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ON THE MONOTONE SEQUENTIAL CLOSURE OF A C^*-ALGEBRA

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1. Introduction.

In this paper axioms are given for "Baire*-algebras", a type of C^*-algebras previously studied by R. V. Kadison [7] and G. K. Pedersen [12], [13], and for "J Baire*-algebras", the Jordan algebras analogously defined. The axioms are modelled after those of [6] in such a way that Baire*-algebras appear as σ-analogues of von Neumann algebras. For countably generated Baire*-algebras we give elements of a structure theory, for example, a comparison lemma and a characterization of modularity, using methods of E. B. Davies [2], D. M. Topping [16], I. Kaplansky [8], [9], [10], and G. K. Pedersen [13].

Baire*-algebras generalize the Σ^*-algebras of E. B. Davies [1], however, it is not known whether every Baire*-algebra is a Σ^*-algebra, cf. [13].

For general information about von Neumann algebras, C^*-algebras and JC-algebras we refer to [3], [4], [15] and [16].

The author is indebted to E. B. Davies and G. K. Pedersen for the pleasure of reading preprints of [2], [12] and [13], and to G. K. Pedersen for numerous illuminating conversations.

2. Baire*-algebras.

Definition 2.1. A Jordan representation of a JC-algebra A is called σ-normal if, for every monotone increasing sequence of elements from A with a least upper bound in A, the image of the least upper bound is the least upper bound of the images.

Analogously we define σ-normality of linear functionals on C^*-algebras and JC-algebras.

Definition 2.2. A C^*-algebra (resp. JC-algebra) A is called a Baire*-algebra (resp. J Baire*-algebra) if every normbounded monotone increasing sequence in A has a least upper bound in A, and A has a sep-

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arating family of σ-normal states. A concrete Baire*-algebra (resp. J Baire*-algebra) is a self-adjoint algebra of (resp. a Jordan algebra of self-adjoint) operators on a Hilbert space containing the limit of any of its weakly convergent monotone sequences.

From the work of R. V. Kadison [7] it follows that the smallest set J_s^m of self-adjoint operators containing a given Jordan algebra J of self-adjoint operators and containing the limit of any of its weakly convergent monotone sequences, is a JC-algebra and hence a J Baire*-algebra. If J is the self-adjoint part A^h of a self-adjoint operator algebra A, then, as shown in [13], $(A^h)_s^m$ is the self-adjoint part of a C^*-algebra, and hence a Baire*-algebra, A_s^m (the notation of [7] is altered slightly) the set of the Baire operators associated with A. Especially, a concrete J Baire*-algebra (resp. Baire*-algebra) is a J Baire*-algebra (resp. Baire*-algebra). Also from [6] and [7] it follows that on a separable Hilbert space we get weakly closed algebras.

Lemma 2.3. *Let A be a JC-algebra, in which every normbounded increasing sequence has a least upper bound. For every separable subset B of A there exists a projection $\mu \in A$ such that $\mu b = b\mu = b$ for every $b \in B$.*

Proof. We may suppose that B is a JC-subalgebra. Let (μ_n) be an increasing sequence from B, with $0 < \mu_n$ and $\|\mu_n\| \leqq 1$, such that (μ_n) is an approximate identity for B. Let μ be the least upper bound of (μ_n). For $x \in B$ we have

$$(\mu-\mu_n)x^*x(\mu-\mu_n) \to (x\mu - x)^*(x\mu - x)$$

uniformly, and for $m \leqq n$

$$0 \leqq (\mu-\mu_n)x^*x(\mu-\mu_n) \leqq \|x\|^2\|\mu\|(\mu-\mu_m) .$$

Since the positive cone is uniformly closed,

$$(x\mu - x)^*(x\mu - x) \leqq \|x\|^2\|\mu\|(\mu-\mu_m)$$

for every m, and $x\mu = x = \mu x$. Further

$$0 \leqq \mu^2-\mu_n = \mu(\mu-\mu_n) \leqq \|\mu\|(\mu-\mu_n)$$

for all n, so $\mu^2-\mu = 0$.

In the same way it can be proved that every C^*-algebra in which every normbounded monotone increasing net has a least upper bound has a unit.

Theorem 2.4. *Every J Baire*-algebra has a faithful Jordan representation as a concrete J Baire*-algebra.*

PROOF (cf. [6]). For any $x \in A$ and any increasing sequence (a_n) from A with least upper bound a we can choose a JC-subalgebra B of A with unit u containing x, (a_n), a, and the least upper bound b of (xa_nx). If x has an inverse y in B, then $(a_n)=(yxa_nxy)$ has a least upper bound a satisfying

$$a \leqq yby \leqq yxaxy = a ,$$

so $xax=b$. In any case there exists $K \in \mathsf{N}$ such that $x+ku$ has an inverse in B for any $k>K$. Then for any σ-normal state f

$$\begin{aligned} &kf(x(a_n-a)+(a_n-a)x) \\ &\quad = f((x+ku)(a_n-a)(x+ku)) + f(x(a-a_n)x)+k^2f(a-a_n) \to f(xax-b) \end{aligned}$$

for any $k>K$, so $f(xax-b)=0$ and $xax=b$.

Let C be a C^*-algebra containing A and generated by A. Any state f of A can be extended to a state of the subspace $\tilde{A}=A+\mathsf{R}1$ of $\tilde{C}$, then to $\tilde{A}+i\tilde{A}$, and by the Hahn–Banach theorem to a functional f on $\tilde{C}$. Since $\|f\|=f(1)$, f is a state of $\tilde{C}$. If f is σ-normal on A and (a_i) is an increasing sequence from A with least upper bound a, then for any $u,v \in C$ of the form

$$u = x_1x_2\ldots x_n, \qquad v = y_1y_2\ldots y_m,$$

with $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m \in A$, we have

$$f(u^*a_iu) \to f(u^*au)$$

and

$$|f(u^*(a-a_i)v)|^2 \leqq f(u^*(a-a_i)u)f(v^*(a-a_i)v) \to 0 .$$

It follows that the restriction to A of the representation associated with f is σ-normal. The direct sum of the representations associated with states of C extending a separating family of σ-normal states of A is σ-normal and faithful on A, and hence maps A onto a concrete J Baire*-algebra.

In the same way we can prove

THEOREM 2.5. *Every Baire*-algebra has a faithful representation as a concrete Baire*-algebra.*

For an abstract C^*-algebra A we define $\mathscr{B}_A$, the enveloping Baire*-algebra or the Baire operators associated with A, as A_s^m computed in the universal representation of A, cf. [12]. In the terminology of [1], $\mathscr{B}_A \subseteqq \tilde{A}$, and $\mathscr{B}_A$ may as well be computed in the reduced atomic representation. If A is G.C.R., then $\mathscr{B}_A=\tilde{A}$, see [13]; especially, if $A=C_0(T)$, where T is a locally compact Hausdorff space, then $\mathscr{B}_A$ is the set of bounded Baire functions on T.

Any representation π of A has a unique extension to a σ-normal representation of $\mathscr{B}_A$, viz. the restriction to $\mathscr{B}_A$ of the extension to a normal representation of A''. This extension maps onto $\pi(A)_s^m$; in fact, more generally any σ-normal (Jordan-) representation of a (J) Baire*-algebra maps onto a (J) Baire*-algebra, see the proof of proposition 4.2 in [12], cf. also [2].

All the work in [14] on applications of Σ^*-algebras in quantum mechanics applies just as well to Baire*-algebras.

Definition 2.6. The universal σ-normal representation of a Baire*-algebra A is the direct sum of all the representations associated with the non-zero positive σ-normal linear functionals on A.

Theorem 2.7. *The set of linear combinations of σ-normal states of a Baire*-algebra A is a uniformly closed subspace P of A'.*

For x and y in A'' and $f \in P$ the functional $a \to f(xay)$, $a \in A$, is in P.

P' is a von Neumann algebra isomorphic to the weak closure of A in its universal σ-normal representation.

The canonical map ι of A into P' is σ-normal and faithful, and to any σ-normal representation φ of A there exists a unique normal representation ψ of P' satisfying $\psi \circ \iota = \varphi$.

Proof. If f is a σ-normal state, $x,y \in A''$, and (a_n) is an increasing sequence from A with least upper bound a, then

$$f(x^* a_n x) = \big(\pi_f(a_n)\pi_f(x)\xi_f \mid \pi_f(x)\xi_f\big) \to f(x^* a x) .$$

By polarization, $a \to f(xay)$ is in P.

Then every $f \in \bar{P}$ is σ-normal, and $a \to f(xay)$ is in $\bar{P}$ for $x,y \in A''$. The intersection of the kernels in A'' of the functionals in $\bar{P}$ is a weakly closed two-sided ideal, so $\bar{P}'$ is a von Neumann algebra. Since the predual of a von Neumann algebra is generated by its positive elements, $\bar{P} = P$. Defining ι by

$$\iota(a)(f) = f(a), \qquad a \in A,\ f \in P ,$$

ι is clearly σ-normal and faithful. Let φ be a σ-normal representation of A, and let $\tilde{\varphi}$ denote the normal extension to A''. If $f(x) = 0$ for all $f \in P$, then $(\tilde{\varphi}(x)\zeta \mid \eta) = 0$ for all $\zeta, \eta \in H_\varphi$, so $\tilde{\varphi}$ can be transferred to a normal representation ψ of P', unique by the uniqueness of $\tilde{\varphi}$. If φ is the universal σ-normal representation, ψ is faithful and hence an isomorphism, since if $x \in P'$ and $\psi(x) = 0$, then $f(x^*x) = 0$ for all $f \in P$.

A Jordanized version of theorem 2.7 can be established on the basis of [5].

COROLLARY 2.8. *If A is the enveloping Baire*-algebra of a C^*-algebra C, then P is isomorphic to C' and P' is isomorphic to C''.*

3. Projections in a J Baire*-algebra.

In this section A is a J Baire*-algebra on a Hilbert space H.

For $x \in A$ the range projection of x is in A; the set of projections in A is a σ-complete lattice.

As in [16] we say that two projections e and f in A can be exchanged (by a partial symmetry) if there exists $s \in A$ with $ses=f$ and $s^2=e\vee f$. For any projection $g \in A$ greater than $e\vee f$, this is the case if and only if there exists $t \in A$ with $t^2=g$ and $te=ft$. Further e and f are in position p' if and only if

$$e \wedge (e\vee f-f) = (e\vee f-e) \wedge f = 0 .$$

We call e and f perspective in A if there exists a projection $g \in A$ such that both e and g, and f and g are in position p'; and e and f are called S-equivalent (resp. projective) if there exist projections $e=e_0, e_1, \ldots, e_n=f$ such that e_{i-1} and e_i can be exchanged (resp. are perspective) for $i=1,\ldots,n$.

PROPOSITION 3.1 (cf. [16, corollary 11]). *S-equivalence is the same as projectivity.*

PROOF (cf. [16, theorems 6 and 7]). If e and f are in position p' and s is the difference between the range projections of the positive and the negative parts of $e+f-e\vee f$, then e and f are exchanged by $s+e\vee f-s^2$. If e and f are exchanged by s, with $s^2=e\vee f$, and

$$g = \tfrac{1}{2}(e\vee f+s+e\wedge f-s(e\wedge f)) ,$$

then both e and g, and f and g are in position p'.

LEMMA 3.2 (cf. [16, theorem 9 and proposition 11]). *Let (e_n) and (f_n) be two sequences of projections such that e_n and f_n can be exchanged for each n, and $e_n\vee f_n$ and $e_m\vee f_m$ are orthogonal for $n\neq m$. Then $\sum e_n$ and $\sum f_n$ can be exchanged.*

PROOF. If e_n and f_n are exchanged by s_n with ${s_n}^2=e_n\vee f_n$, then $\frac{1}{2}(s_n+e_n\vee f_n)$ are pairwise orthogonal projections and $\sum e_n$ and $\sum f_n$ are exchanged by

$$\sum s_n = 2\sum \tfrac{1}{2}(s_n+e_n\vee f_n) - \sum e_n\vee f_n .$$

Note that if e and f are orthogonal and perspective projections in A, they can be exchanged ([16, proposition 10]). Also, that for any projections e and f in A the range projections of ef and fe are in position p' and hence can be exchanged ([16, corollary 7 and lemma 12]).

The following result is implicit in [13].

Lemma 3.3. *Let (e_i) be a sequence of pairwise orthogonal projections in A with sum e. Let x be an operator on H such that $e_i x e_j + e_j x e_i \in A$ for all i,j. Then $exe \in A$.*

Proof. Put $z = \sum 2^{-i} e_i$. Then z has range projection e, and $zxz \in A$ since A is uniformly closed, so $exe \in A$ by a lemma of Kadison (cf. [12, lemma 5]).

Lemma 3.4. *If A contains an infinite sequence (e_i) of pairwise orthogonal non-zero projections with e_i and e_{i+1} perspective for all i, then A contains a J Baire*-subalgebra isomorphic to the Jordan algebra of self-adjoint operators on a separable infinite-dimensional real Hilbert space.*

Proof. We may assume $\sum e_i = 1$. By [16, lemma 21] there exists a sequence of partial symmetries $s_{i1} = s_{1i}$ with $s_{i1}^2 = e_1 \vee e_i$ and

$$s_{i1} e_1 s_{1i} = e_i, \quad s_{11} = e_1 .$$

Define $v_{i1} = e_i s_{i1}$, $v_{1i} = v_{i1}^*$ and $v_{ij} = v_{i1} v_{1j}$, so $v_{ii} = e_i$. Let B be the C^*-algebra generated by A; then (v_{ij}) is a set of matrix units in B, and $v_{ij} + v_{ji} \in A$ for all i,j. Define

$$C = \{x \in \bar{B} \mid \forall i,j \, \exists \lambda_{ij} \in \mathbf{C} : \; e_i x e_j = \lambda_{ij} v_{ij}\};$$

then by [9, lemma 15] C is a weakly closed subalgebra of $\bar{B}$ isomorphic to the algebra of operators on a separable infinite-dimensional complex Hilbert space. Define

$$D = \{x \in C \mid x = x^*, \forall i,j : \; \lambda_{ij} \text{ is real}\};$$

D is a weakly closed Jordan subalgebra of B isomorphic to the Jordan algebra of self-adjoint operators on a separable infinite-dimensional real Hilbert space. For $x \in D$

$$\lambda_{ji} = \overline{\lambda_{ij}} = \lambda_{ij}$$

so

$$e_i x e_j + e_j x e_i = \lambda_{ij}(v_{ij} + v_{ji}) \in A ,$$

and $x \in A$ by lemma 3.3.

A is called invertible (cf. [15]) if for any $a_1, a_2, \ldots, a_n \in A$,

$$a_1a_2\ldots a_n + a_n\ldots a_2a_1 \in A\ .$$

LEMMA 3.5. *If A is invertible, orthogonal projective projections can be exchanged.*

PROOF. Assume e and f projective and orthogonal. Choose projections $e=e_0,e_1,\ldots,e_n=f$ and partial symmetries s_i exchanging e_{i-1} and e_i; we may assume

$$s_i^2 = e_0 \vee e_1 \vee \ldots \vee e_n = g\ .$$

Put $u=es_1s_2\ldots s_n$; then $u^*u=f$, $uu^*=e$, $u^*eu=f$, and $u+u^*$ is a partial symmetry in A exchanging e and f.

We call A modular if the lattice of projections in A is modular, that is, if $e \leqq g$ implies $e\vee(f\wedge g)=(e\vee f)\wedge g$ for any three projections $e,f,g \in A$. We call a projection $e \in A$ modular if eAe is modular.

THEOREM 3.6 (cf. [16, proposition 14]). *The following properties* 1)–3) *of a J Baire*-algebra A are equivalent:*

1) *A is modular.*
2) *If (e_i) is an infinite sequence of pairwise perspective and orthogonal projections in A, then every e_i is* 0.
3) *For any pair e,f of perspective projections in A, $f \leqq e$ implies $f=e$.*

If A is invertible, these properties are equivalent to:

4) *If (e_i) is a sequence of pairwise projective and orthogonal projections in A, then every e_i is* 0.
5) *For any pair e,f of projective projections in A, $f \leqq e$ implies $f=e$.*

PROOF. 3) $\Rightarrow$ 1) since the two sides of the modularity identity are comparable and perspective.

4) $\Rightarrow$ 5): If $f<e, f\neq e$, and f and e are projective, we can find partial symmetries $s_1,s_2,\ldots,s_n$ and $u=s_1s_2\ldots s_n$ such that $u^*eu=f$. Then $(u^{*n}(e-f)u^n)$ provides a counterexample to 4). In the same way 2) $\Rightarrow$ 3).

5) $\Rightarrow$ 3) is trivial, and so is 2) $\Rightarrow$ 4) by lemma 3.5 when A is invertible.

1) $\Rightarrow$ 2) follows from lemma 3.4.

4. Countably generated J Baire*-algebras.

DEFINITION 4.1. A J Baire*-subalgebra of a J Baire*-algebra A is a Jordan subalgebra B containing the least upper bound (computed in A) of each of its normbounded monotone increasing sequences; such a B is a J Baire*-algebra. A J Baire*-algebra A is called countably generated if A has a countable subset B such that A is the smallest JBaire*-subalgebra of A containing B.

In the rest of this section, A denotes a countably generated JBaire*-algebra.

LEMMA 4.2. *A has a unit. Every projection in A has a central support.*

PROOF. See [13] or [2].

LEMMA 4.3. *Let Z be the center of A. Two projections e and f have orthogonal central supports if and only if*

$$eAf = 0,$$

and if and only if

$$eaf+fae = 0 \quad \text{for all } a \in A .$$

The center of eAe is eZe.

PROOF. See [16, lemma 2.4 and theorem 14].

THEOREM 4.4 (cf. [16, theorem 10 and corollaries 18 and 19] and [2, theorem 2.6]). *If e and f are projections in the countably generated J Baire*-algebra A, there exists projections $e_1,e_2,f_1,f_2 \in A$ and a symmetry $s \in A$ with*

$$e = e_1+e_2, \quad f = f_1+f_2, \quad se_1s = f_1 ,$$

and e_2 and f_2 have orthogonal central supports.

For any central projection h with $e_2 \leqq h \leqq 1-f_2$ we have

$$\begin{aligned} shes &\leqq hf , \\ s(1-h)fs &\leqq (1-h)e , \\ s(1-h)(1-e)s &\leqq (1-h)(1-f) . \end{aligned}$$

PROOF. A is generated by a sequence (U_n) of symmetries. In view of lemma 3.2, if $ef=0$, the proof of theorem 2.6 in [2] applies. For the general case, see [16, corollary 18].

THEOREM 4.4 (cf. [11, III, theorem 2.2]). *In a modular invertible countably generated J Baire*-algebra A projective projections can be exchanged, and perspectivity is transitive.*

PROOF. Let e and f be projective projections in A. Choose projections e_1, e_2, f_1, f_2 such that

$$e = e_1+e_2, \quad f = f_1+f_2 ,$$

e_1 and f_1 can be exchanged, and e_2 and f_2 have orthogonal central supports g and h. Then ge and $gf=gf_1$ and ge_1 are projective. By theorem 3.6, $ge=ge_1$, so

$$e_1 \geqq ge \geqq ge_2 = e_2 ,$$

and $e=e_1$; similarly $f=f_1$.

Now the following three propositions can be proved exactly as in [16, pp. 26–28].

PROPOSITION 4.5. *If e and f are modular projections in A, then $e \vee f$ is modular.*

PROPOSITION 4.6. *If e and f are perspective projections in A and e is modular, then $f \leqq e$ implies $f=e$.*

PROPOSITION 4.7. *If A is invertible, then projective modular projections can be exchanged.*

Also, for A invertible the Schröder–Bernstein theorem holds for modular projections.

5. Projections in a Baire*-algebra.

In this section A denotes a Baire*-algebra acting on a Hilbert space H.

As in a Σ^*-algebra (cf. [2]) we call two projections e and f equivalent, $e \sim f$, if there exists $v \in A$ with

$$v^*v = e, \quad vv^* = f .$$

This relation $\sim$ is additive. We call e and f U-equivalent if there exist a projection $g \geqq e \vee f$ and $u \in A$ with

$$u^*u = uu^* = g, \quad u^*eu = f .$$

If this is the case, then for any projection $h \in A$ greater than g there exists $v\ (=u+h-g)$ with $v^*v=vv^*=h$ and $v^*ev=f$, and the relation is transitive. Projective projections are U-equivalent, and U-equivalent projections are equivalent. Orthogonal equivalent projections can be exchanged: if $v^*v=e$, $vv^*=f$, and $e \perp f$, then $v+v^*$ is a partial symmetry exchanging e and f.

The next theorem and proof is due to G. K. Pedersen (oral communication).

THEOREM 5.1. *Let $x=u|x|$ be the canonical polar decomposition of $x \in A$. Then $|x|$ and u belong to A.*

Proof. Since $|x|=(x^*x)^{\frac{1}{2}}$, $|x| \in A$. Also, the range projections u^*u and uu^* of $|x|$ and $|x^*|$ are in A. Further,

$$(x+|x|)(n^{-1}+|x|)^{-2}(x+|x|)^* \rightarrow uu^*+u+u^*+u^*u$$

monotonely, so $u+u^* \in A$. Applying this argument to ix we find $iu-iu^* \in A$, so $u \in A$.

Corollary 5.2. *The equivalence relation is countably additive.*

Proof. Let (v_i) be a sequence of partial isometries in A with $v_i^*v_i = \delta_{ij}e_i$ and $v_iv_j^* = \delta_{ij}f_i$. Then $\sum 2^{-i}v_i \in A$ and has support and range projections $\sum e_i$ and $\sum f_i$. By theorem 5.1 these are equivalent.

Inspecting the polar decomposition of $\sum 2^{-i}v_i$ we find in fact $\sum v_i \in A$. More generally, by using the identity

$$x + x^* = (1+u)|x|(1+u^*) - |x^*| - |x| ,$$

G. K. Pedersen proved: If (x_i) is a sequence from A with $\sum|x_i|$ and $\sum|x_i^*|$ weakly convergent, then $\sum x_i$ is weakly convergent with sum in A.

By the methods used in sections 3 and 4 we get:

Theorem 5.3. *The following properties of a Baire*-algebra A are equivalent:*

1) *A is modular.*
2) *If (e_i) is an infinite sequence of pairwise orthogonal equivalent projections in A, then every e_i is 0.*
3) *A does not contain a Baire*-subalgebra isomorphic to the algebra of bounded operators on an infinite-dimensional Hilbert space.*
4) *A is finite, that is, every isometry in A is unitary.*

Theorem 5.4. *In a finite, countably generated Baire*-algebra equivalent projections can be exchanged. Hence equivalence, U-equivalence, projectivity, and perspectivity coincide.*

Proposition 5.5. *In a countably generated Baire*-algebra the supremum of two finite projections is finite, and equivalent finite projections can be exchanged.*

In the rest of the paper A is supposed to be countably generated.

For reference we note the following easy consequence of the comparison lemma.

Lemma 5.6. *Let e, f and g be projections in A. If $e \prec f$ and $f \prec g$ and $e \leqq g$, then there exists a projection $h \in A$ with $e \leqq h \leqq g$ and $f \sim h$.*

LEMMA 5.7 (cf. [8, lemma 6.4]). *Let (e_i) be a decreasing sequence with infimum e of finite projections in A. If a projection $f \in A$ satisfies $f \prec e_i$ for all i, then $f \prec e$.*

PROOF. Inductively we can choose a decreasing sequence (f_i) with infimum $g \geqq f$ and $e_i \sim f_i$ for all i. Then $f_1 - g \sim e_1 - e$, so by finiteness $g \sim e$.

PROPOSITION 5.8 (cf. [8, theorem 6.5]). *Let (e_i) be a decreasing sequence with infimum e of finite projections in A. For any projection $f \in A$ the infimum of $(e_i \vee f)$ is $e \vee f$.*

PROOF. If h denotes the infimum $- e \vee f$, then

$$h \leqq e_i \vee f - e \vee f \sim e_i - (e \vee f) \wedge e_i \leqq e_i - e ,$$

so $h = 0$.

In the same way we prove: if (e_i) is an increasing sequence with finite supremum e of projections in A and f is any projection in A, then the supremum of $(e_i \wedge f)$ is $e \wedge f$. Summing up we have:

The lattice of projections in a finite, countably generated Baire-algebra satisfies the countable analogues of the axioms for a continuous geometry.*

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