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CONVOLUTIONS OF MEASURES AND SETS OF ANALYTICITY

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0. Introduction and notation.

In a locally compact abelian group, G, with character group Γ, we denote by $A(G)$ the algebra of functions

$$f(g) = \int_\Gamma \hat{f}(\gamma)(-g,\gamma)\, dm ,$$

where $\hat{f} \in L^1(\Gamma, dm)$ with the norm

$$\|f\| = \int |\hat{f}|\, dm ,$$

dm being the Haar measure of Γ. If E is a closed subset of G, we put

$$I(E) = \{f : f \in A(G) \text{ and } f(x)=0,\ \forall x \in E\} .$$

$I(E)$ is a closed ideal of $A(G)$, so we can write

$$A(E) = A(G)/I(E) .$$

Since E is the carrier space of $A(E)$, we can identify elements of $A(E)$ with restrictions to E of functions in $A(G)$.

E is called a *Helson set* if $A(E) = C_0(E)$.

If φ is a complex-valued function defined on $[-1,1]$, φ is said to *operate* on $A(E)$ if $\varphi \circ f \in A(E)$, for all $f \in A(E)$ whose range is $[-1,1]$. E is called a *set of analyticity* if essentially (cf., for example, [8]) only the analytic functions operate on $A(E)$.

The main results of this paper are theorems 2 and 4. We start by proving that if the product of two continuous measures on G charges a set $E \subset G \times G$, then E contains arbitrarily large squares. Then, by relating the product and the convolution of the measures, we achieve an algebraic criterion.

Theorem 1 is a known result [9], but the proof given here is more direct. Theorem 3 was announced in [10, Theorem 9.3.5], but, hitherto, no proof has been published.

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1. Preliminary results.

PROPOSITION 1. *Let K_1, K_2 be locally compact metrisable spaces. Let μ_1, μ_2 be continuous regular bounded Borel measures on K_1, K_2, respectively. Let F be a $\mu_1 \times \mu_2$-measurable closed set in $K_1 \times K_2$ such that $|\mu_1 \times \mu_2|(F) > 0$. Then, for any positive integer n, there exist sets X_n, Y_n in K_1, K_2 respectively with*

$$|X_n| = |Y_n| = n \quad \textit{and} \quad X_n \times Y_n \subset F.$$

PROOF. Without loss of generality, we may assume μ_1, μ_2 positive, K_1, K_2 compact, $\operatorname{supp}(\mu_1) = K_1$, $\operatorname{supp}(\mu_2) = K_2$ and $\mu_1(K_1) = \mu_2(K_2) = 1$.

There exist continuous maps $t_1 \colon K_1 \to I$, $t_2 \colon K_2 \to I$, where I is the unit interval, such that

$$\lambda(B) = \mu_1(t_1^{-1}(B)) = \mu_2(t_2^{-1}(B))$$

for any λ-measureable set $B \subset I$, λ denoting Lebesgue measure. Moreover, we can choose t_1, t_2 to be homeomorphisms of $K_1 \setminus N_1$, $K_2 \setminus N_2$ onto $I \setminus M_1$, $I \setminus M_2$ respectively, where

$$\mu_1(N_1) = \mu_2(N_2) = \lambda(M_1) = \lambda(M_2) = 0$$

and

$$t_1(N_1) \subset M_1, \quad t_2(N_2) \subset M_2$$

(cf. [1]). Put $F_1 = F \setminus (K_1 \times N_2) \cup (N_1 \times K_2)$. Then $(\mu_1 \times \mu_2)(F) > 0$.

Suppose there exist $X_n', Y_n' \subset I$ such that

$$|X_n'| = |Y_n'| \quad \text{and} \quad X_n' \times Y_n' \subset t_1 \times t_2(F_1).$$

Then $X_n' \subset I \setminus M_1$, $Y_n' \subset I \setminus M_2$, so, if we put

$$X_n = t_1^{-1}(X_n'), \quad Y_n = t_2^{-1}(Y_n'),$$

we have the result.

It remains to show the result for $K_1 = K_2 = I$, $\mu_1 = \mu_2 = \lambda$. Let $x \in F$ be a point of density of F with respect to $\mu = \lambda \times \lambda$. There exist intervals $I_n = [a, b]$, $J_n = [c, d]$, $(b-a) = (d-c)$, with x in the interior of $K = I_n \times J_n$, such that

$$\mu(K \cap F) > (1 - 1/n^2)\mu(K). \tag{1.1}$$

Now, writing χ_F for the characteristic function of F and K_{ij} for the squares of area $(b-a)^2/n^2$,

$$K_{ij} = [a + i(b-a)/n,\ a + (i+1)(b-a)/n] \times$$
$$\times [c + j(b-a)/n,\ c + (j+1)(b-a)/n], \quad i,j = 0, \ldots, n-1,$$

we obtain

$$\mu(K\cap F) = \iint_K \chi_F(x,y)\,dx\,dy = \sum_{i,j=0}^{n-1}\iint_{K_{ij}} \chi_F(x,y)\,dx\,dy = \iint_{K_{00}} \Psi(x,y)\,dx\,dy\,,$$

where

$$\Psi = \sum_{i,j=0}^{n-1} \chi_F(x+i(b-a)/n,\ y+j(b-a)/n)\,.$$

Suppose now that

$$\Psi(x,y) \leqq n^2-1 \quad \text{for all } (x,y)\in K_{00}\,.$$

Then

$$\mu(K\cap F) \leqq ((n^2-1)/n^2)\mu(K) = (1-1/n^2)\mu(K)\,,$$

which contradicts (1.1). So,

$$\Psi(x_0,y_0) = n^2 \quad \text{for some } (x_0,y_0)\in K_{00}\,.$$

But this is just the same as $X_n\times Y_n \subset F$, where

$$X_n = \{x_0+i(a-b)/n\}_{i=0}^{n-1}, \qquad Y_n = \{y_0+j(a-b)/n\}_{j=0}^{n-1},$$

which proves the proposition.

We now prove a combinatorial lemma.

LEMMA 1. *Let there exist a collection* $\{(X_s', Y_s')\}_{s=1}^{\infty}$ *of pairs of subsets of a group* G, *with* $|X_s'|=|Y_s'|=s$. *Then there exists a collection* $\{(X_n, Y_n)\}_{n=1}^{\infty}$ *of pairs of subsets of* G, *with each pair satisfying*

a) $X_n\subset X_s'$, $Y_n\subset Y_s'$ *for some* s, *depending on* n,

b) $|X_n|=|Y_n|=n$,

(1.2) c) $X_n\cap Y_n=\emptyset$,

(1.3) d) $x+y=x'+y' \Rightarrow x=x', y=y' \quad \forall x,x'\in X_n\ \forall y,y'\in Y_n$.

PROOF. We shall show that we can choose $X_n\subset X_s'$ and $Y_n\subset Y_s'$ where $s=n^3+1$. The case $n=1$ is clear, so suppose that $n>1$ and that sets $\{x_1,\ldots,x_t\}$, $\{y_1,\ldots,y_t\}$, $t<n$, satisfying (1.2) and (1.3) have been chosen. The relation

$$(1.4) \qquad x+y_i = x_j+y_k, \quad x \neq x_j, \qquad i,j,k=1,\ldots,t\,,$$

is satisfied by $t^2(t-1)$ values of x in G, and the relation

$$(1.5) \qquad x = y_j, \quad j=1,\ldots,t\,,$$

is satisfied by t values in G. The set $X_s'\setminus\{x_1,\ldots,x_t\}$ has more than t^3-t^2+t points, so we can choose x_{t+1} from that set such that x_{t+1} satisfies neither of (1.4), (1.5). Proceeding similarly for Y_s', we obtain

$\{x_1,\ldots,x_{t+1}\}$, $\{y_1,\ldots,y_{t+1}\}$ satisfying (1.2), (1.3). The lemma follows by induction.

PROPOSITION 2. *Let E be a closed subset of a metrisable locally compact abelian group G, and let μ,ν be two continuous bounded regular Borel measures on G, such that $|\mu*\nu|(E)>0$. Then there exists a collection $\{(X_n, Y_n)\}_{n=1}^{\infty}$ of pairs of disjoint subsets of G, with*

$$|X_n| = |Y_n| = n \quad \textit{and} \quad X_n+Y_n \subset E .$$

PROOF. Let $E^*=\{(x,y)\in G\times G:\ x+y\in E\}$. Since $|\mu*\nu|(E)>0$ gives $|\mu\times\nu|(E)>0$, it follows from proposition 1 that for any positive integer n there exist sets $X_n, Y_n\subset G$, with $|X_n|=|Y_n|=n$, such that $X_n\times Y_n\subset E^*$. But $X_n\times Y_n\subset E^*$ means that $X_n+Y_n\subset E$, and, by lemma 1, we can assume X_n, Y_n to be disjoint.

2. Independent sets and Helson sets.

We recall that a set E in a group G is *independent* if

$$n_1x_1+n_2x_2+\ldots+n_kx_k=0 \;\Rightarrow\; n_1x_1=\ldots=n_kx_k=0$$

for any choice of integers $n_1,\ldots,n_k$ and distinct points $x_1,\ldots,x_k$ of E.

THEOREM 1. *Let E be an independent closed subset of a metrisable locally compact abelian group G, and let $\mu,\nu>0$ be continuous measures on G. Then*

$$|\mu*\nu|(E) = 0 .$$

PROOF. Suppose $|\mu*\nu|(E)>0$. By proposition 2 and lemma 1, there exist points $x_1,x_2,y_1,y_2\in G$ such that x_i+y_j, $i,j=1,2$, are distinct points of E. But in view of the identity

$$(x_1+y_1)-(y_1+x_2)+(x_2+y_2)-(y_2+x_1) = 0 ,$$

this contradicts the independence of E.

THEOREM 2. *Let E be a Helson set in the locally compact metrisable abelian group G. Let μ,ν be continuous measures on G. Then*

$$|\mu*\nu|(E) = 0 .$$

PROOF. By a well-known result [7] there exist an integer $K>0$, such that, for all systems $(x_1\ldots x_n)\subseteq G$, and for each inteter $s>0$, there are at most Kns points of E of the form

$$\alpha_1x_1+\ldots+\alpha_nx_n ,$$

the α_i being integers with

$$\sum_{1}^{n} |\alpha_i| < 2^s .$$

If $|\mu \times \nu|(E) > 0$, for any n there exist sets $(x_i), (y_j) \subseteq G$, $i,j = 1,\ldots,n$, such that all n^2 points of the form $x_i + y_j$ are in E. Taking $n > K$, we obtain the required contradiction.

3. Sets of analyticity.

We introduce the following definition, which is a modification of Definition 4.3.1 in [10; Ch. 4, § 3].

DEFINITION. Let X, Y be closed sets of the locally compact abelian group G, and let m_1, m_2 be two elements of the set $\{2,3,\ldots,\infty\}$. Then $f \in C(X \cup Y)$ is called an *$\{m_1,m_2\}$-function* if

$$f^{m_1}(x) = 1, \quad f^{m_2}(y) = 1, \qquad \forall x \in X,\ \forall y \in Y ,$$

where $f^\infty(z)$ denotes $|f(z)|$. The pair (X, Y) is called an *$\{\overline{m_1,m_2}\}$-pair* of the group G if. for every $\{m_1,m_2\}$-function f, there exists a continuous character χ of G with

$$|f(z) - \chi(z)| < \tfrac{1}{3} \quad \forall z \in X \cup Y .$$

PROPOSITION 3. *Let G be a locally compact abelian group. Let $\{(X_n', Y_n'\}_{n=1}^{\infty}$ be a sequence of pairs of disjoint subsets of G with $|X_n'| = |Y_n'| = n$. Then there exist a collection $\{(X_r, Y_r)\}_{r=1}^{\infty}$ of pairs of subsets of G and a collection of pairs of points $\{(x_r, y_r)\}_{r=1}^{\infty}$ with*

$$|X_r| = |Y_r| = r, \quad X_r \subset X_n',\ Y_r \subset Y_n' \quad \textit{for some } n(r) ,$$

and each $(X_r - x_r, Y_r - y_r)$ is an $\{\overline{m_1,m_2}\}$-pair of G for some $m_1(r)$, $m_2(r)$.

We shall postpone the proof of this until section 4, in order to proceed immediately to

THEOREM 3. *Let G be a locally compact abelian group. Let E be a compact subset of G. Let $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of pairs of disjoint subsets of G such that*

$$|X_n| = |Y_n| = n \quad \textit{and} \quad X_n + Y_n \subset E .$$

Then E is a set of analyticity [10; Theorem 9.3.5].

PROOF (cf., for example, [3]). We shall show that, for any $t > 0$, there exists a real function $f \in A(E)$ with

$$\|f\|_A \leqq t \quad \text{and} \quad \|e^{if}\|_A > e^{\alpha t} ,$$

where $\alpha > 0$ is independent of t. This shows (cf. [6]) that E is a set of analyticity. We may suppose, by lemma 1, that

$$x+y=x'+y' \Rightarrow x=x', y=y' \quad \forall x,x' \in X_n,\ \forall y,y' \in Y_n,\ n=1,2,\ldots .$$

We may also suppose, by proposition 3, that there exists a collection $\{(x_n,y_n)\}_1^\infty$ of points of G such that each $\{X_n-x_n, Y_n-y_n\}$ is an $\{\overline{m_1,m_2}\}$-pair of G for some m_1,m_2. Then, by [10, Ch. 4, § 3], [5],

$$C(X_n-x_n) \hat{\otimes} C(Y_n-y_n) \cong A(X_n+Y_n-x_n-y_n)$$

with

$$\|\ldots\|_{\hat{\otimes}} \leqq \|\ldots\|_A ,$$

$$\|\ldots\|_A \leqq 36\|\ldots\|_{\hat{\otimes}} \quad \text{(36 is an arbitrary constant)} .$$

Hence $C(X_n)\hat{\otimes} C(Y_n) \cong A(X_n+Y_n)$ with the same norm equivalence. Let D^∞ denote the product of countably many copies of Z_2, the group of two elements. Given $t>0$, there exists a real $f_1 \in A(D^\infty)$ with

$$\|f_1\|_A \leqq t \quad \text{and} \quad \|e^i f_1\| > e^{\beta t} ,$$

where $\beta > 0$ is independent of t, cf. [2]. So, for r large enough, there exists a real function $h \in A(Z_2{}^r)$ with

$$\|h\| \leqq t \quad \text{and} \quad \|e^{ih}\| > e^{\beta/2t} .$$

The map $M\colon A(Z_2{}^r) \to C(Z_2{}^r)\hat{\otimes} C(Z_2{}^r)$ defined by

$$Mg(x,y) = g(x+y)$$

is an isometric injection [10, Ch. 8]. But $C(Z_2{}^r) \cong C(X_n)$ and $C(Y_n)$ for $n=2^r$, so there exists a real function $k \in A(X_n+Y_n)$ with

$$\|k\| \leqq 36\,t \quad \text{and} \quad \|e^{ik}\| > e^{\beta t/2} .$$

Hence there exists a real function $f \in A(E)$ with

$$\|f\|_{A(E)} < t \quad \text{and} \quad \|e^{if}\| > e^{\alpha t},$$

where $\alpha = 74\beta$.

From theorem 3 and proposition 2 we deduce

THEOREM 4. *Let E be a compact subset of a metrisable locally compact abelian group G. Let μ,ν be continuous regular bounded Borel measures on G. Then*

$$|\mu*\nu|(E) > 0 \Rightarrow E \text{ is a set of analyticity.}$$

We note that theorem 2 is an immediate consequence of theorem 4.

4. The proof of proposition 3.

We require some preliminary notations and lemmas.

T will denote the group of complex numbers of unit modulus under the operation of multiplication. We use the additive convention for all other abelian groups. The symbols X_r, Y_r will denote sets containing precisely r elements.

DEFINITIONS. *A* set $X_r = \{e^{i\pi\alpha_1}, \ldots, e^{i\pi\alpha_r}\} \subset T$, is called *q-lacunary* for some $q > 1$ if

i) $|\alpha_{j+1}/\alpha_j| > q, \quad j = 1, \ldots, r-1$,
ii) $|\alpha_r| < 1/(3\pi)$.

A set $X_r \subset T$ will be called an *∞-set* if, given any function $f\colon X_r \to T$, there exists a continuous character χ of T with

$$|\chi(x) - f(x)| < \tfrac{1}{3} \quad \text{for all } x \in X_r .$$

If G is a locally compact abelian group G, a set $X_r \subset G$ will be called a *K_p-set* for some prime number p, if, given any function $f\colon X_r \to T$ with $f^p(x) = 1$ for all $x \in X_r$, there exists a continuous character χ of G with

$$\chi(x) = f(x) \quad \text{for all } x \in X_r .$$

LEMMA 2. *If* $X_r = \{e^{i\pi\alpha_1}, \ldots, e^{\pi\alpha_r}\} \subset T$ *is a q-lacunary set for an integer* $q > 3\pi + 2$, *then it is an ∞-set.*

PROOF. Let $f\colon X_r \to T$ be a given function. Since $|\alpha_1| < 1/(3\pi)$, there exists a set S_1 of consecutive integers satisfying

$$|e^{i\pi n\alpha_1} - f(e^{i\pi\alpha_1})| < \tfrac{1}{3} \quad \text{for all } n \in S_1$$

and

$$1 \leqq |S_1| \leqq [2/(3\pi|\alpha_1|)] .$$

Now

$$[2/(3\pi|\alpha_1|)]|\alpha_2| \geqq (2/3\pi)|\alpha_2/\alpha_1| - |\alpha_2| \geqq 2q/(3\pi) - |\alpha_2| \geqq 2 ,$$

so there exists a subset S_2 of consecutive integers of S_1 satisfying

$$|e^{i\pi n\alpha_2} - f(e^{i\pi\alpha_2})| < \tfrac{1}{3} \quad \text{for all } n \in S_2$$

and

$$1 \leqq |S_2| \leqq [2/(3\pi|\alpha_2|)] .$$

Continuing thus, any member of S_r defines a character χ such that

$$|\chi(e^{i\alpha_j}) - f(e^{i\alpha_j})| < \tfrac{1}{3}, \qquad j = 1, \ldots, r .$$

LEMMA 3. *Let* $q > 1$ *and* $K > 6\pi$ *be integers. For any integer* $r > 0$, *let* $n = K(2q)^r$ *and let* X_n' *be the set*

$$X_n' = \{x_j\}_1^n = \{e^{in\alpha_j}\}_1^n \subset T ,$$

where the x_j *are distinct. Then we can choose a subset* $X_r = \{x_k\}_{k=1}^r$ *of distinct points of* X_n' *and a point* x_{j_0} *of* X_n' *such that* $X_r x_{j_0}^{-1}$ *is q-lacunary.*

PROOF. Clearly, by picking a subset of X_n' if necessary, we can assume

$$|\alpha_j - \alpha_i| < 1/(3\pi) \quad \forall x_j, x_i \in X_n' .$$

We now proceed by induction, starting trivially.

Suppose that, from any subset of X_n' with $(2q)^{s-1}$ elements, we can pick a subset X_{s-1} and a point $x \in X_n'$ such that $X_{s-1}x^{-1}$ is q-lacunary. Let $Y \subset X_n'$ have $(2q)^s$ elements. We may suppose, without loss of generality, that

$$Y = \{x_j\}_{j=1}^{(2q)^s} \quad \text{with } \alpha_j < \alpha_{j+1} .$$

Partition Y into $2q$ sets of consecutive points by putting

$$Y^k = \{x_j\}_{j=(k-1)(2q)^{s-1}+1}^{k(2q)^{s-1}} \quad k = 1,\ldots,2q .$$

Let h be such that

$$|\alpha_{h(2q)^{s-1}} - \alpha_{(h-1)(2q)^{s-1}+1}| = l, \text{ say }, \tag{4.1}$$

is minimal. Choose a subsequence $\{x_{j_m}\}_{m=1}^{s-1} \subset Y^h$ and a point $x_{j_0} \in Y^h$ such that $\{x_{j_m} x_{j_0}^{-1}\}_{m=1}^{s-1}$ is q-lacunary. If $h \leqq q$, put $x_{j_s} = x_{(2q)^s}$; if $h > q$ put $x_{j_s} = x_1$. Then

$$|\alpha_{j_s} - \alpha_{j_0}| > ql \quad \text{and} \quad |\alpha_{j_{s-1}} - \alpha_{j_0}| \leqq l$$

by (4.1), so $\{z_{j_m} z_{j_0}^{-1}\}_{m=1}^{s}$ is q-lacunary. This completes the induction.

LEMMA 4. *Let* X_{3r}, Y_{3r} *be q-lacunary subsets of T. Then we can pick* $X_r \subset X_{3r}$, $Y_r \subset Y_{3r}$ *such that* $X_r \cup Y_r$ *is q-lacunary.*

PROOF. Put $X_{3r} = \{x_j\}_1^{3r} = \{e^{i\pi\alpha_j}\}_1^{3r}$, $Y_{3r} = \{y_j\}_1^{3r} = \{e^{i\pi\beta_j}\}_1^{3r}$. We choose $\{\gamma_j\}_1^{2r}$ by the following method. Choose

$$\begin{aligned} \gamma_1 &= \alpha_1 \quad \text{if} \quad |\alpha_2| \leqq |\beta_2| , \\ &= \beta_1 \quad \text{if} \quad |\alpha_2| > |\beta_2| . \end{aligned}$$

Suppose now that $\gamma_1,\ldots,\gamma_t$, $t < 2r$, have been chosen and that, without loss of generality, $\gamma_t = \alpha_k$, and the last β chosen was β_l (the symbol β_0 denoting that no β has as yet been chosen). There are four cases:

i) if r elements of $\{\alpha_j\}$ have been chosen put $\gamma_{t+1} = \beta_{l+3}$,
ii) if r elements of $\{\beta_0\}$ have been chosen put $\gamma_{t+1} = \alpha_{k+1}$,

and if neither i) nor ii), then

iii) if $|\alpha_{k+2}| \leqq |\beta_{l+3}|$ put $\gamma_{t+1} = \alpha_{k+1}$,
iv) if $|\alpha_{k+2}| > |\beta_{l+3}|$ put $\gamma_{t+1} = \beta_{l+3}$.

It is clear that this process is possible and yields the desired result.

Now we introduce a few more notations and definitions.

$$T^{\omega} = \prod_{j=1}^{\infty} T_j$$

will denote the direct product of countably many copies of T.

$$\pi_j\colon \; T^{\omega} \to T_j$$

will denote the canonical projections and $\bar{\pi}_j\colon T_j \to T^{\omega}$, the canonical injections.

$Z_j(q)$ will denote the cyclic group of order q embedded in $\bar{\pi}_j(T_j)$. If $q = \infty$, $Z_j(q)$ will denote an arbitrary but fixed embedding of the group of integers in $\bar{\pi}_j(T_j)$.

Let S be the set of all numbers of the form p^h where p runs through the primes and $h = h(p)$ is the maximal integer with $p^h < 3\pi$. Then H will denote the group

$$\prod_{S} \left(\prod_{j=1}^{\infty} Z_j(p^h) \right),$$

and f_p will denote the canonical projection

$$f_p\colon \; H \to \prod_{j=1}^{\infty} Z_j(p^h) .$$

Since

$$H = \prod_{j=1}^{\infty} \left(\prod_{S} Z_j(p^h) \right),$$

we shall identify H with its natural image in T^{ω} (as explained above).

Definition. A set $X_r = \{x_1, \ldots, x_r\} \subset T^{\omega}$ will be called *∞-triangular* if there exist $i_1, \ldots, i_r$ such that for $1 \leqq j < t \leqq r$, $1 \leqq m \leqq [3\pi]$,

(4.2)
$$\text{i)} \; \pi_{i_j}(x_t) = 1 ,$$
$$\text{ii)} \; \{\pi_{i_t}(x_t)\}^m \neq 1 .$$

A set $X_r = \{x_1, \ldots, x_r\} \subset H \subset T^{\omega}$ will be called *p^k-triangular* for some prime p and some $k \leqq h(p)$, if

$$f_p(X_r) \subset \prod_{j=1}^{\infty} Z_j(p^k)$$

and if there exist $i_1, \ldots, i_r$ such that for $1 \leqq j < t \leqq r$,

$$(4.3) \qquad \begin{aligned} &\text{i)} \ \pi_{i_j} f_p(x_t) = 1 , \\ &\text{ii)} \ \pi_{i_t} f_p(p^{k-1}x_t) \neq 1 \end{aligned}$$

A set $X_r \subset T^\omega$ will be called *triangular* if it is either ∞-triangular or p^k-triangular, for some p and k, and $i_1, \ldots, i_r$ will be called the *determining co-ordinates* of X_r.

LEMMA 5. a) *Let $X_r = \{x_1, \ldots, x_r\}$ be an ∞-triangular set in T^ω. Then X_r is an ∞-set.*

b) *Let $X_r = \{x_1, \ldots, x_r\}$ be a p^k-triangular set in T^ω. Then X_r is a K_p-set.*

PROOF. a) Let $f\colon X_r \to T$ be a given function. Writing χ_i for $\chi \circ \bar{\pi}_i \circ \pi_i\colon T^\omega \to T_i$, we can choose a character χ of T^ω with

$$\chi_i \equiv 1 \quad \text{for } i \neq i_1, \ldots, i_r ,$$

$$|\chi_{i_r}(x_r) - f(x_r)| < \tfrac{1}{3},$$

by ii) of (4.2),

$$|\chi_{i_{r-1}}(x_{r-1})\, \chi_{i_r}(x_{r-1}) - f(x_{r-1})| < \tfrac{1}{3} ,$$

and so on. Thus, taking into account i) of (4.2),

$$|\chi(x) - f(x)| < \tfrac{1}{3} \quad \text{for all } x \in X_r .$$

b) Let $f\colon X_r \to T$ satisfy $f^p(x) = 1$ for all $x \in X_r$. Then we choose χ such that

$$(4.4) \qquad \begin{aligned} \chi_{i_r}(x_r) &= f(x_r) , \\ \chi_{i_{r-1}}(x_{r-1})\, \chi_{i_r}(x_{r-1}) &= f(x_{r-1}) , \end{aligned}$$

and so on, where

$$\text{if } \chi \bar{\pi}_{i_j}(e^{i\pi\alpha}) = e^{i2\pi n\alpha}, \quad \text{then } n \equiv 0 \ (\text{mod}\, q),$$

where $q = p_1^{h_1} \ldots p_l^{h_l} p^{k-1}$, $p_i^{h_i} \in S$ and $p_i \neq p$. This is possible, by (4.3), and since every factor of (4.4) is a pth root of unity.

LEMMA 6. *Let k be a fixed integer and let $X_n' \subset T^\omega$ be fixed sets of n points of T^ω satisfying*

$$|\pi_i(X_n')| < k \quad \textit{for all } i, n = 1, 2, \ldots .$$

Then, given $r > 0$, and for large enough n, there exists a subset X_r of r points of X_n' and $x \in T^\omega$ such that $X_r - x$ is triangular.

PROOF. Suppose there exists an i_1 such that there exist $x, x' \in \pi_{i_1}(X_n')$ with $\pi_i(m(x - x')) \neq 1$ for any integer $m < 3\pi$. Select $X^1 \subset X_n'$ with

$$\pi_{i_1}(x) = x^1 \in T_{i_1} \quad \text{for all } x \in X^1$$

and

$$|X^1| > n/k .$$

Choose $x_1 \in (X_n' \setminus X^1) - \bar{\pi}_{i_1}(x^1)$ such that

$$\pi_{i_1}(mx_1) \neq 1 \quad \text{for all integers } m < 3\pi .$$

Now, if we can, we select i_2, X^2, $x^2(\in T_{i_2})$ and

$$x_2 \in (X^1 \setminus X^2) - \bar{\pi}_{i_1}(x^1) - \bar{\pi}_{i_2}(x^2),$$

where

$$\pi_{i_2}(mx_2) \neq 1 \quad \text{for all integers } m < 3\pi$$

and $|X^2| > n/k^2$. Then

$$\pi_{i_1}(x_2) = 1 .$$

So we either obtain a set $X_r \subset X_n' - (\bar{\pi}_{i_1}(x^1) + \dots + \bar{\pi}_{i_{r-1}}(x^{r-1}))$ which is ∞-triangular, or, for some t, we can no longer find j such that there exist $x, x' \in X^t$ with $\pi_j(m(x-x')) \neq 1$ for all integers $m < 3\pi$. Then, by translation, we can assume $X^t \subset H$. To simplify the notation we shall omit the index t.

For some prime p, $|f_p(X)| = n^1 \geqq |X|^{\frac{1}{4}}$. As before, let h be the largest integer such that $p^h < 3\pi$.

If $h = 1$, it is clear that we can use the method of the earlier part of this lemma to choose a subset X_r of X and $x \in X$ such that $X_r - x$ is p-triangular.

If $h > 1$, consider the map

$$g\colon \prod_{j=1}^{\infty} Z_j(p^h) \to \prod_{j=1}^{\infty} Z_j(p^h) \Big/ \prod_{j=1}^{\infty} Z_j(p^{h-1}) \cong \prod_{j=1}^{\infty} Z_j(p) .$$

If $|gf_p(X)| \geqq k^r$, we choose $X^0 \subseteq X$ with

$$gf_p(x) \neq gf_p(x') \text{ for all } x \neq x' \text{ in } X^0 , \tag{4.5}$$

and $|X^0| \geqq k^r$. We can then, as before, choose $X_r \subset X^0$ and $x \in T^\omega$ such that $X_r - x$ is p^h-triangular.

If $|gf_p(X)| < k^r$, then there exists a subset $\bar{X}$ of X such that $|f_p(\bar{X})| > n^1/k^r$ and

$$gf_p(x) = gf_p(x') \quad \text{for all } x, x' \in \bar{X} .$$

Let $x'' \in \bar{X}$, then

$$f_p(\bar{X} - x'') \subset \prod_1^{\infty} Z_j(p^{h-1}) .$$

Continuing we obtain a set X^0 with

$$f_p(X^0 - \bar{x}) \subset \prod_1^{\infty} Z_j(p^t) ,$$

some t, $1 \leqq t \leqq h-1$, some $\bar{x}$, with $|f_p(X^0)| > n^1/k^{2r}$ and

$$gf_p(x) \neq gf_p(x') \quad \text{for all } x \neq x' \in X^0 - \bar{x} ,$$

where g denotes the canonical map

$$g\colon \prod_{j=1}^{\infty} Z_j(p^t) \to \prod_{j=1}^{\infty} Z_j(p) .$$

If n is large enough, $|f_p(X^0)| > k^r$, so we can choose $X_r \subset X^0$ and $x \in T^\omega$ such that $X_r - x$ is p^t-triangular.

For convenience of reference, we now repeat the statement of

PROPOSITION 3. *Let G be a locally compact abelian group. Let $\{(X_n', Y_n')\}_{n=1}^{\infty}$ be a sequence of pairs of disjoint subsets of G with $|X_n'| = |Y_n'| = n$. Then there exists a collection $\{(X_r, Y_r)\}_{r=1}^{\infty}$ of pairs of subsets of G and a collection of pairs of points $\{(x_r, y_r)\}_{r=1}^{\infty}$ with*

$$|X_r| = |Y_r| = r, \qquad X_r \subset X_n',\ Y_r \subset Y_n' \quad \textit{for some } n(r) ,$$

and each $(X_r - x_r, Y_r - y_r)$ is an $\{\overline{m_1, m_2}\}$-pair of G for some $m_1(r)$, $m_2(r)$.

PROOF. Let G_n be the subgroup of G generated by $X_n' \cup Y_n'$. Since G_n is finitely generated, it is of the form $\prod_{j=1}^{m} Z_j(q_j)$. By embedding $Z_j(q_j)$ in T_j, we achieve an (algebraic) embedding of G_n in T^ω. We shall identify X_n', Y_n' with their images in T^ω. There are three cases:

1. For any $s > 0$, there exist n, i, j such that $|\pi_i(X_n')| > s$, $|\pi_j(Y_n')| > s$.
2. For any $s > 0$, there exists n, i such that $|\pi_i(X_n')| > s$, but for all j, m, $|\pi_j(Y_m')| < k$ where k is an integer independent of s, n.
3. There exist a $k > 0$ such that $|\pi_i(X_n')| < k$ and $|\pi_i(Y_n')| < k \ \ \forall i, n$.

These cases are exhaustive (after possible interchange of X_n' and Y_n').
Case 1 divides into two subcases.

1a. For any $s > 0$, there exist i, n such that $|\pi_i(X_n')| > s$ and $|\pi_i(Y_n')| > s$. By Lemmas 3 and 4, we find $X_r \subset X_n'$, $Y_r \subset Y_n'$ and $x_r, y_r \in G_n \subseteq T^\omega$ such that

$$\{\pi_i(X_r - x_r)\} \cup \{\pi_i(Y_r - y_r)\} \quad \text{is } q\text{-lacunary} .$$

By lemma 2, $(X_r - x_r, Y_r - y_r)$ is an $\{\overline{\infty, \infty}\}$-pair of T^ω.

1b. For any $s > 0$, there exist i, n such that $|\pi_i(X_n')| < s$, but there is a $k > 0$ such that

$$|\pi_i(X_n')| > s \ \Rightarrow\ |\pi_i(Y_n')| < k .$$

However, there exist j, n such that

$$|\pi_i(X_n')| > s \quad \text{and} \quad |\pi_j(Y_n')| > s, \qquad i \neq j .$$

Taking s large enough, and fixing i and j, we choose a subset $Y \subset Y_n'$ such that

$$\pi_i(y') = \pi_i(y) = y^1 \text{ say}, \quad \forall y,y' \in Y ,$$

and

$$|\pi_j(Y)| > s .$$

Then, by lemma 3, we can find $X_r \subset X_n'$, $Y_r \subset Y_n$, $x \in T_i$, $y \in T_j$ such that

$$\pi_i(X_r - x_r),\ \pi_j(Y_r - y_r) \quad \text{are } q\text{-lacunary} ,$$

where

$$x_r = \bar{\pi}_i(x), \quad y_r = \bar{\pi}_i(y_i') + \bar{\pi}_j(y) .$$

Then, if f: $(X_r - x_r) \cup (Y_r - y_r) \to T$ is a given function, there exists, by lemma 2, a character χ on T^ω such that

$$|\chi_j(y) - f(y)| < \tfrac{1}{3} \quad \forall y \in Y_r - y_r$$

and

$$|\chi_i(x)\chi_j(x) - f(x)| < \tfrac{1}{3} \quad \forall x \in X_r - x_r .$$

So $(X_r - x_r, Y_r - y_r)$ is an $\{\overline{\infty, \infty}\}$-pair of T^ω.

Case 2. Taking s large enough, we can find arbitrary large n such that, for some $i(n)$, $|\pi_i(X_n')| > s$. As in 1b we select $Y \subset Y_n'$ such that

$$\pi_i(y) = \pi_i(y') \quad \forall y,y' \in Y$$

and

$$|Y| \geqq n/k .$$

By Lemmas 3 and 6, we select subsets $X_r \subset X_n'$, $Y_r \subset Y$ and $x_r, y_r \in G_n \subset T^\omega$ such that $\pi_i(X_r - x_r)$ is q-lacunary, $Y_r - y_r$ is triangular and, in addition,

$$\pi_i(y) = 1 \quad \forall y \in Y_r - y_r .$$

Then, by lemmas 2 and 5, the pair $(X_r - x_r, Y_r - y_r)$ is an $\{\overline{\infty, m_2}\}$-pair for some m_2.

Case 3. Using lemma 6, it is easy to see that, for n large enough we can choose sets $X_r \subset X_n'$, $Y_r \subset Y_n'$ and points $x_r, y_r \in G_n \subseteqq T^\omega$ such that $X_r - x_r$, $Y_r - y_r$ are triangular, with the additional condition that, if $i_1, \ldots, i_r, j_1, \ldots, j_r$, are the determining coordinates for $X_r - x_r$, $Y_r - y_r$ respectively, then

$$\pi_{i_t}(y) = 1 \quad \forall y \in Y_r - y_r, \quad t = 1, \ldots, r ,$$
$$\pi_{j_t}(x) = 1 \quad \forall x \in X_r - x_r .$$

This ensures, by lemma 6, that $(X_r - x_r, Y_r - y_r)$ forms an $\{\overline{m_1, m_2}\}$-pair of T^ω for some m_1, m_2.

To conclude the proof of the proposition, we need only remark that, if $(X_r, Y_r) \subset G_n \subset T^\omega$ is an $\{\overline{m_1, m_2}\}$-pair for T^ω, it is an $\{\overline{m_1, m_2}\}$-pair for G. This follows since, if X is a finite set in G, and χ is an (algebraic) character on G, there exist, for any $\varepsilon > 0$, a continuous character $\bar{\chi}$ on G, with

$$|\chi(x) - \bar{\chi}(x)| < \varepsilon \quad \forall x \in X ,$$

cf. [4].

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