MATH. SCAND. 24 (1969), 232–238

A NOTE ON DESINTEGRATION, TYPE AND GLOBAL TYPE OF VON NEUMANN ALGEBRAS

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Introduction.

A von Neumann algebra (v.N. algebra) which satisfies

(i) the center Z is σ-finite,

(ii) the commutant Z' and the v.N. algebra are both generated by Z and a countable set of operators,

has a central decomposition associated with a finite measure space. In theorem 1.4 we show, that this decomposition reduces the type-classification problem to that of factors on a separable Hilbert space.

Proposition 2.1 and 2.2 could be called a sort of Fubini theorem for direct integrals of v.N. algebras. These results might be a help in understanding the globally central decomposition, which is introduced in section 3.

In analogy with the usual central decomposition, theorems 3.2 and 3.5 show that we can reduce global classification problems to those of global factors.

In the special case of a centrally smooth v.N. algebra we end up by refinding the canonical decomposition from [4].

1.

Let H_n be a fixed Hilbert space of dimension n for $n=1,2,\ldots,\aleph_0$, and let A_n be the set of all v.N. algebras on H_n. Following [3] and [4] we have a standard Borel structure on

$$A = \bigcup \{A_n \mid n=1,2,\ldots,\aleph_0\} .$$

The relative structure on the set F of factors is also standard.

PROPOSITION 1.1. *The set F_{II} of type* II *factors is an analytic set in A.*

PROOF. A v.N. algebra is semi-finite if and only if it is algebraically isomorphic to the commutant of a finite v.N. algebra. The set F_{sf} of semi-

Received September 27, 1968.

finite factors is thus the saturation with respect to algebraic isomorphism of the set F_f' of the commutants of finite factors. F_f is a Borel set [4, theorem 2.8] and thus F_f' is Borel. F_{sf} is then analytic by the proof of [4, theorem 3.4], and we get that $F_{\mathrm{II}} = F_{sf} \setminus F_{\mathrm{I}}$ is an analytic set.

COROLLARY 1.2. *Let $x \to B(x)$ be a Borel field of factors on a finite measure space (X,μ). The set $\{x \in X \mid B(x) \in F_{\mathrm{II}}\}$ is measurable.* (*X need not be countably separated.*)

The next theorem due to R. J. Aumann [1] is a generalization of "The principle of measurable choice" [2, appendix V].

THEOREM 1.3. *Let (T,μ) be a finite measure space and X a standard Borel space. Let G be a Borel set in $T \times X$ such that the projection into T is the whole of T. Then there exists a Borel null set $N \subset T$ and a Borel map $g: T \setminus N \to X$ such that $(t,g(t)) \in G$ for all $t \in T \setminus N$.*

This theorem enables us to improve some well-known results. We have for example:

THEOREM 1.4. *Let (X,μ) be equivalent to a finite measure space, $x \to B(x)$ a Borel field of factors, and $B = \int_X B(x)\,d\mu(x)$. Then B is of type* I (*resp.* I_n, II, II_1, II_∞, III) *if and only if $B(x)$ is of type* I (*resp.* I_n, II, II_1, II_∞, III) *for almost all $x \in X$.*

The theorem is well known in the case where X is a standard Borel space ([2, chap. II and III] and [5]). The proof of the theorem follows the same line as the proof of this special case. The improvement is mainly relying on theorem 1.3. However, in order to use this theorem we must know that the sets $X_{\mathrm{I}} = \{x \in X \mid B(x) \in F_{\mathrm{I}}\}$, X_{I_n}, X_{II_1}, X_{II_∞}, X_{III} are measurable, but this follows from [4] and corollary 1.2.

2.

PROPOSITION 2.1. *Let (X,μ) and (Y,ν) be standard Borel spaces with measures. Let $\mu \otimes \nu$ be the product measure, and $(x,y) \to B(x,y)$ a Borel field of v.N. algebras on $X \times Y$. The v.N. algebras*

$$\int_{X\times Y} B(x,y)\, d\mu\otimes\nu(x,y) \quad \text{and} \quad \int_X^{\beta} \int_Y B(x,y)\, d\nu(y)\, d\mu(x)\,,$$

where β is a suitable coherence, are spatially isomorphic.

PROOF. This follows immediately from the Fubini theorem and [4, lemma 4.5].

PROPOSITION 2.2. *Let* (X,μ) *and* (Y,ν) *be as in proposition* 2.1 *and* $x \to B(x)$ *and* $y \to C(y)$ *Borel fields of v.N. algebras. Then*

(i) $(x,y) \to B(x) \otimes C(y)$ *is a Borel field,*

(ii) *the v.N. algebras*

$$\int_X B(x)\,d\mu(x) \otimes \int_Y C(y)\,d\nu(y) \quad \textit{and} \quad \int_{X\times Y} B(x)\otimes C(y)\,d\mu\otimes\nu(x,y)$$

are spatially isomorphic.

PROOF. When we have shown (i), we get (ii) from proposition 2.1 and [2, chap. II, § 3, proposition 3].

Identify $H_n \otimes H_m$ and H_{nm} by suitable fixed isometries. Then $\otimes\colon A_n \times A_m \to A_{nm}$ is well defined for all n and m in $\{1,2,\ldots,\aleph_0\}$. Let I_n denote the set of scalar multiples of the unit operator on H_n. By [4, remark after lemma 2.1],

$$B \to B \otimes I_m \quad \text{and} \quad C \to I_n \otimes C \quad \text{are Borel maps}\,.$$

Further, $(B,C) \to (B \cup C)''$ is a Borel map : $A_{nm} \times A_{nm} \to A_{nm}$ [3, theorem 3, corollary 2]. Combining the above we see that

$$(B,C) \;\to\; B \otimes C = (B \otimes I_n \cup I_m \otimes C)''$$

is a Borel map. Now (i) follows easily.

3.

In the following let B denote a v.N. algebra on a separable Hilbert space H, let Z be the center of B and Z° the lattice of projections in Z. Let $\sim$ denote spatial equivalence in Z°, that is, $E \sim F$ if and only if B_E and B_F are spatially isomorphic, and if and only if there exists a partial isometry $U \in L(H)$ such that $UBU^* \subset B$ and $U^*BU \subset B$, $U^*U = E$ and $UU^* = F$. (See [4, § 6].) Let $Z_G{}^\circ$ be the set of globally central projections, and $Z_G = (Z_G{}^\circ)''$. We define

$$G(B) \;=\; \{U \in L(H) \mid \; U \text{ is unitary, } UBU^* = B\}\,.$$

Then we have $G(B) = G(B')$.

LEMMA 3.1. $Z_G = G(B)'$, *and* $Z_G{}^\circ$ *is the lattice of projections in* Z_G.

PROOF. $G(B)$ contains all unitaries in $B \cup B'$, thus $G(B)' \subset Z$. From [4, lemma 6.1], we have $Z_G{}^\circ = Z^\circ \cap G(B)'$ and we get indeed $Z_G = (Z^\circ \cap G(B)')'' = G(B)'$.

For $E \in Z^\circ$ let $\hat{E}$ denote the minimal projection in $Z_G{}^\circ$ greater than E; $\hat{E}$ is called the globally central support of E.

Define global type as in [4, § 6], and define B to be of global type I_n, denoted $\mathrm{I}_n{}^G$, if there exist $E_i \in Z^\circ$, $i = 1,2,\ldots,n$, pairwise orthogonal, spatially equivalent and globally multiplicity free such that $I = \sum E_i$.

It is a well-known procedure to show that a v.N. algebra B has a unique decomposition into a direct sum of v.N. algebras of the types $\mathrm{I}_n{}^G$, $n = 1,2,\ldots,\aleph_0$, $\mathrm{II}_1{}^G$, $\mathrm{II}_\infty{}^G$, III^G, such that the projections $E_{\mathrm{I}_n},\ldots, E_{\mathrm{III}}$ determining it are globally central.

It is quite easy to prove that any v.N. algebra B has a greatest central projection E_C such that B_{E_C} is centrally smooth. It follows that B_F is not centrally smooth for any $F \in Z^\circ$ with $F \leqq I - E_C$, and that $E_C \in Z_G{}^\circ$.

We know [4, § 6], that $E_C \perp E_{\mathrm{II}}$, but which relations there are between E_C and E_{I} (resp. E_{III}) or whether $E_C = I$ in general is not known.

We have $Z_G \subset B \subset Z_G{}'$. We can take Z_G as the set of diagonal operators $Z_G(\mu)$ on a direct integral of Hilbert spaces $H = \int_X H(x)\,d\mu(x)$. Then B is decomposable: $B = \int_X B(x)\,d\mu(x)$. This decomposition of B over Z_G is essentially unique. We call it the globally central decomposition of B.

THEOREM 3.2. (i) *Let $B = \int_X B(x)\,d\mu(x)$ be the globally central decomposition of B. Then $B(x)$ is a global factor for almost all $x \in X$.*

(ii) *If $B = \int_X B(x)\,d\mu(x)$, and $B(x)$ is a global factor for almost all $x \in X$, and the set of diagonal operators $Z(\mu)$ is contained in Z_G, then $Z(\mu) = Z_G$, and we have the globally central decomposition.*

PROOF. (i). The space $G(B)$ is a Polish space in the strong operator topology. Let T_k, $k = 1,2,\ldots$, be a strongly dense sequence in $G(B)$. Let $T_k = \int_X T_k(x)\,d\mu(x)$. Since the set of T_k's generates $G(B)'' = Z_G{}'$, the set of $T_k(x)$'s generates $L(H(X))$ for almost all $x \in X$.

For all k we have $T_k B T_k{}^* = B$, and thus $T_k(x)B(x)T_k(x)^* = B(x)$ for almost all $x \in X$. From this we deduce that, for almost all $x \in X$,

$$\{T_k(x) \mid k = 1,2,\ldots\} \subset G(B(x)),$$

and therefore $L(H(x)) = G(B(x))''$, which shows that $B(x)$ is a global factor for almost all $x \in X$.

(ii). Take E in $Z_G{}^\circ$. Since $Z(\mu) \subset Z_G$, we can write $E = \int_X E(x)\,d\mu(x)$. We can assume that, for all $x \in X$, $B(x)$ is a global factor and $E(x)$ is a central projection in $B(x)$. The set $X' = \{x \in X \mid E(x) \neq 0, I\}$ is Borel.

Suppose that $\mu(X') \neq 0$. We can assume that $H(x) = H_n$ for a fixed n and all $x \in X'$.

Since $B(x)$ is a global factor, and $E(x) \neq 0, I$ for all $x \in X'$, there exists for all $x \in X'$ a unitary operator in $L(H_n)$ such that $UB(x)U^* = B(x)$ and $UE(x)U^* \neq E(x)$.

The set G_n of unitaries on H_n is a Polish space in the strong topology. Take the set

$$M = \{(x, U) \in X' \times G_n \mid UB(x)U^* = B(x),\ UE(x)U^* \neq E(x)\}$$

which is a Borel set [4, lemma 2.1].

From theorem 1.3 we get a Borel null set $N \subset X'$ and a Borel map $x \to U'(x)$ from $X' \setminus N$ into $L(H_n)$ such that, for all $x \in X' \setminus N$,

$$U'(x) \in G(B(x)) \quad \text{and} \quad U'(x)E(x)U'(x)^* \neq E(x) .$$

Now extend U' to a Borel field U on the whole of X by defining $U(x)$ to be the unit operator on $H(x)$ for x not in $X' \setminus N$. For $U = \int_X U(x)\, d\mu(x)$ we have $U \in G(B)$ and $UEU^* \neq E$, which is a contradiction since $E \in Z_G$. Thus we have $\mu(X') = 0$, and therefore $E \in Z(\mu)$.

This completes the proof.

Proposition 3.3. *Let E and F be central projections and $E \sim F$, let U be a partial isometry determining the equivalence. Then $U \in Z_G'$.*

Proof. It is sufficient to show, that $UG = GU$ for all $G \in Z_G^\circ$.

It is obvious that $GE \sim UGEU^* \leqq F$. Since $G \in Z_G^\circ$, we have $UGEU^* \leqq G$, and thus $UGEU^* \leqq GF$. In the same way we may obtain $U^*GFU \leqq GE$. Combining these inequalities we get $UGEU^* = GF$ or $UGU^* = GUU^*$, and thus $UG = UEG = UGE = GFU = GU$.

Proposition 3.4. *Let $B = \int_X B(x)\, d\mu(x)$ be the globally central decomposition of B. Let $E = \int_X E(x)\, d\mu(x)$ and $F = \int_X F(x)\, d\mu(x)$ be central projections in B. Then*

(i) *$E \sim F$ if and only if $E(x) \sim F(x)$ for almost all $x \in X$,*

(ii) *$\hat{E} = \int_X \widehat{E(x)}\, d\mu(x)$.*

Proof. (i). Let U be a partial isometry giving the equivalence. By proposition 3.3, U is decomposable: $U = \int_X U(x)\, d\mu(x)$. For almost all $x \in X$ we have that $U(x)$ is a partial isometry such that

$$U(x)^* U(x) = E(x), \quad U(x)U(x)^* = F(x) ,$$

$$U(x)^* B(x) U(x) \subset B(x), \quad U(x) B(x) U(x)^* \subset B(x) ,$$

and thus $E(x) \sim F(x)$ for almost all $x \in X$.

On the other hand, suppose that $E(x) \sim F(x)$ for almost all $x \in X$, that is, $B(x)_{E(x)}$ is spatially isomorphic to $B(x)_{F(x)}$. It follows from [2, chap. II, § 3, Proposition 6] and [4, lemma 4.1] that B_E is spatially isomorphic to B_F, and thus $E \sim F$.

(ii). $E = \int_X E(x) d\mu(x)$ is a globally central projection if and only if $\mu(\{x \in X \mid E(x) \neq 0, I\} = 0$. Since $B(x)$ is a global factor, $\widehat{E(x)} = 0$ if $E(x) = 0$ and $\widehat{E(x)} = I$ if $E(x) \neq 0$. From this (ii) follows easily.

THEOREM 3.5. *If B is a v.N. algebra on a separable Hilbert space and $\int_X B(x) d\mu(x)$ is its globally central decomposition, then we have:*

(i) *B is of type I^G (resp. $\mathrm{I}_n{}^G$, $\mathrm{II}_1{}^G$) if and only if $B(x)$ is of type I^G (resp. $\mathrm{I}_n{}^G$, $\mathrm{II}_1{}^G$) for almost all $x \in X$.*

(ii) *If B is of type $\mathrm{II}_\infty{}^G$, then $B(x)$ is of type $\mathrm{II}_\infty{}^G$ for almost all $x \in X$.*

(iii) *If $B(x)$ is of type III^G for almost all $x \in X$, then B is of type III^G.*

We omit the proof of this theorem. It is quite extensive and involves a thorough discussion of the spatial equivalence relation on Z°. However, it uses the same technic and is rather similar to the proof of theorem 1.4, and the basic difficulties are tackled in propositions 3.3 and 3.4.

It is not known whether (ii) or (iii) has a converse. This problem might be accessible similar to [5], by some kind of trace argument.

REMARK 3.6. If B is centrally smooth and

$$B = \sum_{n=0}^{\infty} \int_{W_n} B(x)\, d\mu_n(x) \otimes B_n = \sum_{n=0}^{\infty} \int_{W_n} B(x) \otimes B_n\, d\mu_n(x)$$

is the canonical decomposition, as described in [4, § 5], it is easily seen from theorem 3.2 (ii) that this gives the globally central decomposition by taking (X, μ) to be the disjoint sum of (W_n, μ_n), $n = 0, 1, 2, \ldots, \aleph_0$. Also, the type $\mathrm{I}_n{}^G$ part of B is the part associated with W_n for $n = 1, 2, \ldots, \aleph_0$, and the type III^G part of B is that associated with W_0.

If every v.N. algebra on a separable Hilbert space is centrally smooth, then theorem 3.5 almost trivially follows from the above remark.

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