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VECTORVALUED DISTRIBUTIONS COVARIANT UNDER ALGEBRAIC REPRESENTATIONS OF AN ORTHOGONAL GROUP OF ARBITRARY SIGNATURE

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0. Introduction.

Let G be a Lie group acting on a C^∞-manifold X and let A be a C^∞-representation of G as linear mappings on a finite-dimensional linear space L. A distribution T on X with values in L is called covariant under A if

$$A(g)T(x) = T(gx)$$

for every $g \in G$. Suppose that X locally is a product $V \times Y$ where

$$Y = [0,1] \times \ldots \times [0,1] \subseteqq R^s$$

and that $V \times \{y\}$ for every $y \in Y$ is an open part of an orbit in X. In this case we are going to prove that, at least locally, a covariant distribution can be written

$$T(x) = \textstyle\sum_j f_j(x)\, \psi_j(x)\ , \tag{1}$$

where f_j are distributions on X with values in C which are invariant under G, that is $f_j(gx)=f_j(x)$ for every $g \in G$ and ψ_j are C^∞-functions from X to L which are covariant. Then we use this result to get an explicit description of the covariant distributions when $X = \mathsf{R}^n$, $G = SO(p,q)$ and A is an algebraic representation of $SO(p,q)$. Here $SO(p,q)$ is the connected component of the unit e of the orthogonal group which leaves the quadratic form

$$\langle x,x \rangle = \sum_{\nu=1}^{n=p+q} \varepsilon(\nu)\, x_\nu^{\,2}, \quad \varepsilon(1) = \ldots = \varepsilon(p) = 1,\ \ \varepsilon(p+1) = \ldots = \varepsilon(p+q) = -1$$

invariant. It turns out that the covariant distributions can be written as (1). The invariant distributions f are described in [2] for $p=1$ and in [5] for the remaining p and q.

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1. Covariant distributions on manifolds which locally are product manifolds.

Let the Lie group G act on the C^∞-manifold X and write $x_1 \sim x_2$ if and only if there is $g \in G$ such that $gx_1 = x_2$. In the following we suppose that the following condition is satisfied:

(*) To every $x_0 \in X$ there exist neighbourhoods $U \subseteq X$ and $V \subseteq Gx_0$ and a diffeomorphism $\varphi = \varphi_U : U \to V \times Y$ such that $p_Y \varphi(x_1) = p_Y \varphi(x_2)$ if and only if $x_1 \sim x_2$. (Here p_Y is the projection $V \times Y \to Y$.)

It is easily seen that φ can be chosen so that $\varphi(x) = (x, y_0)$ if $x \in V$ and $\varphi(x_0) = (x_0, y_0)$.

Lemma 1. *To every $x_0 \in X$ there is a neighbourhood $W \subseteq X$ and a C^∞-function $h : W \times W \to G$ such that $h(x_1, x_2) x_1 = x_2$ if $x_1, x_2 \in W$ and $x_1 \sim x_2$.*

Proof. Choose neighbourhoods $U \subseteq X$, $V \subseteq Gx_0$ and a diffeomorphism $\varphi = \varphi_U$ as above. It is easily seen that there is a neighbourhood $O \subseteq G$ of e such that $gx \in U$ if $\varphi(x) = (x_0, y)$. Define $\alpha : O \times Y \to U$ by setting $\alpha(g, y) = g\varphi^{-1}(x_0, y)$. The restriction of α to $O \times \{y_0\}$ is then equal to $\varphi^{-1} \circ (\pi \times 1_Y)$, where $\pi : O \to V$ and $\pi g = gx_0$. It is well-known that $\operatorname{rank} d\pi(e) = \dim V$ (see, for example, Helgason [3, pp. 110–113]) and this implies that

$$\operatorname{rank} d\alpha(e, y_0) = \dim V + \dim Y = \dim V \times Y .$$

Now it follows that, there is a C^∞-function $\beta : W \to O \times Y$ such that $\alpha \circ \beta = id_W$ and we can put $h(x_1, x_2) = \gamma(x_2)\gamma(x_1)^{-1}$, where $\gamma = p_O \circ \beta$ and p_O is the projection $O \times Y \to O$.

If A is a representation of G as linear mappings of the finite-dimensional linear space L we put

$$L^x = \{v \in L \,;\ A(g)v = v \text{ for every } g \in G_x\},$$

where $G_x = \{g \in G \,;\ gx = x\}$.

Let U be an open part of X with compact closure and $C^\infty(U, L)$ the set of all C^∞-functions from U to L. We put

$$N(U) = \{f \in C^\infty(U, L) \,;\ f(x) \in L^x \text{ for every } x \in U\} .$$

$N(U)$ is a module with coefficients in $C^\infty(U, C)$ which is spanned by a finite number of functions ψ_j. Evidently it is sufficient to prove this statement in a neighbourhood of every point $x_0 \in X$ since U has compact closure. Choose a neighbourhood W of x_0 and a function h as in lemma 1 and let $\{e_j\}$ be a basis of L^{x_0}. Then

$$\psi_j(x) = A(h(x,x_0))\, e_j \in C^\infty(W,L)$$

is a basis of L^x for every $x \in W$.

If g is sufficiently close to $e \in G$ and if x belongs to a small neighbourhood of x_0 we have

$$\psi_j(gx) = A(h(gx,x_0))\, e_j = A(h(gx,x))\, \psi_j(x) = A(g)\, \psi_j(x)$$

since $g^{-1}h(gx,x) \in G_x$. Suppose that G has the Lie-algebra Γ. If T is covariant, we have

$$e^{-\gamma' t}T(e^{\gamma t}x) = T(x) \quad \text{for every } \gamma \in \Gamma ,$$

where $\gamma' = dA(\gamma)$. By differentiation with respect to t we get

$$\gamma' T(x) = \gamma^+ T(x) , \tag{2}$$

where γ^+ is the vectorfield on X generated by γ. On the other hand (2) implies that $A(g)T = T \circ g$ for every g in a neighbourhood of e and consequently for all g if G is connected. It is also clear that $\gamma' \psi_j(x) = \gamma^+ \psi_j(x)$ when $x \in W$.

THEOREM 1. *Let T be a distribution on X with values in L such that*

$$\gamma' T = \gamma^+ T \quad \text{for all } \gamma \in \Gamma \tag{3}$$

in the relatively compact open set $U \subseteq X$. Then

$$T(x) = \textstyle\sum_j f_j(x)\, \psi_j(x) \qquad \text{in } U ,$$

where f_j are distributions on X with values in $\mathbf{C}$ such that $\gamma^+ f_j = 0$ in U and ψ_j belongs to $C^\infty(U,L)$ and has the property $\gamma' \psi_j = \gamma^+ \psi_j$ in U.

PROOF. It is evident that it suffices to prove the theorem in a neighbourhood of every point in U. Let $x_0 \in U$ and choose a neighbourhood $U' \subseteq X$ of x_0 according to (*) and a neighbourhood $O \subseteq G$ of e such that $OU' \subseteq U$. From (3) it is easily seen that $A(g)\, T(x) = T(gx)$ in U' for every $g \in O$. Let V be a neighbourhood of x_0 in Gx_0 and $\varphi = \varphi_{U'}: U' \to V \times Y$ a diffeomorphism (see *). Clearly we can choose U' so that there is a function $h: U' \times U' \to G$ as in lemma 1. Now we put

$$\bar{T} = T \circ \varphi^{-1}, \quad \bar{g} = \varphi \circ g \circ \varphi^{-1} \quad \text{for } g \in O, \quad \bar{h} = h \circ (\varphi^{-1} \times \varphi^{-1}) .$$

$\bar{T}$ is a distribution on $V \times Y$ such that

$$A(\bar{g}) \circ T = \bar{T} \circ \bar{g} \qquad (A(\bar{g}) = A(g)) .$$

If R_ε^Y is a regulisator on Y (see de Rham [4]), then for every $y \in Y$

$\overline{T}_{\varepsilon,y}=R_\varepsilon{}^Y\overline{T}(y)$ is a distribution on $V\times\{y\}$ which satisfies

$$A(\overline{h}(z_1,z_2))\,\overline{T}_{\varepsilon,y} = \overline{T}_{\varepsilon,y}\circ\overline{h}(z_1,z_2)$$

for every z_1 and z_2 in V.

Let ω^α be an infinitely differentiable m-form on V ($m=\dim V$) such that

$$\omega^\alpha \to \delta_{z_0} \quad \text{as } \alpha\to 0$$

in distribution-sense. If $\omega_z{}^\alpha=\omega^\alpha\circ\overline{h}(z_0,z)$ it is easily seen that $\omega_z\to\delta_z$ in distribution-sense and if $S=\overline{T}_{\varepsilon,y}$ we have

$$\langle S,\,\omega_z^\alpha\rangle \to S(z) \quad \text{as } \alpha\to 0$$

in distribution-sense. For every z_1 and z_2 in V we have

$$A(\overline{h}(z_1,z_2))\,\langle S,\,\omega_z^\alpha\rangle = \langle S\circ\overline{h}(z_1,z_2),\,\omega_z^\alpha\rangle\,.$$

Here the function on the left is a C^∞-function and we can put $z_2=z$ and get

$$A(\overline{h}(z_1,z))\,\langle S,\,\omega_z^\alpha\rangle = \langle S,\,\omega_z^\alpha\circ\overline{h}(z_1,z)^{-1}\rangle\,.$$

But as $\omega_z^\alpha\circ\overline{h}(z_1,z)^{-1} = \omega^\alpha\circ\overline{h}(z_0,z)\overline{h}(z_1,z)^{-1} = \omega_{z_1}^\alpha$ we have

$$A(\overline{h}(z_1,z)\langle S,\,\omega_z^\alpha\rangle = \langle S,\,\omega_{z_1}^\alpha\rangle\,.$$

Here the function on the left converges to $A(\overline{h}(z_1,z)\circ S(z)$ which is independent of z because the function on the right is independent of z. Furthermore $A(\overline{h}(z_1,z))\circ S(z)$ is a C^∞-function in z_1 and as $\langle S,\omega_{z_1}^\alpha\rangle\to S(z_1)$ we conclude that $S\in C^\infty(V,L)$. Now we have proved that the distribution $S=\overline{T}_{\varepsilon,y}$ is infinitely differentiable in z and y. The function $T_\varepsilon=\overline{T}_{\varepsilon,y}\circ\varphi$ is an element of the module $N(U)$ and consequently we can write

$$T_\varepsilon(x) = \textstyle\sum_j f_{j,\varepsilon}(x)\,\psi_j(x)\,,$$

where ψ_j are the functions above. As ψ_j are linear independent it follows that $f_{j,\varepsilon}$ converges in $\mathscr{D}'(U)$ to distributions f_j. We have proved that

$$T(x) = \textstyle\sum_j f_j(x)\,\psi_j(x)$$

in a neighbourhood of every point in U and consequently in all of U.

From the construction of h in lemma 1 and from the proof above it follows that $\bar{f}_{j,\varepsilon}$ only depend on y and consequently $\gamma^+f_{j,\varepsilon}=0$ and then $\gamma^+f_j=0$.

Remark. The theorem holds for an arbitrary open set $U\subseteq X$ if there is a finite number of linear independent covariant functions ψ_j which span $N(U)$. If $U=X$ this implies that the distributions f_j are invariant under G.

2. Description of L^x when $G=SO(p,q)$.

Let A be an irreducible representation of $G=SO(p,q)$ as linear mappings of a finite-dimensional linear space L. For every $x \in \mathsf{R}^n$, L^x is the direct sum of the one-dimensional subspaces of L which is invariant under $A(G_x)$. From Weyl [6] it follows that the representations of G_x, which is isomorphic to $SO(p',q')$ where $p'+q'=n-1$, have degree 1 if and only if they have the weight $(0,\ldots,0)$. If A is irreducible it follows from Boerner ([1], p. 251–254) that there are one-dimensional subspaces of L invariant under $A(G_x)$ if and only if A has the weight $(k,0,\ldots,0)$ where k is a positive integer and that there is only one such subspace. If the weight of A is $(k,0,\ldots,0)$ L is isomorphic to $\pi_k{}^\circ$, where $\pi_k{}^\circ$ is the space of all homogeneous polynomials

$$p(\xi) = \textstyle\sum_\alpha a_\alpha \xi_\alpha = \sum_\alpha a_{\alpha_1\ldots\alpha_k}\xi_{\alpha_1}\cdots\xi_{\alpha_k}$$

in $\xi=(\xi_1,\ldots,\xi_n)$ of degree $=k$ and where a_α are symmetric in α and

$$\operatorname{Tr}(p) = \left(\textstyle\sum_j \varepsilon(j)\, a_{jj\alpha'}\right)\xi_{\alpha'} = 0,$$

where $\alpha'=(\alpha_3,\ldots,\alpha_k)$. (See Weyl [6].)

Evidently the polynomials $\langle\xi,\xi\rangle^j=(\sum_{\nu=1}^n \varepsilon(\nu)\xi_\nu{}^2)^j$ are invariant under G and the polynomials $\langle x,\xi\rangle^s=(\sum_{\nu=1}^n \varepsilon(\nu)x_\nu\xi_\nu)^s$ are invariant under G_x. We put

$$P_{j,k}(x,\xi) = c_{j,k}\langle\xi,\xi\rangle^j\langle x,\xi\rangle^{k-2j}, \qquad 0 \leqq 2j \leqq k\,,$$

where

$$c_{j,k} = \binom{k}{2j}(2j)!/2^n\,.$$

Then $P_{j,k}$ is homogeneous in ξ of degree k and after a calculation we get

$$\operatorname{Tr}(P_{j,k}) = \begin{cases} (n+1)P_{j-1,k-2}+\langle x,x\rangle P_{j,k-2} & \text{if } 2j\leqq k-2\,,\\ (n+1)P_{j-1,k-2} & \text{if } 2j=k-1\,,\\ nP_{j-1,k-2} & \text{if } 2j=k\,. \end{cases}$$

Here $P_{-1,k-2}=0$.

Now it is easily seen that, if $u_j=(-1/n+1)^j$ when $j<2k$ and $u_j=(-1/n+1)^j(n+1)/n$ when $j=2k$, we have

$$\operatorname{Tr}\sum_{0\leqq 2j\leqq k} u_j\langle x,x\rangle^j P_{j,k}(x,\xi) = 0\,.$$

We have proved the following.

Lemma 2. *If A is an irreducible representation of $SO(p,q)$ with the weight $(k,0,\ldots,0)$, then L^x is spanned by the polynomial*

$$P_k(x,\xi) = \sum_{0\leqq 2j\leqq k} u_j c_{jk} \langle x,x\rangle^j \langle \xi,\xi\rangle^j \langle x,\xi\rangle^{k-2j} .$$

For other irreducible representations $L^x=\{0\}$.

3. Distributions covariant under algebraic representations of $SO(p,q)$. Now we can use the results in section 1 and 2 to characterize the distributions on $\mathsf{R}_0^n=\mathsf{R}^n\setminus\{0\}$ which are covariant under irreducible representations of $SO(p,q)$.

Lemma 3. *Let A be an irreducible representation of $SO(p,q)$ as linear mappings on the finite dimensional linear space L. Then there are covariant distributions on R_0^n if and only if A has the weight $(k,0,\ldots,0)$ and in that case T is covariant if and only if*

$$T(x)(\xi) = f(x)\, P_k(x,\xi),$$

where f is an invariant distribution on R^n with values in C.

Remark. If A has the weight $(k,0,\ldots,0)$ every distribution with values in L can be written as a homogeneous polynomial in ξ of degree k with distributions T_α as coefficients that is

$$T(x)(\xi) = \textstyle\sum_\alpha T_\alpha(x)\, \xi_\alpha,$$

where T_α is symmetric.

Proof. From theorem 1 and lemma 2 it follows that it is sufficient to prove that R_0^n has the property $(*)$. If $x_0=(x_1{}^0,\ldots,x_n{}^0)\in\mathsf{R}^n$, then, for example, $x_1{}^0\neq 0$ and the function $\varphi\colon \mathsf{R}^n\to\mathsf{R}^n$ defined by

$$y_1 = \varphi_1(x) = \langle x,x\rangle, \qquad y_k = \varphi_k(x) = x_k \quad \text{if} \quad k\geqq 2$$

is a diffeomorphism in a neighbourhood of x_0. Furthermore $\{y' \mid |y'-x_0'|<\varepsilon\}$ where $y'=(y_2,\ldots,y_n)$, $x_0'=(x_2,\ldots,x_n)$ is diffeomorphic to an open part of Gx_0 if $\varepsilon>0$ is sufficiently small.

Now it remains to characterize the distributions with support in the origin which are covariant under an irreducible representation A. If the distribution T has values in L and support in the origin we can write

$$T(x)(\xi) = P(D^\varepsilon,\xi)\,\delta(x),$$

where $D^\varepsilon=(D_1{}^\varepsilon,\ldots,D_n{}^\varepsilon)$, $D_k=\varepsilon(k)(\partial/\partial x_k)$. Furthermore T is covariant if and only if

$$P(g^{-1}D^{\varepsilon}, g\xi) = P(D^{\varepsilon}, \xi) \quad \text{for every } g \in SO(p,q),$$

that is, when the polynomial $P(x,\xi)$ is covariant under A. Then evidently $P(x,\xi) \in L^x$ for every x and consequently

$$P(x,\xi) = Q(\langle x,x\rangle) P_k(x,\xi)$$

with P_k as in lemma 2. But as P is a polynomial it follows if we observe the construction of P_k that Q also is a polynomial and we have proved

LEMMA 4. *Let A be an irreducible representation of $SO(p,q)$ as linear mappings on the finite-dimensional linear space L. There are covariant distributions T with values in L and support in the origin if and only if A has the weight $(k,0,\ldots,0)$ and in that case a distribution T is covariant if and only if*

$$T(x)(\xi) = Q(\square)\, P_k(D^{\varepsilon}, \xi)\, \delta(x)$$

where

$$\square = \sum_{\nu=1}^{n} \varepsilon(\nu)\, \partial^2/\partial^2 x_\nu .$$

In order to combine lemmas 3 and 4 we now prove

LEMMA 5. $P_k(x,\xi)\, \square^{k+j} \delta(x) = k!\, P(D_k^{\varepsilon}, \xi)\, (1 + \square + \square^2 + \ldots + \square^j)\, \delta(x)$.

PROOF. At first we observe that $P_k(x,\xi) = P_k(\xi,x)$ and consequently we have $\mathrm{Tr}_x\, P_k(x,\xi) = 0$ if we regard P_k as a polynomial in x with polynomials in ξ as coefficients. We put $P_k(x,\xi) = \sum_\alpha a_\alpha(\xi)\, x_\alpha$ where $a_\alpha(\xi)$ is symmetric in α. If $\varphi \in D(\mathsf{R}^n)$ we get after a calculation

$$\square(P_k(x,\xi)\varphi(x)) = \binom{k}{2} \mathrm{Tr}_x\, P_k(x,\xi)\, \varphi(x) + \sum_{j=1}^{n} \sum_\alpha a_\alpha(\xi)\, (x_\alpha/x_{\alpha_j})\, D_{\alpha_j}\varphi(x) + $$
$$+ P_k(x,\xi)\square\varphi(x) .$$

Now if we use that $\mathrm{Tr}_x\, P_k(x,\xi) = 0$ it follows by induction that if $r \leqq k$, then

$$\square^r(P_k(x,\xi)\, \varphi(x)) = \sum_{\mu=0}^{r} c_\mu \sum_{j_1,\ldots,j_\mu} a_\alpha(x_\alpha/x_{\alpha_{j_1}} \ldots x_{\alpha_{j_\mu}}) D_{\alpha_{j_1}} \ldots D_{\alpha_{j_\mu}} \square^{r-\mu}\varphi(x),$$

where c_μ are combinatorial coefficients with $c_r = 1$. Now the lemma easily follows.

We have now proved the following theorem.

THEOREM 2. *If A is an irreducible representation of $SO(p,q)$ as linear mappings on the finite-dimensional linear space L then there are distributions on R^n with values in L which are covariant under A if and only if A has the weight $(k,0,\ldots,0)$. In that case a distribution T is covariant if and only if it can be written*

$$T(x)(\xi) = f(x)\,P_k(x,\xi)\,,$$

where f is an invariant distribution with values in C.

If A is an algebraic representation of $SO(p,q)$ as linear mappings on the finite-dimensional linear space L it is well known that A is reducible (see, for example, Boerner [1]). Then we can write

$$L = \oplus_m a_m\, L_m\,,$$

where $m=(m_1,\ldots,m_p)$, $p=[n/2]$, L_m is the invariant subspace which belongs to the irreducible representation A_m with weight m, and a_m are non-negative integers a finite number of which are >0. It is easily seen that

$$L^x = \oplus_m a_m\, L_m{}^x = \oplus\, a_{(k,0..,0)}\, L^x{}_{(k,0..,0)} = \oplus\, a_k\, L_k{}^x$$

because $L_m{}^x=\{0\}$ if $m \neq (k,0,\ldots,0)$. The space L^x is spanned by $a_k\, P_k(x,\xi^{j_k})$, where $\xi^{j_k}=(\xi_1{}^{j_k},\ldots,\xi_n{}^{j_k})$ and $1 \leqq j_k \leqq a_k$.

Theorem 3. *If A is an algebraic representation of $SO(p,q)$ as linear mappings on the finite-dimensional linear space L then the distribution T on* R^n *with values in L is covariant under A if and only if*

$$(T(x))(\xi) = \sum_k \sum_{j_k=1}^{a_k} a_k\, f_{k,j_k}(x)\, P_k(x,\xi^{j_k})\,,$$

where f_{k,j_k} are distributions in R^n *with values in* C *which are invariant under $SO(p,q)$ and $\xi=(\xi_1{}^1,\ldots,\xi_n{}^1,\,\xi_1{}^2,\ldots,\xi_n{}^2,\ldots)$.*

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