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HYPERPLANES AND LINES ASSOCIATED WITH FAMILIES OF COMPACT SETS IN LOCALLY CONVEX SPACES

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1. Introduction.

Let $\mathscr{A}$ be a finite family of two or more nonempty disjoint compact sets in a linear topological space X. Suppose $B = \bigcup \{A : A \in \mathscr{A}\}$ is an infinite set which is not contained in any straight line of X. A theorem established in [2] for X any real normed space and, previously, in [1] for $X = E^n$ states that then a closed hyperplane exists which intersects exactly two members of $\mathscr{A}$. (Additional references and some motivation for the interest in the above result are also to be found in [1] and [2].) It is the main purpose of the present note to extend the above theorem to the case when X is an arbitrary real Hausdorff locally convex space. The proof of our main result (Theorem 4) is based on certain facts pertaining to the intersections of single members of $\mathscr{A}$ by hyperplanes or lines. These facts, which seem to be of independent interest, are discussed in Section 3.

2. Preliminaries.

2.1 Notation. For S an arbitrary set in a linear topological space $\overline{\text{co}}S$, $X \setminus S$ and S' denote the closed convex hull, the complement and the set of all accumulation points of S respectively.

2.2 Lemma. *Let C be a compact convex set in a real locally convex Hausdorff linear topological space E, $W \subset E$ a compact set with $W' \subset C$ and L a straight line such that $L \cap (W \cup C) = \emptyset$. Then a closed hyperplane π exists such that $L \subset \pi$ and $\pi \cap (W \cup C) = \emptyset$.*

Proof. By a standard separation theorem [3, p. 119] there exists a hyperplane π_1 strongly separating L and C. Let π_2 be a hyperplane parallel to π_1 through an arbitrary point of L. Then, as can be easily seen, $L \subset \pi_2$ and $C \cap \pi_2 = \emptyset$. Clearly $W \cap \pi_2$ is finite (or empty). If empty,

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as is always the case when $\dim E \leqq 2$, then π_2 itself is as desired. If not, let σ be a closed linear variety of codimension 2 such that $L \subset \sigma \subset \pi_2$ and $\sigma \cap W = \emptyset$. Let $x \in C$, $y \in \pi_2 \setminus \sigma$ and suppose λ_t is the hyperplane spanned by σ and the point $tx + (1-t)y$, $0 < t < 1$. It is easily seen that the family $\{E_t^+ \colon 0 < t < 1\}$ of all open halfspaces E_t^+ determined by λ_t and containing x is an open cover of C. This implies the existence of a $\bar{t}$ with $0 < \bar{t} < 1$ such that $E_t^+ \supset C$ for all t with $0 < t < \bar{t}$. Our assumptions on W, the fact that $\sigma \cap W = \emptyset$ and the definition of the E_t^+ now clearly imply that only finitely many of these intersect W. Hence, for some t, $\pi = \lambda_t$ is as desired.

2.3 Lemma. *Let A be a compact set in a complete locally convex Hausdorff linear topological space. Then the set of all extreme points of $\overline{\text{co}}\, A$ is contained in A.*

Proof. The set $\overline{\text{co}}\, A$ is compact [5, p. 60]; hence, by a theorem of Milman [4, p. 335] the set of all extreme points of $\overline{\text{co}}\, A$ is contained in A, as asserted.

3. Hyperplanes and lines intersecting one set only.

3.1 Theorem 1. *Let $\mathscr{A}$ be a finite nonempty family of disjoint nonempty compact subsets of a real locally convex Hausdorff linear topological space X. Then there exists a closed hyperplane α in X which is a support hyperplane of $B = \bigcup \{A \colon A \in \mathscr{A}\}$ and is disjoint from all members of $\mathscr{A}$ but one.*

Proof. Since all members of $\mathscr{A}$ are complete it suffices to find a hyperplane $\hat{\alpha}$ in the completion $\hat{X}$ of X with the analogous properties (in other words, no restriction of generality would result from assuming X to be complete); for then $\alpha = \hat{\alpha} \cap X$ would be as desired. In $\hat{X}$ then, let e be an extreme point of $\overline{\text{co}}\, B$. By the previous lemma $e \in B$. Hence e belongs to exactly one member of $\mathscr{A}$, say A_1. Let $B_1 = B \setminus A_1$. Then $e \notin \overline{\text{co}}\, B_1$, for $e \in \overline{\text{co}}\, B_1$ implies, as before, $e \in B_1$ against the disjointness of members of $\mathscr{A}$. Let f be a continuous linear functional with $f(e) > f(z)$ for all $z \in \overline{\text{co}}\, B_1$ and suppose $m = \max\{f(x) \colon x \in A_1\}$. Then clearly, $f^{-1}(m) = \{y \colon f(y) = m\}$ may serve as $\hat{\alpha}$.

3.2 Theorem 2. *Let $\mathscr{A}$ be a finite nonempty family of disjoint nonempty compact sets in a real locally convex Hausdorff space X. Let $x \in X \setminus \overline{\text{co}}\, B$ where $B = \bigcup \{A \colon A \in \mathscr{A}\}$ and suppose $B \cup \{x\}$ is not contained in any straight line of X. Then a nonempty set E consisting of at most* *four points of B and a neighbourhood $U \subset X \setminus \overline{\text{co}}\, B$ of x exist such that for each $u \in U \setminus \{x\}$ there is an $e \in E$ with the property that the straight line L through u and e does not contain x and, if $e \in A_e \in \mathscr{A}$, $L \cap \overline{\text{co}}\,(B \setminus A_e) = \emptyset$.*

PROOF. Let α_0 be a closed hyperplane strictly separating x and $\overline{\text{co}}\, B$ and α_1 a hyperplane parallel to α_0, intersecting the cone C spanned by x and $\overline{\text{co}}\, B$ and such that x and B lie on the same side of it. As in the proof of Theorem 1 no restriction of generality results from assuming X to be complete, and this we wish to do. Thus $\overline{\text{co}}\, B$ is compact and so is $Z = \alpha_1 \cap C$. (For this and other facts pertaining to extreme points, cones and extreme rays used in this proof see e.g. [4, pp. 333–345].)

Being the closed convex hull of its extreme points, Z must have at least two such points z_1, z_2 not on the same line through x. The rays

$$R_i = \{y\colon\ y = x + t(z_i - x),\ t \geqq 0\}\ , \quad i = 1,2\ ,$$

are then both extreme rays. Hence each one of them contains at least one, and at most two, extreme points of $\overline{\text{co}}\, B$. We will show that the set consisting of these points may serve as E. Let L_i be the straight line containing R_i, $i = 1,2$. To prove the theorem it suffices to show that to each L_i there is a neighbourhood U_i of x such that if $u \in U_i \setminus L_i$ then there is an $e \in E \cap R_i$ such that the straight line L through u and e is disjoint from $\overline{\text{co}}\,(B \setminus A_e)$; for then $U = U_1 \cap U_2$ would clearly satisfy the conclusion of the theorem. For the construction of U_i (i will be taken fixed in the sequel) we distinguish between the cases when $E \cap R_i$ consists of one or two points. Without loss of generality we may, in the first case, assume that the origin 0 is the single extreme point on R_i. Then 0 belongs to a unique member A_0 of $\mathscr{A}$ and $\overline{\text{co}}\,(B \setminus A_0) \cap R_i = \emptyset$. Let $w_0 = \frac{1}{2}(x + z_0)$, where $\{z_0\} = R_1 \cap \alpha_0$ and $w_1 = 2z_1$; let λ be a closed hyperplane separating $\overline{\text{co}}\,(B \setminus A_0)$ and L_i and λ^- the open halfspace determined by λ which contains L_i. Let V be a convex neighbourhood of 0 such that $V \subset \lambda^-$ and $(w_j + V) \cap \alpha_j = \emptyset$ for $j = 0, 1$. It is readily seen that

$$U_i = \{y\colon\ y = tv,\ v \in w_0 + V,\ t \geqq 0\} \cap \{y\colon\ y = tv,\ v \in w_1 + V,\ t \leqq 0\}$$

is as desired.

In a similar manner one applies the above argument to the somewhat more general case when $R_i \cap \overline{\text{co}}\, B$ is a subset of a single member of $\mathscr{A}$.

Suppose then r and s are two distinct members of E on R_i with r belonging to the open interval (x,s) and let $r \in A_r \in \mathscr{A}$, $s \in A_s \in \mathscr{A}$

(where A_r and A_s are not necessarily distinct members of $\mathscr{A}$). Let $K_r=\overline{\mathrm{co}}(B\setminus A_r)$ and $K_s=\overline{\mathrm{co}}(B\setminus A_s)$; let L_r be the closed ray emanating from r and containing x, L_s the closed ray emanating from s, contained in R_i and not containing x. We clearly have $L_r\cap K_r=L_s\cap K_s=\emptyset$. Hence closed hyperplanes λ_r and λ_s exist strictly separating L_r from K_r and L_s from K_s respectively. If λ_r^- is the open halfspace determined by λ_r which is disjoint from K_r then, clearly, $x\in\lambda_r^-$. Since, by assumption, $L_i\cap\overline{\mathrm{co}}B$ is not contained in A_s, clearly λ_s separates s and x. Hence, if λ_s^+ is the open halfspace determined by λ_s which contains K_s then $x\in\lambda_s^+$. Thus $\lambda_r^-\cap\lambda_s^+\cap\alpha_0^-$, where α_0^- is the open halfspace determined by α_0 which contains x, is a neighbourhood W of x. Let u be any point in $W\setminus L_i$ and consider the straight lines $\overline{ur},\overline{us}$ joining u with r and s respectively. We claim that either the first is disjoint from K_r or the second is disjoint from K_s. Suppose this is not so. Then, since $u\in\lambda_r^-$, r is between u and $\overline{ur}\cap K_r$ on the straight line $\overline{ur}$; on the other hand, since $u\in\lambda_s^+$, $\overline{us}\cap K_s$ is contained in the open interval (u,s). Thus in the (two-dimensional) plane spanned by L_i and u points of $\overline{\mathrm{co}}B$ are to be found on both sides of L_i which is clearly incompatible with the fact that $R_i\subset L_i$ is an extreme ray. This contradiction shows that W may serve as U_i completing the proof of the theorem.

3.3 As an immediate consequence of Theorem 2 and known separation theorems (or, alternatively, by 2.2) we have the following.

THEOREM 1′. *Let X and $\mathscr{A}$ be as in Theorem 2. Then the set of all points in $X\setminus\overline{\mathrm{co}}B$ through which there is no closed hyperplane intersecting exactly one member of $\mathscr{A}$ is discrete (or empty).*

3.4 THEOREM 3. *Let $X,\mathscr{A},B$ and x be as in Theorem 2, and let M be a subset of X with $x\in M'$. Then there exist a set $A\in\mathscr{A}$, a point $e\in A$ and an infinite set Λ of straight lines L passing through points of M satisfying*

$$L\cap A\neq\emptyset\,, \tag{1}$$

$$L\cap\overline{\mathrm{co}}(B\setminus A)=\emptyset\,, \tag{2}$$

$$\bigcap\{L: L\in\Lambda\}=\{e\}\,. \tag{3}$$

PROOF. Let U and E be as in the conclusion of Theorem 2 and suppose Λ_e is the collection of all straight lines L, with $x\notin L$, joining points of $M\cap U$ with $e\in E$ and satisfying (1) and (2) (for $A=A_e$ where $e\in A_e$). Should Λ_e be finite for each $e\in E$ then, clearly, a neighbourhood $V\subset U$ of x could be found such that no straight line in $\bigcup\{\Lambda_e: e\in E\}$ would pass through any of the points of $M \cap V$. By Theorem 2 this is impossible.

4. The main result.

THEOREM 4. *Let $\mathscr{A}$ be a finite family of two or more nonempty disjoint compact sets in a real locally convex Hausdorff linear topological space X. Suppose $B = \bigcup \{A : A \in \mathscr{A}\}$ is an infinite set which is not contained in any straight line of X. Then there exist a hyperplane π and two members A_1, A_2 of $\mathscr{A}$ so that*

$$\pi \cap A_i \neq \emptyset, \quad i = 1,2, \tag{4}$$

$$\pi \cap A = \emptyset, \quad A \in \mathscr{A} \setminus \{A_1, A_2\}, \tag{5}$$

and

$$\pi \cap \overline{\mathrm{co}}(B' \setminus (A_1 \cup A_2)) = \emptyset. \tag{6}$$

PROOF. As before, we may assume X to be complete. Let $\mathscr{A}' = \{A' : A \in \mathscr{A}, A' \neq \emptyset\}$. By Theorem 1 there exists a closed hyperplane α which supports B' and intersects only one member of $\mathscr{A}'$ say A_1'. Let $a \in \alpha \cap A_1'$ and put $B_1 = \bigcup \{A : A \in \mathscr{A}, A \neq A_1\}$. The theorem is easily seen to be true in the special case when B_1' is a subset (possibly empty) of a straight line through a. Let α^+ be the closed halfspace determined by α which contains B_1' and let

$$\mathscr{A}_1^+ = \{A \cap \alpha^+ : A \in \mathscr{A}, A \neq A_1, A \cap \alpha^+ \neq \emptyset\}.$$

We note that the members of $\mathscr{A}_1^+$ are all closed, hence compact. Since

$$B_1^+ = B_1 \cap \alpha^+ = \bigcup \{A^+ : A^+ \in \mathscr{A}_1^+\}$$

is a compact subset of α^+ we have $a \notin \overline{\mathrm{co}} B_1^+$ by Lemma 2.3. Hence, by Theorem 3 there is an infinite collection Λ of straight lines L through points of A_1 intersecting a single member A_e^+ of $\mathscr{A}^+$ such that

$$\bigcap \{L : L \in \Lambda\} = \{e\} \subset A_e^+ \quad \text{and} \quad L \cap \overline{\mathrm{co}}(B_1^+ \setminus A_e^+) = \emptyset.$$

Since $B_1 \setminus B_1^+$ is finite only finitely many members of Λ can intersect this last set so that a $L \in \Lambda$ exists which satisfies (4), (5) and (6) with $A_2 = A_e$ and π replaced by L. The conclusion of the theorem now follows from 2.2 upon setting $C = \overline{\mathrm{co}}(B' \setminus (A_1 \cup A_2))$ and $W = B \setminus (A_1 \cup A_2)$.

REFERENCES

1. M. Edelstein, F. Herzog and L. M. Kelly, *A further theorem of the Sylvester type*, Proc. Amer. Math. Soc. 14 (1963), 359–363.
2. M. Edelstein and L. M. Kelly, *Bisecants of finite collections of sets in linear spaces*, Canad. J. Math. 18 (1966), 375–380.

3. J. L. Kelley, I. Namioka et al., *Linear topological spaces*, Princeton, 1963.
4. G. Köthe, *Topologische lineare Räume* I, Berlin, 1960.
5. A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge, 1964.

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