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POWER-BOUNDED MATRICES OF FOURIER-STIELTJES TRANSFORMS II

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0. Introduction.

A well-known theorem by Beurling and Helson [1] says that a Fourier–Stieltjes transform f has bounded powers if and only if

$$(*) \qquad f(y) \,=\, c\,\exp(i\langle x,y\rangle), \qquad y \in \mathsf{R}^n ,$$

where $|c|=1$ and $x \in \mathsf{R}^n$.

The purpose of this note is to give a corresponding necessary and sufficient condition for a matrix of Fourier–Stieltjes transforms to have bounded powers in $\mathscr{B}$, the non-commutative Banach algebra of matrices of Fourier–Stieltjes transforms, in a natural norm (for definitions, see Section 1).

We will prove (Theorem 1 in Section 2) that φ has bounded powers in $\mathscr{B}$ if and only if there exists an invertible $p \in \mathscr{B}$ such that $p^{-1}\varphi p$ is diagonal, with diagonal entries of the form $(*)$.

As a corollary of this result we get a characterization of those homogeneous matrix functions P on R^n, with real eigenvalues, for which $\exp(iP) \in \mathscr{B}$ (Corollary 2 in Section 3). We will prove that $\exp(iP) \in \mathscr{B}$ if and only if P is of the form

$$P(y) \,=\, \sum_{j=1}^{n} A_j y_j, \qquad y \in \mathsf{R}^n ,$$

where $A_1,\ldots,A_n$ are commuting, diagonable matrices with real eigenvalues. This is a general form of the multiplier theorem (Theorem 1) proved in [3], in the extreme cases $p=1$ and $p=\infty$.

In [3] the multiplier theorem was used to get necessary and sufficient conditions for the Cauchy problem for symmetric hyperbolic systems to be well posed in L_p. A generalization of the necessity part of Theorem 2 in [3], in the case $p=\infty$ (and $p=1$), for certain systems of pseudo-differential operators is contained in Theorem 2 in Section 3 below.

The main tools are the results obtained in [4], especially Theorem 3

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of that note, and a characterization of the idempotents in $\mathscr{B}$, carried out in Section 1. In the proof of Theorem 1 we use a device, formally the same as that used by Strang [5]. As a by-product of this we will be able to prove that if φ has bounded powers in $\mathscr{B}$, there exists an hermitian, invertible H in $\mathscr{B}$ such that $H^{-1}\varphi H$ is unitary (Corollary 1 in Section 2).

1. Idempotents in $\mathscr{B}$.

We will take over the notions and results of [4], mainly those presented in Sections 2 and 4 of that paper. For convenience we repeat the most central notions.

For complex N-vectors v, $|v|$ will be the Euclidean norm. For an $N \times N$-matrix A, $|A|$ will denote the corresponding operator norm.

By $\mathscr{L}_1(\mathsf{R}^n)$, or simply $\mathscr{L}_1$, we mean the complex N-vector functions with components in L_1 with norm

$$\|v\|_1 = \int_{R^n} |v(x)| \, dx \, .$$

In the same way $\mathscr{M}(\mathsf{R}^n)$ denotes the set of $N \times N$-matrices with elements in $M(\mathsf{R}^n)$, that is, elements which are bounded measures on R^n. For $\mu \in \mathscr{M}(\mathsf{R}^n)$ the norm is defined by

$$\|\mu\| = \sup \{\|\mu * v\|_1 \, ; \; v \in \mathscr{L}_1, \; \|v\|_1 \leqq 1\} \, ,$$

where $\mu * v$ is defined in the obvious way (see [4]). We also define the convolution between elements in $\mathscr{M}$ via the usual matrix multiplication, and have

$$\|\mu * \nu\| \leqq \|\mu\| \, \|\nu\|, \qquad \mu, \nu \in \mathscr{M} \, .$$

$\mathscr{M}(\mathsf{R}^n)$ is then a Banach algebra with unit (non-commutative for $N > 1$). For matrices and vectors with elements in M and L_1 we define the Fourier–Stieltjes and Fourier transform by taking the transform element-wise. For $\mu \in \mathscr{M}$, with transform $\hat{\mu}$, we define $\|\hat{\mu}\| = \|\mu\|$.

In this way the Fourier–Stieltjes transforms of elements in $\mathscr{M}$ is a Banach algebra, denoted $\mathscr{B}(R^n)$ or, for $N = 1$, $B(\mathsf{R}^n)$, with unit E under pointwise (matrix-) multiplication and addition.

Let $\mathfrak{m}$ be the maximal ideal space of $B(\mathsf{R}^n)$. Of course, $\mathfrak{m}$ is also the maximal ideal space of $M(\mathsf{R}^n)$. Every function in $\mathscr{B}$ can be extended to a continuous function on $\mathfrak{m}$, the Gelfand transform of the element in $\mathscr{B}$. Below the transform will get the same notion as the element. This will simplify the formulations of the propositions and will cause no confusion.

For facts about $\mathfrak{M}$ and the Gelfand transform we refer to [2]. Some of the properties of $\mathfrak{M}$ are collected in the following lemma.

LEMMA 1. *Let $\mathfrak{M}$ be the maximal ideal space of $B(\mathsf{R}^n)$. Then:*

(i) *$\mathfrak{M}$ is connected and compact.*

(ii) *$\mathfrak{M}$ contains R^n as a determining subset for B, that is if $f \in B$ and $f=0$ on R^n, then $f=0$ on $\mathfrak{M}$.*

(iii) *If $f_1,\ldots,f_k \in B$ and F is analytic in a neighborhood of*

$$\{(f_1(y),\ldots,f_k(y))\ ;\ y \in \mathfrak{M}\},$$

then $F(f_1,\ldots,f_k) \in B$.

That E_0 is idempotent merely means that $E_0{}^2=E_0$, on all of R^n.

LEMMA 2. *Let $E_0 \in \mathscr{B}$ be an idempotent. Then the eigenvalues of E_0 can be choosen to be $=1$ or $=0$ on $\mathfrak{M}$, and to have constant multiplicity on $\mathfrak{M}$.*

PROOF. Since E_0 can be extended to a continuous function on $\mathfrak{M}$, and since R^n is a determining subset of $\mathfrak{M}$, E_0 is a continuous idempotent matrix function on $\mathfrak{M}$. Then the eigenvalues of E_0 can be choosen continuous on $\mathfrak{M}$ and they only assume the values 0 and 1 there. As $\mathfrak{M}$ is connected this proves the lemma.

We can now give the following characterization of the idempotent elements in $\mathscr{B}(\mathsf{R}^n)$.

PROPOSITION 1. *Let $E_0 \in \mathscr{B}$ be idempotent. Then there exists an $S \in \mathscr{B}$, invertible in $\mathscr{B}$, such that $S^{-1} E_0 S$ is diagonal, with diagonal elements $=1$ or $=0$.*

PROOF. Let $E_1=E-E_0$. Then E_1 is also an idempotent element in $\mathscr{B}$. Since the eigenvalues of E_0 and E_1 have constant multiplicity on $\mathfrak{M}$, we can find eigenvectors to the non-zero eigenvalues of E_0 and E_1 that are analytic functions of the eigenvalues and the elements of E_0 respectively E_1. As analytic functions operate on B, the elements of these eigenvectors belong to B. Furthermore they can be choosen linearly independent on $\mathfrak{M}$. Let S be the matrix with these eigenvectors as column vectors. Then $\det S \neq 0$ on $\mathfrak{M}$, and so the inverse of S is elementwise an analytic function of the elements of S on $\mathfrak{M}$, and so $S^{-1} \in \mathscr{B}$. It is easy to verify that S is the matrix asked for. The structure of the diagonal matrix follows from Lemma 2.

By induction we obtain in the same way.

COROLLARY 1. *Let* $E_1, \ldots, E_l$ *be orthogonal idempotents in* $\mathscr{B}$. *Then there exists an invertible element* S *in* $\mathscr{B}$, *such that* $S^{-1} E_j S$ *is diagonal, with diagonal entries* $=1$ *or* $=0$, $j=1,\ldots,l$.

2. Power bounded elements in $\mathscr{B}$.

In this section we give (Theorem 1 below) necessary and sufficient conditions for a matrix function φ to have bounded powers in $\mathscr{B}$.

In the case $N=1$, Theorem 1 reduces to the Beurling–Helson theorem [1], a local form of which was used in [4] (and so in the proof of Lemma 3 below). For $N>1$ the non-commutativity of $\mathscr{B}$ mainly adds a similiarity transformation, bounded in $\mathscr{B}$. The theorem can easily be extended to hold, with obvious changes, on $\mathscr{B}(\Omega)$, where Ω is an open, connected subset of a *LCA* group (cf. [4]). For the applications we have in mind the version proved in Theorem 1 will be sufficient.

As a corollary (Corollary 1) we notice that if φ has bounded powers in $\mathscr{B}$, then φ is close to unitary in $\mathscr{B}$.

THEOREM 1. *Let* φ *have bounded powers in* $\mathscr{B}$. *Then there exists a* $p \in \mathscr{B}$, *invertible in* $\mathscr{B}$, *such that* $p^{-1} \varphi p$ *is diagonal, with diagonal elements of the form*

$$\lambda_j(y) = c_j \exp(i\langle x_j, y\rangle), \qquad y \in \mathsf{R}^n, \tag{1}$$

where $|c_j|=1$ *and* $x_j \in \mathsf{R}^n$, $j=1,\ldots,N$. *This condition is also sufficient.*

For the proof of the necessity part of the theorem we need some lemmas which we state and prove before we give the proof of Theorem 1.

LEMMA 3. *Let* φ *have bounded powers in* $\mathscr{B}$. *Then there exist functions* $\lambda_1,\ldots,\lambda_N$ *of the form*

$$\lambda_j(y) = c_j \exp(i\langle x_j, y\rangle), \qquad y \in \mathsf{R}^n, \tag{1}$$

where $|c_j|=1$ *and* $x_j \in \mathsf{R}^n$, $j=1,\ldots,N$, *such that* $\lambda_1(y),\ldots,\lambda_N(y)$ *are the eigenvalues of* $\varphi(y)$, $y \in \mathsf{R}^n$, *counted with proper multiplicities.*

This is Theorem 3 in [4], where a proof can be found.

LEMMA 4. *Let* $\varphi \in \mathscr{B}$ *and suppose that*

$$\sup\{\|\varphi^m\|\,;\; m=1,2,\ldots\} = C < +\infty .$$

Then for every power-bounded ψ *in* B *we have*

$$\|(r-1)(r\psi-\varphi)^{-1}\| \leqq C, \qquad r>1 . \tag{2}$$

PROOF. Since φ has bounded positive powers, all the eigenvalues of φ have modulus $\leqq 1$. As $|r\psi| = r > 1$, $(r\psi - \varphi)^{-1}$ exists and can be expanded in a convergent power series

$$(r\psi - \varphi)^{-1} = \sum_{j=0}^{\infty} (r\psi)^{-j-1} \varphi^j .$$

Since ψ has bounded powers in B, $\|\psi\| = 1$ (see [1]). Hence by the assumptions

$$\|(r\psi)^{-j-1} \varphi_j\| \leqq C\, r^{-j-1} .$$

It follows that $(r\psi - \varphi)^{-1} \in \mathscr{B}$ for $r > 1$ and that

$$\|(r\psi - \varphi)^{-1}\| \leqq C(r-1)^{-1}, \qquad r > 1 ,$$

which proves (2).

The next lemma is well known (cf. Strang [5]).

LEMMA 5. *Let φ be a matrix function with eigenvalues of modulus* 1 *on $\mathfrak{m}$. Assume that for some constant C*

$$|(Z - \varphi(y))^{-1}| \leqq C\,(|Z| - 1)^{-1}, \qquad |Z| > 1,\ y \in \mathfrak{m} .$$

Then φ has linear factors on $\mathfrak{m}$, that is there exist idempotent orthogonal matrix functions $E_1, \ldots, E_l$ on $\mathfrak{m}$, such that

$$\varphi = \lambda_1 E_1 + \ldots + \lambda_l E_l,$$

where $\lambda_1, \ldots, \lambda_l$ are eigenvalues of φ, and $E_1 + \ldots + E_l = E$.

PROOF OF THEOREM 1. Lemma 4 and Lemma 5 together with the fact that

$$|\Phi(y)| \leqq \|\Phi\|, \qquad y \in \mathfrak{m},\ \Phi \in \mathscr{B} ,$$

proves that for $y \in \mathfrak{m}$,

$$\varphi(y) = y_1(y)\, E_1(y) + \ldots + \lambda_l(y)\, E_l(y),$$

where $E_1, \ldots, E_l$ are orthogonal idempotent matrices, with sum E, on $\mathfrak{m}$. By Lemma 3, λ_j can be taken to have the form

(1) $$\lambda_j(y) = c_j \exp(i\langle x_j, y\rangle), \qquad y \in \mathsf{R}^n .$$

In particular the λ_j's have bounded powers in B and are continuous on $\mathfrak{m}$.

Next we observe that

(3) $$(r-1)\,\lambda_k\,(r\lambda_k - \varphi)^{-1} = (r-1)\,\lambda_k \sum_{j=1}^{l} (r\lambda_k - \lambda_j)^{-1} E_j \to E_k$$

as $r \to 1$, pointwise on R^n. By Lemma 4 the left hand side of (3) is uniformly bounded for $r>1$ in $\mathscr{B}$. Hence (cf. Lemma 2 in [4]) there exists an $\tilde{E}_k \in \mathscr{B}$ such that $E_k = \tilde{E}_k$ a.e. on R^n. This holds for $k=1,\ldots,l$ and so

$$\varphi = \lambda_1 \tilde{E}_1 + \ldots + \lambda_l \tilde{E}_l \,. \tag{4}$$

Both sides of (4) are continuous and hence (4) holds on all of R^n. In the same way we see that $\tilde{E}_1,\ldots,\tilde{E}_l$ are orthogonal idempotents in $\mathscr{B}$. Corollary 1 in Section 1 then shows that there exists an invertible $S \in \mathscr{B}$ such that $S^{-1}\tilde{E}_j S$ is diagonal, $j=1,\ldots,l$. Take $p=S$. Then

$$p^{-1}\varphi\, p = \lambda_1 S^{-1}\tilde{E}_1 S + \ldots + \lambda_l S^{-1}\tilde{E}_l S$$

is a sum of diagonal matrices, hence a diagonal matrix of the form stated in Theorem 1. This proves the necessity part of the theorem.

For the proof of the sufficiency part we observe that a diagonal matrix Λ with diagonal entries of the form (1) has bounded powers in $\mathscr{B}$. But

$$\varphi^m = p\Lambda^m p^{-1}$$

and so

$$\|\varphi^m\| \leqq \|p\|\,\|p^{-1}\|\,\|\Lambda^m\| \leqq C, \qquad m = \pm 1, \pm 2, \ldots ,$$

which completes the proof of the theorem.

Let φ have bounded powers in $\mathscr{B}$. Using the matrices $\tilde{E}_j$ constructed in the proof of Theorem 1, we will construct an invertible hermitian matrix H in $\mathscr{B}$ such that $H^{-1}\varphi H$ is unitary. This is formally done as the corresponding construction by Strang [5].

Corollary 1. *Let φ have bounded powers in $\mathscr{B}$. Then there exists an invertible hermitian matrix H in $\mathscr{B}$, such that $H^{-1}\varphi H$ is unitary.*

Proof. First let

$$H_0(y) = \textstyle\sum_j \tilde{E}_j(y)\,\tilde{E}_j^*(y), \qquad y \in \mathfrak{m} \,.$$

Then $H_0 = H_0^*$ on $\mathfrak{m}$ and for any N-vector v and any $y \in \mathfrak{m}$

$$\langle H_0(y)v, v\rangle \geqq |v|^2 \,.$$

This follows at once from the fact that the $\tilde{E}_k$'s are orthogonal idempotents with sum E (see Strang [5]). Further

$$\begin{aligned}\varphi^{-1} H_0 &= \varphi^{-1}\sum_j \tilde{E}_j \tilde{E}_j^* = \sum_{j,k} \bar{\lambda}_k \tilde{E}_k \tilde{E}_j \tilde{E}_j^* \\ &= \sum_k \bar{\lambda}_k \tilde{E}_k \tilde{E}_k^* = \sum_{j,k} \tilde{E}_j \tilde{E}_j^* \bar{\lambda}_k \tilde{E}_k^* = H_0\, \varphi^*\end{aligned}$$

and so

$$\varphi^* = H_0^{-1}\,\varphi^{-1}\,H_0\,. \tag{5}$$

Now let H be the hermitian square-root of H_0. Since $H_0 \geqq E$ on $\mathfrak{m}$ the elements of H can be taken to be analytic functions of the elements of H_0 on $\mathfrak{m}$. For example, let

$$H(y) = \frac{1}{2\pi i}\int_C \lambda^{\frac{1}{2}}\,(\lambda - H_0(y))^{-1}\,d\lambda, \qquad y \in \mathfrak{m}\,,$$

where C is a suitable curve in the right half-plane surrounding the spectrum of H_0. From (5)

$$(H^{-1}\varphi\, H)^* = H\,\varphi^*\,H^{-1} = H\,H_0^{-1}\,\varphi^{-1}\,H_0\,H^{-1} = H^{-1}\varphi^{-1}H = (H^{-1}\varphi\, H)^{-1},$$

and the corollary is proved.

3. Homogeneous matrices and the Cauchy problem for systems in L_∞.

In this section we will give necessary and sufficient conditions for $\exp(iP)$ to be in $\mathscr{B}$, when P is a homogeneous matrix function on R^n with real eigenvalues.

As an application of this we will then give necessary conditions that the Cauchy problem for systems of constant-coefficient pseudo-differential operators should be well posed in L_∞ (and in L_1, by duality).

COROLLARY 2. *Let P be a homogeneous matrix function of positive degree on R^n with real eigenvalues. Then* $\exp(iP) \in \mathscr{B}$ *if and only if*

$$P(y) = \sum_{j=1}^{n} A_j\,y_j, \qquad y \in \mathsf{R}^n\,, \tag{6}$$

where $A_1,\ldots,A_n$ are diagonable, commuting matrices with real eigenvalues.

Before the proof we remark that the corollary generalizes Theorem 1 in [3] in the extreme cases $p=1$ and $p=\infty$.

PROOF OF COROLLARY 2. Let $\varphi = \exp(iP)$ and assume that $\varphi \in \mathscr{B}$. For $m>0$ then

$$\varphi^m(y) = \exp(imP(y)) = \exp(iP(sy)) = \varphi(sy)$$

where $s^k = m$, $k>0$ being the order of P. From this follows that

$$\|\varphi^m\| = \|\varphi\| = C, \qquad m=1,2,\ldots\,.$$

That φ has bounded powers in $\mathscr{B}$ then follows from the fact that

$|\det\varphi| = 1$ as in Corollary 1 in Section 4 in [4]. Hence, by Theorem 1, there exists a $p \in \mathscr{B}$, invertible in $\mathscr{B}$, such that $p^{-1}Pp$ is diagonal with real linear diagonal elements. From this

$$P(y) = p(y)\left(\sum_{j=1}^{n} D_j\, y_j\right) p^{-1}(y)$$

where D_j are real diagonal matrices. But then P has linear eigenvalues on R^n. From this follows that P is homogeneous of degree 1, and so

$$tP(y) = P(ty) = p(ty)\left(t\sum_{j=1}^{n} D_j\, y_j\right) p^{-1}(ty)$$

for $t > 0$. If we divide both members by t, we obtain

$$P(y) = p(ty)\left(\sum_{j=1}^{n} D_j\, y_j\right) p^{-1}(ty)\,.$$

Let $t \to 0$. As both p and p^{-1} are continuous on R^n we get

$$P(y) = \sum_{j=1}^{n} p(o)\, D_j\, p^{-1}(o)\, y_j$$

and so the necessity part is proved. The sufficiency follows from the well-known fact that by the assumptions on $A_1,\ldots,A_n$ we can find an invertible (constant) matrix S such that $S^{-1} A_j\, S = D_j$, $j = 1,\ldots,n$, are real diagonal matrices. Then

$$\exp\bigl(iP(y)\bigr) = S\, \Lambda(y)\, S^{-1}$$

where $\Lambda(y)$ is a diagonal matrix with diagonal entries of the form $\exp(i\langle x_j,y\rangle)$, $y \in \mathsf{R}^n$, where $x_j \in \mathsf{R}^n$, $j = 1,\ldots,N$. This ends the proof of the Corollary.

Let $P(D)$ be a matrix of pseudo-differential operators with constant coefficients. By this we mean that for C^∞ N-vector functions u with compact support (then we say that $u \in \mathscr{D}$)

$$\widehat{P(D)\, u}(y) = P(y)\, \hat{u}(y)\,,$$

where we merely assume that $|P(y)|$ do not grow faster than some power of y. The order k (here assumed > 0) is defined so that

$$s^{-k}\, P(sy) \to P_0(y)$$

as $s \to \infty$, uniformly on compact subsets of R^n. Here P_0, the principal part of P, is a non-zero homogeneous matrix function of the degree k on R^n.

Let us consider the Cauchy problem

$$\text{(7)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = P(D)\,u, & x \in \mathsf{R}^n, \\ u(x,o) = u_0(x), & 0 \leqq t \leqq T, \end{cases}$$

where P is a matrix pseudo-differential operator as defined above, and where $u_0 \in \mathscr{D}$. We say that (7) is well posed in L_∞ if there is a constant $C = C(T)$ such that for any solution u of (7) we have

$$\text{(8)} \qquad \|u(\cdot,t)\|_\infty \leqq C\,\|u_0\|_\infty, \qquad 0 \leqq t \leqq T,\ u_0 \in \mathscr{D}$$

and

$$\|h^{-1}(u(o,t+h) - u(\cdot,t)) - P(D)\,u(\cdot,t)\|_\infty \to 0, \quad h \to 0\,.$$

If P_0 has imaginary eigenvalues on R^n we will give the following necessary condition for (7) to be well posed in L_∞.

THEOREM 2. *Assume that* (7) *is well posed in* L_∞, *and suppose that the principal part* P_0 *of* P *has imaginary eigenvalues on* R^n. *Then* P_0 *has the form*

$$P_0(D) = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j},$$

where $A_1,\ldots,A_n$ *are diagonable, commuting matrices with real eigenvalues.*

This theorem strengthens the necessity part of Theorem 2 in [3] in the extreme cases $p=1$ and $p=\infty$. In view of Corollary 2 above it will be sufficient to prove the following lemma.

LEMMA 6. *Assume that* (7) *is well posed in* L_∞. *Then* $\exp(P_0) \in \mathscr{B}$.

PROOF. From (7) and the definition of P we obtain (cf. [3]) the Fourier transform of the solution of the Cauchy problem as

$$\hat{u}(y,t) = \exp(tP(y))\,\hat{u}_0(y)\,.$$

For any $v \in \mathscr{L}_1$ we get by (8) and Parsevals formula that

$$\text{(9)} \qquad \left|\int (\exp(tP(y))\,\hat{u}_0(y),\ \hat{v}(y))\,dy\right| \leqq C\,\|u_0\|_\infty\,\|v\|_1, \qquad 0 \leqq t \leqq T\,.$$

But this implies that $\varphi_t = \exp(tP)$ belongs to $\mathscr{B}$, and that

$$\text{(10)} \qquad \|\varphi_t\| \leqq C, \qquad 0 \leqq t \leqq T\,.$$

If k is the order of P, let $s^k = t$, $s > 0$. Since affine transformations of the variable are isometries on $\mathscr{B}$ and since

$$\varphi_t(s^{-1}y) = \exp\bigl(P_0(y)+o(1)\bigr), \qquad s \to 0,$$

uniformly on compact subsets of R^n, it follows from (10) that $\exp(P_0) \in \mathscr{B}$ (cf. Lemma 2 in [4]).

The L_1-version of Theorem 2 follows from (9), assuming that (8) holds in L_1-norm.

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