hence $\alpha \simeq \beta$.

We postpone showing that the sum is regressive as this will be obtained as an immediate corollary of our main result.

We claimed above that our definition of the sum was a "natural" one. This is the case for the following reason. It is well known [1] that if f is increasing, recursive, then for each k

$$f_\Lambda(T-(k+1)) = \Sigma_{T-k}\, e_f(n) .$$

Moreover, by the definition of a series of non-negative integers given in [2],

$$\Sigma_{T-k} e_f(n) = \text{Req} \bigcup_n j\left(t_{k+n}, \nu(e_f(n))\right) . \tag{4}$$

Hence, for each k, the sets appearing on the right of (4) are "natural" representatives of the terms of the series $\Sigma_T f_A(T-(k+1))$.

THEOREM 2. *Let f be an increasing, recursive function and T an infinite, regressive isol. Then*

$$\Sigma_T f_A(T-(n+1)) = \Sigma_T f(n) . \tag{5}$$

PROOF. It suffices to prove that

$$\bigcup_k j\left[t_k, \bigcup_n j\left(t_{k+n}, \nu(e_f(n))\right)\right] \simeq \bigcup_m j(t_m, \nu(f_m)) , \tag{6}$$

where k, n, m run through the non-negative integers. Let the sets on the left and right of (6) be denoted by δ and γ respectively. In order to show that δ and γ are recursively equivalent, we describe a uniform, effective procedure which pairs the members of δ with those of γ. To facilitate this, we rewrite δ as follows:

$$\begin{aligned}\delta &= \bigcup_k \bigcup_n j\left[t_k, j\left(t_{k+n}, \nu(e_f(n))\right)\right] \\ &= \bigcup_m \bigcup_{k=0}^m j\left[t_k, j\left(t_m, \nu(e_f(m-k))\right)\right] .\end{aligned}$$

Consider the following infinite array containing the members of δ. In each row of this array, the values of m and k are constant and the rows are ordered according to the lexicographical ordering of the pairs (k,m). Moreover, the elements in each row are arranged according to the size of the second component.

$$\begin{array}{lll} j(t_0, j(t_0, 0)), & \ldots, & j\left(t_0, j(t_0, e_f(0)-1)\right) \\ j(t_0, j(t_1, 0)), & \ldots, & j\left(t_0, j(t_1, e_f(1)-1)\right) \\ j(t_1, j(t_1, 0)), & \ldots, & j\left(t_1, j(t_1, e_f(0)-1)\right) \\ j(t_0, j(t_2, 0)), & \ldots, & j\left(t_0, j(t_2, e_f(2)-1)\right) \\ j(t_1, j(t_2, 0)), & \ldots, & j\left(t_1, j(t_2, e_f(1)-1)\right) \\ \ldots & & \end{array}$$

It is assumed that any row for which $e_f(m-k)=0$ has been deleted from the array. Note that the number of elements in the row for which $m=m_1$ and $k=k_1$ is equal to $e_f(m_1-k_1)$ and hence the totality of elements having a fixed value of m, say $m=m_1$ is given by

$$e_f(m_1-0)+e_f(m_1-1)+\ldots+e_f(m_1-m_1) = f(m_1) .$$

To each such group of elements of δ, we let correspond, in order, the elements

$$j(t_{m_1}, 0), j(t_{m_1}, 1), \ldots, j(t_{m_1}, f(m_1) - 1) \quad \text{of } \gamma .$$

Since f is recursive and t is regressive, it is clear that this correspondence is effective. Hence $\delta \simeq \gamma$.

COROLLARY. *For every increasing, recursive function f and every infinite, regressive isol T,*

$$\Sigma_T f_\Lambda(T - (n+1)) \in \Lambda_R .$$

PROOF. In [1] is proved that $\Sigma_T f(n) \in \Lambda_R$. Hence the Corollary follows from (5) in Theorem 2.

With this corollary, we have completed the proof of Theorem 1.

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