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LINEAR INEQUALITIES AND THE ABSTRACT MOMENT PROBLEM IN TOPOLOGICAL VECTOR SPACES

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Introduction.

In this paper we shall be concerned with the following two problems, one being the dual of the other.

THE FIRST INEQUALITY PROBLEM. *Given a set of elements x_ι of a real topological vector space E and a corresponding set of scalars α_ι, we seek an element f of E', the (topological) dual of E, such that*

$$f(x_\iota) \geqq \alpha_\iota$$

holds for all ι, and f satisfies a given topological condition.

THE SECOND INEQUALITY PROBLEM. *Given a set of elements f_ι of E' and a corresponding set of scalars α_ι, we seek an element x of E such that*

$$f_\iota(x) \geqq \alpha_\iota$$

holds for all ι, and x satisfies a given topological condition.

The purpose of the paper is to review results already established, and to present a new result on the second inequality problem. Notice that, if we replace the inequalities by equalities, and allow E to be real or complex, we are then considering the abstract moment problem. In fact, as we shall see, we may regard this as a special case of the inequality problem; for brevity let us call it the moment case (the moment form of the theorems will bear the number of the corresponding inequality theorem with a prime).

I wish to thank Dr. M. R. Mehdi for introducing me to this topic and for his subsequent help and encouragement.

Normed space.

The following theorem provides the full solution of the first inequality problem for the case of the normed space.

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THEOREM 1. *Let $\{x_\iota : \iota \in I\}$ be any set of elements of the real normed space E and $\{\alpha_\iota : \iota \in I\}$ any corresponding set of scalars. Suppose ϱ to be a given positive real number. A necessary and sufficient condition for the existence of u of E' satisfying the relations*

(i) $u(x_\iota) \geqq \alpha_\iota, \quad \iota \in I,$
(ii) $\|u\| \leqq \varrho$

is that

$$\sum_{\iota \in \Omega} \lambda_\iota \alpha_\iota \leqq \varrho \|\sum_{\iota \in \Omega} \lambda_\iota x_\iota\| \tag{1}$$

should hold for every finite subset Ω of I and every set of non-negative scalars λ_ι, $\iota \in \Omega$.

This result was given by K. Fan [2, p. 124], though it is contained in an earlier paper of S. Mazur and W. Orlicz [5, p. 152].

The corresponding theorem for the moment case is well known; it will now be given to serve as the representative example of the moment form.

THEOREM 1′. *Let $\{x_\iota : \iota \in I\}$ be any set of elements of the real or complex normed space E and $\{\alpha_\iota : \iota \in I\}$ any corresponding set of scalars. Suppose ϱ to be a given positive real number. A necessary and sufficient condition for the existence of u of E' satisfying the relations*

(i) $u(x_\iota) = \alpha_\iota, \quad \iota \in I,$
(ii) $\|u\| \leqq \varrho$

is that

$$|\sum_{\iota \in \Omega} \lambda_\iota \alpha_\iota| \leqq \varrho \|\sum_{\iota \in \Omega} \lambda_\iota x_\iota\| \tag{2}$$

should hold for every finite subset Ω of I and every set of scalars λ_ι, $\iota \in \Omega$.

Theorem 1′ was proved in particular cases by F. Riesz and E. Helly (c. 1912) and in the general case by H. Hahn (1927); a proof is to be found, for instance, in Banach's book [1, p. 55]. But it is quite easily deduced from theorem 1 (I am indebted to H. P. Mulholland for pointing this out to me), as we shall now see, the proof being adaptable to each of the subsequent theorems in the moment form. The necessity is trivial; to prove the sufficiency we consider first the case when E is real. From (2) we obtain the inequality

$$\sum_{\iota \in \Omega} (\lambda_\iota - \lambda_\iota') \alpha_\iota \leqq \varrho \|\sum_{\iota \in \Omega} (\lambda_\iota - \lambda_\iota') x_\iota\|,$$

where the λ_ι and λ_ι' are arbitrarily chosen non-negative scalars. Hence

$$\sum_{\iota \in \Omega} \lambda_\iota \alpha_\iota + \sum_{\iota \in \Omega} \lambda_\iota' (-\alpha_\iota) \leqq \varrho \|\sum_{\iota \in \Omega} \lambda_\iota x_\iota + \sum_{\iota \in \Omega} \lambda_\iota' (-x_\iota)\|,$$

and the result now follows from an application of theorem 1 to the sets $\{x_\iota, -x_\iota : \iota \in I\}$ and $\{\alpha_\iota, -\alpha_\iota : \iota \in I\}$. Now let E be complex, and denote by E_r the normed space E with the scalars restricted to the real numbers. From (2) we obtain the inequality,

$$|\sum_{\iota\in\Omega}(\lambda_\iota - i\lambda_\iota')\alpha_\iota| \leqq \varrho\|\sum_{\iota\in\Omega}(\lambda_\iota - i\lambda_\iota')x_\iota\|,$$

where the λ_ι and λ_ι' are arbitrarily chosen real numbers. Hence

$$|\sum_{\iota\in\Omega}\lambda_\iota\beta_\iota + \sum_{\iota\in\Omega}\lambda_\iota'\gamma_\iota| \leqq \varrho\|\sum_{\iota\in\Omega}\lambda_\iota x_\iota + \sum_{\iota\in\Omega}\lambda_\iota'(-ix_\iota)\|,$$

where we define $\beta_\iota + i\gamma_\iota \equiv \alpha_\iota$. By what we have proved there exists f of $(E_r)'$ such that

$$f(x_\iota) = \beta_\iota, \quad f(-ix_\iota) = \gamma_\iota, \quad \iota \in I,$$

$$\|f\| \leqq \varrho \,.$$

Then u given by $u(x) = f(x) + if(-ix)$ is a continuous linear functional on E; further, since we may write $u(x) = re^{i\theta}$ (r, θ real numbers), then $|u(x)| = f(e^{-i\theta}x) \leqq \varrho\|x\|$, so that $\|u\| \leqq \varrho$. Hence u fulfils the requirements of theorem 1'.

The second inequality problem presents more difficulties in its solution, as the following well known example belonging to the moment case illustrates.

Example. Let $E = C[0,1]$, the space of real continuous functions on $[0,1]$ with the least upper bound norm. We are given a set

$$\{\varphi_n : n = 1, 2, \ldots\}$$

of functions on $[0,1]$, where each φ_n is constant save for a jump of 1 at $t = 1/n$. Then it is easy to see that there is no x of $C[0,1]$ for which

$$\int_0^1 x(t)\, d\varphi_n(t) = (-1)^n, \quad n = 1, 2, \ldots \,.$$

even though the dual form of the inequality (2), namely inequality (4), which we shall presently meet, is satisfied.

The following result, which in its moment form is due to Helly (1921), partially solves the second inequality problem.

Theorem 2. *Let $\{u_\iota : \iota \in I\}$ be any set, the linear hull of which is finite-dimensional, in E', the dual of the real normed space E; let $\{\alpha_\iota : \iota \in I\}$ be any corresponding set of scalars. Suppose ϱ to be a given positive real* *number. A necessary and sufficient condition that corresponding to each* $\varepsilon > 0$ *there should exist* x_ε *of* E *satisfying the relations*

(i) $u_\iota(x_\varepsilon) \geqq \alpha_\iota, \quad \iota \in I,$

(ii) $\|x_\varepsilon\| \leqq \varrho + \varepsilon$

is that

$$\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota \leqq \varrho \|\sum_{\iota\in\Omega} \lambda_\iota u_\iota\| \tag{3}$$

should hold for every finite subset Ω *of* I *and every set of non-negative scalars* λ_ι, $\iota \in \Omega$.

To arrive at the moment form of this theorem, namely theorem 2′ (Helly), one replaces condition (3) by

$$|\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota| \leqq \varrho \|\sum_{\iota\in\Omega} \lambda_\iota u_\iota\|, \tag{4}$$

the λ_ι being real or complex.

A proof of theorem 2′ is given, for instance, by E. Hille and R. Phillips [3, p. 31]. Theorem 2 will be deduced as a corollary of a theorem to appear later in this paper.

We now come to the full solution of the second inequality problem, and therefore of the second moment problem, a result believed to be new. Essentially what is done to obtain this is the replacing of condition (3) in theorem 2 by a stronger condition.

THEOREM 3. *Let* $\{u_\iota : \iota \in I\}$ *be any set of elements of* E', *the dual of the real normed space* E, *and let* $\{\alpha_\iota : \iota \in I\}$ *be any corresponding set of scalars. Suppose* ϱ *to be a given positive real number. A necessary and sufficient condition for the existence of* x *of* E *satisfying the relations*

(i) $u_\iota(x) \geqq \alpha_\iota, \quad \iota \in I,$

(ii) $\|x\| \leqq \varrho$

is that

$$\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota \leqq \varrho \sup_{k=1,\ldots,m} |\sum_{\iota\in\Omega} \lambda_\iota u_\iota(y_k)| \tag{5}$$

should hold for some finite subset $\{y_1, \ldots, y_m\}$, $0 < \|y_k\| \leqq 1$, $k = 1, \ldots, m$, *of* E, *for every finite subset* Ω *of* I *and for every set of non-negative scalars* λ_ι, $\iota \in \Omega$.

PROOF. *Necessity.* Suppose there exists x of E satisfying relations (i) and (ii). If $x = \theta$ (the origin), then (5) is trivially true. Suppose therefore that $x \neq \theta$; then for any finite subset Ω of I and any set $\{\lambda_\iota : \iota \in \Omega\}$ of non-negative scalars we have

$$\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota \leqq \sum_{\iota\in\Omega} \lambda_\iota u_\iota(x) = \|x\| \sum_{\iota\in\Omega} \lambda_\iota u_\iota(x/\|x\|) \leqq \varrho |\sum_{\iota\in\Omega} \lambda_\iota u_\iota(y)|,$$

where $y = x/\|x\|$.

Sufficiency. Put $B=\{x\in E : \|x\|\leqq \varrho\}$ and let F be the linear hull of $\{y_1,\ldots,y_m\}$. Then $X=F\cap B$ is absolutely convex and compact. We first show that the set,

$$G_\Omega = \bigcap_{\iota\in\Omega}\{x\in B : u_\iota(x)\geqq \alpha_\iota\}$$

is non-empty for any finite subset Ω of I. Let n be the cardinality of such a set Ω and put $u_k=u_{\iota_k}$ for brevity. Define the mapping $T\colon E\to \mathsf{R}^n$ by

$$T(x) = (u_1(x),\ldots,u_n(x)) .$$

T is clearly linear and continuous; so the set $D=T(X)$ is absolutely convex and compact.

Now suppose $G_\Omega=\emptyset$ and therefore $G_\Omega\cap F=\emptyset$; then the closed convex cone,

$$C = (\alpha_1,\ldots,\alpha_n) + \{(\xi_1,\ldots,\xi_n) : \xi_i\geqq 0,\ i=1,\ldots,n\}$$

and D are disjoint. Therefore a hyperplane in R^n strictly separates them; that is, there is a linear functional φ on R^n and a real number γ satisfying

$$\sup_{z\in D}\varphi(z) < \gamma < \varphi(y)$$

for every y of C. D being balanced, it follows that

$$\sup_{z\in D}|\varphi(z)| < \varphi(y)$$

holds for every y of C. This means that there are real number $\lambda_1,\ldots,\lambda_n$ for which

$$|\textstyle\sum_{k=1}^n \lambda_k u_k(x)| < \sum_{k=1}^n \lambda_k(\alpha_k+\eta_k)$$

is true for every x of X and for all $\eta_k\geqq 0$, $k=1,\ldots,n$. From this we draw two conclusions:

(i) $\lambda_k\geqq 0, \quad k=1,\ldots,n$,
(ii) $\varrho|\sum_{k=1}^n \lambda_k u_k(y_i)|<\sum_{k=1}^n \lambda_k\alpha_k, \quad i=1,\ldots,m,$

and thus arrive at a contradiction. Hence $G_\Omega\cap F$ is not empty.

Now let, $\mathscr{F}=\{\Omega\colon \Omega$ is a finite subset of $I\}$. The intersection $G_\Omega\cap F$, being equivalent to

$$\bigcap_{\iota\in\Omega}\{x : u_\iota(x)\geqq\alpha_\iota\}\cap X ,$$

is a closed subset of X. Then the set $\mathscr{G}=\{G_\Omega\cap F : \Omega\in\mathscr{F}\}$ is a family of closed subsets of the compact set X. It is clear that

$$\begin{aligned}\bigcap_{k=1}^n G_{\Omega_k} &= \bigcap_{k=1}^n\bigcap_{\iota\in\Omega_k}\{x\in B : u_\iota(x)\geqq\alpha_\iota\}\\ &= \bigcap_{\iota\in N}\{x\in B : u_\iota(x)\geqq\alpha_\iota\}, \qquad N = \bigcup_{k=1}^n\Omega_k ,\end{aligned}$$

where $\Omega_k \in \mathscr{F}$. We have therefore seen that any finite intersection of $\mathscr{G}$ is non-empty; it follows from the finite-intersection property of $\mathscr{G}$ that $\bigcap_{\Omega \in \mathscr{F}} G_\Omega \cap F \neq \emptyset$ and hence $\bigcap_{\Omega \in \mathscr{F}} G_\Omega \neq \emptyset$ holds. This completes the proof.

In the moment form of this theorem, namely theorem 3′, condition (5) is replaced by

$$(6) \qquad |\textstyle\sum_{\iota \in \Omega} \lambda_\iota \alpha_\iota| \leqq \varrho \sup_{k=1,\ldots,m} |\sum_{\iota \in \Omega} \lambda_\iota u_\iota(y_k)| ,$$

the λ_ι being real or complex.

REMARKS. I. Theorem 3′ can be proved in a manner similar to that of theorem 3 just given, the non-negative cone in R^n being replaced by the set $\{(\alpha_1,\ldots,\alpha_n)\}$.

II. The proof of theorems 1 and 1′ rely on Zermelo's axiom through the agency of the theorem on the weak-star compactness of the unit ball in E' and the Hahn–Banach theorem respectively. The deepest theorem depended on in the proof of theorems 3 and 3′ is that of a continuous function's assuming its supremum on a compact set in the medium of the separation theorem in R^n.

By means of the following lemma we shall deduce theorems 2 and 2′ from theorems 3 and 3′ respectively.

LEMMA. *Let $\{u_1,\ldots,u_n\}$ be a linearly independent set in E', the dual of a normed space E; let $\delta > 0$ be given. Then there exists a finite set $\{y_1,\ldots,y_m\}$, $0 < \|y_k\| \leqq 1$, $k = 1,\ldots,m$, in E for which*

$$\|\textstyle\sum_{j=1}^n \mu_j u_j\| \leqq (1+\delta) \sup_{k=1,\ldots,m} |\sum_{j=1}^n \mu_j u_j(y_k)|$$

holds for every set of scalars μ_j, $j = 1,\ldots,n$.

PROOF. Put $B = \{x \in E : \|x\| \leqq 1\}$ and let K denote the ground field of E (either R or C). Further write

$$\Lambda = \{\lambda = (\lambda_1,\ldots,\lambda_n) \in K^n : \|\lambda\| = 1\}$$

(the Euclidean norm is meant) and $u_\lambda = \sum_{i=1}^n \lambda_i u_i$ for each λ of K^n. Choose any λ from Λ, but x_0 from B such that $u_\lambda(x_0)$ differs from zero; put $\varepsilon = \delta |u_\lambda(x_0)|$. By the definition of norm in E' there is some x_λ of B such that

$$\|u_\lambda\| - \varepsilon < |u_\lambda(x_\lambda)|$$

holds. Then we have

$$\|u_\lambda\| < |u_\lambda(x_\lambda)| + \delta\, |u_\lambda(x_0)| ,$$

which we now write as

$$\|u_\lambda\| < (1+\delta)|u_\lambda(y_\lambda)|$$

where

$$y_\lambda = \begin{cases} x_0 & \text{if } |u_\lambda(x_0)| \geqq |u_\lambda(x_\lambda)| \text{ holds ,} \\ x_\lambda & \text{otherwise .} \end{cases}$$

Clearly the condition $0 < \|y_\lambda\| \leqq 1$ is met.

For each λ of Λ define the mapping $T_\lambda : K^n \backslash \{\theta\} \to \mathsf{R}$ by

$$T_\lambda(\varkappa) = (1+\delta)|u_\varkappa(y_\lambda)|/\|u_\varkappa\|.$$

T_λ is continuous, since $\{u_1, \ldots, u_n\}$ is linearly independent. Hence to each λ of Λ there corresponds an open neighbourhood U_λ of λ in K^n that is contained within the set $\{\varkappa : T_\lambda(\varkappa) > 1\}$. Being compact Λ is covered by a finite subfamily of $\{U_\lambda : \lambda \in \Lambda\}$, say by $\{U_{\lambda_1}, \ldots, U_{\lambda_m}\}$. Hence we have

$$\|u_\varkappa\| \leqq (1+\delta)|u_\varkappa(y_{\lambda_k})|$$

for each $\varkappa$ of U_{λ_k} and for $k = 1, \ldots, m$, and therefore

$$\|u_\lambda\| \leqq (1+\delta) \sup_{k=1,\ldots,m} |u_\lambda(y_k)|$$

for each λ of Λ (we put $y_k = y_{\lambda_k}$ for brevity). Now Λ being absorbent in K^n, there is some $\varrho > 0$ for which $\varrho\mu \in \Lambda$ for each μ of K^n; we therefore arrive at the inequality

$$\|\textstyle\sum_{j=1}^n \varrho\mu_j u_j\| \leqq (1+\delta) \sup_{k=1,\ldots,m} |\textstyle\sum_{j=1}^n \varrho\mu_j u_j(y_k)| ,$$

which, divided through by ϱ, yields the required result.

A PROOF OF THEOREMS 2 AND 2′. Let $\{u_1, \ldots, u_n\}$ be a base of the linear hull of the given u_ι, $\iota \in I$. By the lemma there exists a finite set $\{y_1, \ldots, y_m\}$, $0 < \|y_k\| \leqq 1$, $k = 1, \ldots, m$, in E such that

$$\|\textstyle\sum_{j=1}^n \lambda_j u_j\| \leqq (1+\varepsilon/\varrho) \sup_{k=1,\ldots,m} |\textstyle\sum_{j=1}^n \lambda_j u_j(y_k)|$$

for every set of scalars λ_j, $j = 1, \ldots, n$. With condition (3) and condition (4) respectively we obtain

$$\textstyle\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota \leqq (\varrho+\varepsilon) \sup_{k=1,\ldots,m} |\textstyle\sum_{\iota\in\Omega} \lambda_\iota u_\iota(y_k)|$$

and

$$|\textstyle\sum_{\iota\in\Omega} \lambda_\iota \alpha_\iota| \leqq (\varrho+\varepsilon) \sup_{k=1,\ldots,m} |\textstyle\sum_{\iota\in\Omega} \lambda_\iota u_\iota(y_k)|$$

for every finite subset Ω of I and every set of the appropriate scalars λ_ι, $\iota \in \Omega$. The result now follows immediately from theorems 3 and 3′ respectively.

We have seen that the counter-example given above satisfied condition (4) of Helly's theorem. Let us now verify that it does not satisfy condition (6). We show that

$$|\Sigma_{k\in\Omega}\lambda_k(-1)^k| \leqq \varrho\,|\Sigma_{k\in\Omega}\lambda_k y(1/k)| \tag{7}$$

fails to hold for every y of E and every $\varrho > 0$, for some finite subset Ω of the positive integers and some scalars λ_k, $k \in \Omega$.

Let y be any element of E. If $y(1/k) = 0$ for some positive integer k, then (7) is violated for every $\varrho > 0$ if we take $\Omega = \{k\}$ and $\lambda_k = 1$. Suppose therefore that $y(1/k) \neq 0$, $k = 1, 2, \ldots$. If $y(1/k)$ has the same sign for two consecutive values n, $n+1$ of k, (7) is again violated for every $\varrho > 0$ if we take $\Omega = \{n, n+1\}$ and put

$$\lambda_k = (-1)^k/y(1/k), \quad k \in \Omega .$$

Hence we may suppose now that the sign of $y(t)$ alternates at $t = 1, \frac{1}{2}, \ldots$. But then by the continuity of y on $[0,1]$, it is clear that $y(1/k) \to 0$ must hold as $k \to \infty$. Let $\varrho > 0$ be given, and take $\Omega = \{k_1\}$ where k_1 is a positive integer for which $|y(1/k_1)| < 1/\varrho$ holds. Then (7) is violated if we take $\lambda_{k_1} = 1$, and we have now exhausted the possible elements of E that y could be.

Locally convex space.

The generalisation of theorem 1 to a locally convex space [6] (understood throughout to be Hausdorff) is as follows.

THEOREM 4. *Let $\{x_\iota : \iota \in I\}$ be any set of elements in a real locally convex space E, the topology of which is generated by the set S of semi-norms and $\{\alpha_\iota : \iota \in I\}$ any corresponding set of scalars. Suppose that ϱ is a given positive real number and $\{p_1, \ldots, p_m\}$ (p_1 non-zero) a given finite subset of S. A necessary and sufficient condition for the existence of u of E' satisfying the relations*

(i) $u(x_\iota) \geqq \alpha_\iota, \quad \iota \in I$,
(ii) $|u(x)| \leqq \varrho \sup_{j=1,\ldots,m} p_j(x)$

for every x of E is that

$$\Sigma_{\iota\in\Omega}\lambda_\iota\alpha_\iota \leqq \varrho \sup_{j=1,\ldots,m} p_j(\Sigma_{\iota\in\Omega}\lambda_\iota x_\iota) \tag{8}$$

should hold for every finite subset Ω of I and for every set of non-negative scalars λ_ι, $\iota \in \Omega$.

Theorem 4 has been given by O. Hustad [4, p. 397], though as he points out in a later paper the result is essentially contained in [5, p. 147]. Hustad deduces an interesting corollary, from which I first deduced theorem 3.

The moment form of theorem 4 (with (8) replaced by

$$|\sum_{\iota\in\Omega}\lambda_\iota\alpha_\iota| \leqq \varrho \sup_{j=1,\ldots,m} p_j\left(\sum_{\iota\in\Omega}\lambda_\iota x_\iota\right))$$

is deducible from it by the argument used in deducing theorem 1′ from theorem 1, since we used only the semi-norm properties of the norm there. For the same reason theorem 4′ can be proved directly by the standard proof of theorem 1′ mutatis mutandis.

Scrutiny of theorems 1, 3 and 4 reveals that each can be considered as a particular case of the following general theorem.

THEOREM 5. *Let (E_1,E_2) be a dual pair of real vector spaces, E_1 provided with a topology that makes it a locally convex space. Let $\{x_\iota : \iota\in I\}$ be any set of elements of E_1, and $\{\alpha_\iota : \iota\in I\}$ any corresponding set of scalars. Suppose U to be an absolutely convex neighbourhood of the origin in E_1, and let $\varrho>0$ be given. A necessary and sufficient condition for the existence of u of E_2 satisfying the relations*

(i) $\langle x_\iota,u\rangle \geqq \alpha_\iota, \quad \iota\in I$,
(ii) $u\in\varrho U^\circ$, *U° the polar of U,*

is that

$$\sum_{\iota\in\Omega}\lambda_\iota\alpha_\iota \leqq \varrho \sup_{v\in V}|\langle\sum_{\iota\in\Omega}\lambda_\iota x_\iota, v\rangle| \tag{9}$$

should hold for some absolutely convex $\sigma(E_2,E_1)$-compact subset V of U°, for every finite subset Ω of I and for every set of non-negative scalars λ_ι, $\iota\in\Omega$.

PROOF. *Necessity.* If $u=\theta$, then (9) is trivially true. Suppose therefore that $u\neq\theta$. For each finite subset Ω of I, and for every set of non-negative scalars λ_ι, $\iota\in\Omega$,

$$\sum_{\iota\in\Omega}\lambda_\iota\alpha_\iota \leqq \sum_{\iota\in\Omega}\lambda_\iota\langle x_\iota,u\rangle \leqq \varrho|\langle\sum_{\iota\in\Omega}\lambda_\iota x_\iota, u/\varrho\rangle| \leqq \varrho\sup_{v\in V}|\langle\sum_{\iota\in\Omega}\lambda_\iota x_\iota, v\rangle|,$$

where V is the absolutely convex hull of $\{u/\varrho,\theta\}$, which is clearly a $\sigma(E_2,E_1)$-compact subset of U°.

Sufficiency. We first show that the set

$$G_\Omega = \bigcap_{\iota\in\Omega}\{u\in\varrho U^\circ : \langle x_\iota,u\rangle \geqq \alpha_\iota\}$$

is non-empty for any finite subset Ω of I, which is of cardinality n, say.

Denote by $E_2(\sigma)$ the locally convex space E_2 with the $\sigma(E_2,E_1)$ topology. Define the mapping $T: E_2(\sigma) \to \mathsf{R}^n$ by

$$T(u) = (\langle x_1,u\rangle,\ldots,\langle x_n,u\rangle)$$

(we write x_{ι_n} as x_n for brevity). T is clearly linear; it is continuous (E_1 is the dual of $E_2(\sigma)$). Hence $D=T(V)$ is absolutely convex and compact.

Now suppose $G_\Omega=\emptyset$; then the closed convex cone

$$C = (\alpha_1,\ldots,\alpha_n)+\{(\xi_1,\ldots,\xi_n) : \xi_i \geqq 0,\ i=1,\ldots,n\}$$

and D are disjoint. Therefore, as in the proof of theorem 3, there are real numbers λ_k, $k=1,\ldots,n$, satisfying the relations

(i) $\lambda_k \geqq 0, \quad k=1,\ldots,n$,
(ii) $\varrho|\langle\sum_{k=1}^n \lambda_k x_k, v\rangle| < \sum_{k=1}^n \lambda_k\alpha_k$

for every v of V. This contradicts the hypothesis and so we have shown that G_Ω is not empty.

Now let

$$\mathscr{F} = \{\Omega : \Omega \text{ is a finite subset of } I\} .$$

Then

$$\mathscr{G} = \{G_\Omega \cap V : \Omega \in \mathscr{F}\}$$

is a family of $\sigma(E_2,E_1)$-closed subsets of the $\sigma(E_2,E_1)$-compact set V. Precisely as in the proof of theorem 3, by the finite-intersection property of $\mathscr{G}$ it follows that $\bigcap_{\Omega\in\mathscr{F}} G_\Omega \neq \emptyset$, and so the proof is complete.

The moment form of this, theorem 5′, in which condition (9) would be replaced by

$$|\textstyle\sum_{\iota\in\Omega}\lambda_\iota\alpha_\iota| \leqq \varrho \sup_{v\in V} |\langle\sum_{\iota\in\Omega}\lambda_\iota x_\iota, v\rangle| ,$$

λ_ι being real or complex, can be proved directly in a manner similar to the proof above (cf. the first remark following theorem 3).

That theorems 1 and 4 are a particular case of theorem 5 is a consequence of the Alaoglu–Bourbaki theorem on the weak-star compactness of the polar of a neighbourhood in a locally convex space, and the Hahn–Banach theorem. To see this one takes

$$U = \{x \in E : \sup_{j=1,\ldots,m} p_j(x) \leqq 1\}$$

for theorem 4; it is easily verified that ϱU° is precisely the set of u of E' that satisfy relation (ii). One takes $V=U^\circ$ and then has only to show that

$$\sup_{v\in U^\circ} |v(x)| = \sup_{j=1,\ldots,m} p_j(x)$$

holds for each x of E; clearly it is sufficient to show that

$$\sup_{v \in U^\circ} |v(x)| \geqq \sup_{j=1,\ldots,m} p_j(x) \equiv p(x) \tag{10}$$

holds for each x of E, where p is clearly a semi-norm. We may obviously suppose that $x \neq \theta$; then define the linear functional f on the linear hull L of $\{x\}$ by

$$f(\lambda x) = \lambda p(x) .$$

By the Hahn–Banach theorem there is some u of E' such that

$$u(y) = f(y), \quad y \in L, \qquad \text{and} \qquad |u(y)| \leqq p(y), \quad y \in E,$$

hold, so that $u \in U^\circ$. Then $|u(x)| = p(x)$ follows and hence relation (10).

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