MATH. SCAND. 25 (1969), 71–93

MEASURE THEORY FOR C^* ALGEBRAS III

GERT KJÆRGÅRD PEDERSEN

Introduction.

The idea that a functional on a C^* algebra A should be an analogue of an integral, that is, a Radon measure, a point of view which has dominated [12] and [13], will also govern this paper. The functionals in question need not be bounded, hence are not everywhere defined, and this naturally gives rise to a study of non-closed order ideals or two-sided ideals in the algebra. In [16] E. Størmer proved that the class of positive parts of closed two-sided ideals is a lattice in its natural partial ordering. Since then we have presented a simple proof of this result [14], and in section 1 we show how our method applies to a larger class of order ideals called strongly invariant. In section 2 we show that the notion of bounded variation known in measure theory admits a generalization to the non-commutative case, in such a way that exactly the C^* integrals are singled out as the functionals with bounded variation. We also give a characterization in terms of seminorms of the cumbersome topology on K_A introduced in [13]. Section 3 is devoted to traces and the functionals they majorize, and we prove by a generalization of the Riesz decomposition theorem that the traces have a lattice structure.

The problem of decomposing a functional with respect to other functionals, has proved to be considerably more complex than the corresponding problem in the commutative case. In particular the investigations of H. Dye [8] have shown, that a Radon–Nikodym theorem is not likely to hold, and especially that it fails for the class of C^* algebras which we usually regard as the more sympathetic – the type I class. Additionally the problem of decomposing a functional in terms of Radon measures on the pure states is likely to be unsolvable when the algebra is not separable.

Concentrating on the type I class we notice that, if the C^* algebra consists of the compact operators $B_0(H)$ on some Hilbert space H, then any bounded functional has the form $x \to \mathrm{tr}(bx)$, $x \in B_0(H)$, for some

Received March 27, 1968; in revised form November 15, 1968.

operator b of trace class, and if the algebra is a $C_0(X)$, X a locally compact Hausdorff space, then any bounded functional has the form

$$x \to \int_X x(t)\, d\mu(t), \qquad x \in C_0(X)$$

for some bounded Radon measure μ on X. It is shown in section 5 that for any C^* algebra A of type I, the natural combination of these two examples exhaust all bounded functionals on A. For any bounded functional f this gives a decomposition of f in terms of an operator b in the enveloping von Neumann algebra A'', and a bounded Radon measure μ on $\hat{A}$. In section 4 we have exhibited a certain monotone class of operators in A'', called the Baire operators, studied in detail by Kadison [10] for other reasons, and we now show that if A is separable, then b may be chosen as a Baire operator.

The terminology is the standard terminology from [6] with minor modifications from [12] and [13]. For a C^* algebra A we write

A'' for the enveloping von Neumann algebra of A,
$\tilde{A}$ for the algebra obtained when an identity is adjoined to A,
A^R for the self-adjoint elements in A, and
A_1 for the unit sphere in A.

I take this opportunity to thank F. Combes who initiated the paper, and E. T. Kehlet who guided my efforts at the final stage of the work.

1. Strongly invariant order ideals.

In [14, Corollary 2] we proved a Riesz decomposition property for C^* algebras. For the purpose of this paper we shall need the following stronger version, which is also well known in function theory:

PROPOSITION 1.1. *If $\{x_n\}$ and $\{y_m\}$ are sequences in A such that*

$$\sum x_n x_n^* = \sum y_m^* y_m \in A ,$$

then there exists a sequence $\{z_{nm}\}$ in A with

$$\sum_m z_{nm}^* z_{nm} = x_n^* x_n, \quad \sum_n z_{nm} z_{nm}^* = y_m y_m^* .$$

PROOF. Set $x=\sum x_n x_n^*$ and observe that (as in the proof of [14, Theorem 1]) the sequences with kth element $y_m(k^{-1}+x)^{-\frac{1}{2}}x_n$ converge for all n and m to elements $z_{nm} \in A$. We have

$$\sum_m z_{nm}^* z_{nm} = \lim_k x_n^*(k^{-1}+x)^{-1} x x_n = x_n^* x_n$$

and similarly

$$\sum_n z_{nm} z_{nm}^* = \lim_k y_m(k^{-1}+x)^{-1} x y_m^* = y_m y_m^* .$$

By deleting some of the x_n and the corresponding z_{nm} we immediately get the following useful

COROLLARY 1.2. *If* $\sum_n x_n x_n^* \leqq \sum y_m^* y_m$, *there exist* $z_{nm} \in A$ *such that*

$$\sum_m z_{nm}^* z_{nm} = x_n^* x_n, \quad \sum_n z_{nm} z_{nm}^* \leqq y_m y_m^* .$$

An order ideal J of A^+ is called ***strongly invariant*** if $x^*x \in J$ implies $xx^* \in J$ for all $x \in A$.

In [12, Section 1], following the terminology of [9], an order ideal J of A^+ was called invariant if $x^*Jx \subset J$ for every $x \in A$ or, equivalently, if $u^*Ju = J$ for every unitary $u \in \tilde{A}$. Obviously strong invariance implies invariance. If A is commutative, the terms are equal (and empty); and if A is a von Neumann algebra, then since x^*x is carried into xx^* by transformation with a partial isometry in the algebra the two terms again are equal.

If J is an invariant order ideal with roots, that is, $x \in J$ implies $x^{\frac{1}{2}} \in J$, then J is strongly invariant since

$$x^*x \in J \Rightarrow x(x^*x)x^* = (xx^*)^2 \in J \Rightarrow xx^* \in J .$$

Hence in particular $K_A{}^+$ is strongly invariant by [14, Proposition 4]. However we have the following

PROPOSITION 1.3. *There exists a C^* algebra A and an invariant, but not strongly invariant order ideal J of A^+.*

PROOF. Let A be the set of sequences of 2×2 matrices over $\mathbf{C}$, tending to zero at infinity. Obviously, A is a C^* algebra, and is an ideal among all bounded sequences. Define

$$v = \{(v_n)\} = \left\{\begin{pmatrix} 0 & 0 \\ n^{-1} & 0 \end{pmatrix}\right\}, \quad p = \{(p_n)\} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

and let J be the smallest invariant order ideal containing v^*v. If $vv^* \in J$, there would exist, by the minimality of J, a finite set $\{x_i\} \subset A$ and $\alpha \in \mathbf{R}^+$ such that

$$vv^* \leqq \sum x_i^* v^* v x_i + \alpha v^* v .$$

Since $vp = 0$ and $pv = v$, we have with $y_i = x_i p \in A$

$$vv^* = pvv^*p \leqq \sum y_i^* v^* v y_i .$$

Hence, for any n in the sequence,

$$v_n v_n^* \leqq \sum y_{in}^* v_n^* v_n y_{in} ,$$

and since the norm is order preserving on A^+,

$$\|v_n\|^2 \leqq \|v_n\|^2 \sum \|y_{in}\|^2 ,$$

a contradiction since $y_{in} \to 0$ as $n \to \infty$.

Since for any invariant order ideal J the set

$$I = \{x \in A \mid x^*x \in J\}$$

is a two-sided ideal in A, we have

Corollary 1.4. *There exists a two-sided ideal in A which is not self-adjoint.*

The preference of strongly invariant order ideals to invariant order ideals stems from the observation that only the former may serve as ideals of definition for traces on a C^* algebra A. However, as the following theorems show, these ideals also form a very well behaved class of objects in themselves.

Proposition 1.5. *If J_1 and J_2 are strongly invariant order ideals of A^+, then so is J_1+J_2.*

Proof. Suppose $xx^* \leqq y_1+y_2$, $y_i \in J_i$. By corollary 1.2. there exist z_1 and z_2 such that

$$x^*x = z_1^*z_1+z_2^*z_2, \quad z_iz_i^* \leqq y_i .$$

Since J_1 and J_2 are strongly invariant, we conclude that $x^*x \in J_1+J_2$. If $x=x^*$, this proves that J_1+J_2 is an order ideal; and therefore the same equations with general x prove that J_1+J_2 is strongly invariant.

Theorem 1.6. *The class of strongly invariant order ideals of A^+ is a distributive lattice under sum and intersection.*

Proof. It suffices to prove that

$$(J_1+J_2) \cap J_3 = J_1 \cap J_3 + J_2 \cap J_3 .$$

But if $x_i \in J_i$, $i=1,2$, and $x_1+x_2 \in J_3$, then also $x_i \in J_3$, hence

$$(J_1+J_2) \cap J_3 \subset J_1 \cap J_3 + J_2 \cap J_3 ,$$

and since the converse inclusion is evident, the theorem is proved.

2. Functionals with bounded variation.

A convex functional is a function $\varrho \colon A^+ \to [0,\infty]$ satisfying

$$\varrho(\alpha x) = \alpha \varrho(x) \quad \text{for } \alpha \in \mathsf{R}^+ , \tag{1}$$

$$\varrho(x+y) \leqq \varrho(x)+\varrho(y) . \tag{2}$$

It is called invariant if moreover

(3) $$\varrho(x^*x) = \varrho(xx^*) \quad \text{for all } x \in A,$$
(4) $$\varrho(x) \leqq \varrho(y) \quad \text{for } x \leqq y.$$

LEMMA 2.1. *An invariant convex functional ϱ which is finite on A^+ is bounded.*

PROOF. Otherwise we could find $x_n \in A_1^+$, $\varrho(x_n) > n2^n$, hence $x = \sum 2^{-n} x_n \in A_1^+$ but $\varrho(x) \geqq 2^{-n}\varrho(x_n) > n$, a contradiction.

For any convex functional ϱ on A^+ define the *variation* of ϱ at $x \in A^+$ by
$$\tilde{\varrho}(x) = \sup\{\varrho(y^*y) \mid yy^* \leqq x\}.$$

LEMMA 2.2. *$\tilde{\varrho}$ is an invariant convex functional.*

PROOF. Only conditions (2) and (3) need verification. If x^*x and y^*y are given, there exists for any $\alpha \in \mathsf{R}^+$ smaller than $\tilde{\varrho}(x^*x+y^*y)$ an element $z \in A$ such that
$$\alpha \leqq \varrho(z^*z), \quad zz^* \leqq x^*x+y^*y.$$
By corollary 1.2 there exist x_1 and y_1 such that
$$z^*z = x_1^*x_1+y_1^*y_1, \quad x_1x_1^* \leqq xx^*, \quad y_1y_1^* \leqq yy^*.$$
Since ϱ is convex, this gives
$$\alpha \leqq \varrho(x_1^*x_1) + \varrho(y_1^*y_1) \leqq \tilde{\varrho}(xx^*) + \tilde{\varrho}(yy^*),$$
hence
$$\tilde{\varrho}(x^*x+y^*y) \leqq \tilde{\varrho}(xx^*) + \tilde{\varrho}(yy^*).$$
With $x = x^*$ and $y = y^*$ this gives (2), and with $y = 0$ it gives (3).

THEOREM 2.3. *There is a one-to-one correspondence between densely defined, lower semi-continuous invariant convex functionals on A^+, and finite invariant convex functionals on K_A^+.*

PROOF. If ϱ is a finite invariant convex functional on K_A^+, then for any $x \in A^+$ define
$$\tilde{\varrho}(x) = \sup\{\varrho(y) \mid y \leqq x,\ y \in K_A^+\}.$$
Obviously $\tilde{\varrho}$ is a convex extension of ϱ, and since by corollary 1.2 for any $y \in K_A^+$ and $x \in A$ there exist $z \in A$ such that
$$y \leqq x^*x \Rightarrow y = z^*z, \quad zz^* \leqq xx^*,$$
we have $\tilde{\varrho}(x^*x) = \tilde{\varrho}(xx^*)$, because K_A^+ is strongly invariant.

Suppose $x \in A^+$ is a limit point of elements y with $\tilde{\varrho}(y) \leqq 1$. For any $\alpha \in \mathsf{R}^+$ smaller than $\tilde{\varrho}(x)$ there exists $z \in K_A^+$ such that $\alpha \leqq \varrho(z)$, $z \leqq x$. By [14, Proposition 4] the order-related C^* algebra B generated by z is contained in K_A, hence by lemma 2.1 ϱ has a bound β on $B_1{}^+$. Now for any $\varepsilon > 0$ choose $y \in A^+$ with $\tilde{\varrho}(y) \leqq 1$ such that

$$z \leqq y + \beta^{-1}\varepsilon .$$

Then by corollary 1.2

$$z = z_1{}^*z_1 + z_2{}^*z_2, \quad z_1z_1{}^* \leqq y, \quad z_2z_2{}^* \leqq \beta^{-1}\varepsilon .$$

Since $z_2{}^*z_2 \in B^+$ and $\|z_2{}^*z_2\| \leqq \beta^{-1}\varepsilon$, we have

$$\varrho(z) \leqq \tilde{\varrho}(y) + \varepsilon \leqq 1 + \varepsilon ,$$

hence $\tilde{\varrho}(x) \leqq 1$, and $\tilde{\varrho}$ is lower semi-continuous.

If, conversely, ϱ is a densely defined, lower semi-continuous invariant convex functional on A^+, then the elements for which ϱ is finite clearly form a strongly invariant order ideal of A^+ which is dense, hence includes $K_A{}^+$. The restriction of ϱ to $K_A{}^+$ admits a lower semi-continuous extension, which is clearly unique, hence equals ϱ.

In [13, Section 2] we introduced a vector space topology τ on K_A, and defined the C^* integrals of A as the dual of (K_A, τ). With the aid of invariant convex functionals, this topology can be described in a perhaps more accessible form. We adopt the terminology of [13, Section 2] and have

THEOREM 2.4. *The topology τ is the weakest locally convex topology on K_A for which all faithful finite invariant convex functionals on $K_A{}^+$ are continuous.*

PROOF. If ϱ is any finite invariant convex functional on $K_A{}^+$, then by lemma 2.1 the number

$$\beta = \sup \{\varrho(x) \mid x \in B(p)_1{}^+\}$$

is finite for every $p \in Y$ and depends only on the unitary equivalence class $\tilde{p}$. Hence the definition

$$\delta(\tilde{p}) = (\beta \vee 1)^{-1}$$

gives a $\delta \in \Delta$, hence a τ-neighbourhood E_δ. For any $x \in E_\delta{}^+$ we have

$$x \leqq \sum \alpha_n x_n, \quad \sum \alpha_n = 1, \quad x_n \in B(p_n)^+_{\delta(\tilde{p}_n)} ,$$

whence

$$\varrho(x) \leqq \sum \alpha_n \varrho(x_n) \leqq \sum \alpha_n \|x_n\| \beta_n < 1 ,$$

and we have proved that ϱ is τ-continuous.

Conversely, suppose E_δ given. Since τ is stronger than the norm topology, we may as well assume $E_\delta \subset A_1$. Now since E_δ is absorbing, the definition

$$\varrho(x) = \inf\{\alpha \mid \alpha^{-1}x \in E_\delta{}^+\}, \quad x \in K_A{}^+,$$

gives a function on $K_A{}^+$ which is convex since $E_\delta{}^+$ is hereditary-convex. Since $\alpha^{-1}x \in E_\delta{}^+$ implies $\|x\| \leqq \alpha$, we have $\|x\| \leqq \varrho(x)$, hence ϱ is faithful. We know that $E_\delta{}^+$ is invariant under unitary transformations, hence ϱ is unitarily invariant. We strongly suspect that ϱ is also invariant, but for our purpose it will suffice to show, that the variation $\tilde{\varrho}$ defined by lemma 2.2 is finite on $K_A{}^+$. For this it is enough to show that $\tilde{\varrho}$ is finite at every element $x \in K_A{}^+$ such that there exists $p \in Y$ and $y \in K_A{}^+$ with $x \leqq p \leqq y$, since by the remarks preceding Proposition 4 of [14] K_A is generated by linear combinations of such elements.

Now for any $\alpha \in \mathsf{R}^+$ smaller than $\tilde{\varrho}(x)$ there exists by the definition of ϱ a $z \in K_A$ with

$$\alpha \leqq \varrho(z^*z), \quad zz^* \leqq x\,.$$

Since $zz^* \leqq p$, we have $z^*z = z^*pz$, and from $\|z\| \leqq 1$ it follows that $2z$ is the sum of 4 unitary operators u_i. Hence

$$z^*z = \tfrac{1}{4}(\Sigma u_i)^*\, p(\Sigma u_i) \leqq \Sigma\, u_i{}^*pu_i \leqq \Sigma\, u_i{}^*yu_i\,.$$

Since ϱ is unitarily invariant, we conclude $\varrho(z^*z) \leqq 4\varrho(y)$, hence

$$\tilde{\varrho}(x) \leqq 4\varrho(y) < \infty\,.$$

For any $x \in K_A{}^+$

$$\tilde{\varrho}(x) < 1 \Rightarrow \varrho(x) < 1 \Rightarrow x \in E_\delta{}^+\,.$$

Hence the locally convex topology on K_A generated by the faithful invariant convex functionals on $K_A{}^+$ is stronger than τ, hence equals τ.

For any linear functional f on K_A the absolute value of f is a convex functional on $K_A{}^+$, hence the variation ϱ of f defined by lemma 2.2 is an invariant convex functional. The functional f is called of bounded variation if ϱ is finite on K^+.

Clearly this is a possible non-commutative generalization of the conventional use of the term. However it appears to be far more restrictive; for instance it is no longer true that a positive functional is automatically of bounded variation. Nevertheless it seems to be the proper generalization as shown by

THEOREM 2.5. *The functionals on K_A with bounded variation are the C^* integrals.*

PROOF. If f is a functional with bounded variation ϱ, then since $|f(x)| \leqq \varrho(x)$ for any $x \in K_A{}^+$ and since by theorem 2.4 ϱ is τ-continuous, we conclude that also f is τ-continuous, that is, is a C^* integral. Conversely, if f is a C^* integral, then there exists a τ-neighbourhood E_δ such that $x \in E_\delta$ implies $|f(x)| \leqq 1$. But again by theorem 2.4 there exists an invariant convex functional ϱ on $K_A{}^+$ such that $\varrho(x) < 1$ implies $x \in E_\delta{}^+$. It follows that the variation of f, which is clearly the smallest invariant convex functional majorizing f on $K_A{}^+$, is majorized by ϱ, hence is finite on $K_A{}^+$.

3. Functionals majorized by a trace.

The Riesz decomposition property can be used to give a direct proof of the theorem of Thoma [17, p. 116] that the bounded invariant functionals on a C^* algebra form a vector lattice. However the method gives just as easy the following generalizations of the result.

THEOREM 3.1. *The invariant self-adjoint C^* integrals form a vector lattice.*

PROOF. It suffices to show that if f_1 and f_2 are invariant self-adjoint C^* integrals, then the definition, for any $x \in K_A{}^+$

$$f(x) = \inf\{f_1(y_1)+f_2(y_2) \mid y_1+y_2=x\},$$

gives an invariant self-adjoint C^* integral, which then clearly equals $f_1 \wedge f_2$.

It is immediately verified that $f(\alpha x)=\alpha f(x)$ for $\alpha \in \mathsf{R}^+$, and that $f(x_1)+f(x_2) \geqq f(x_1+x_2)$. If $x_1{}^*x_1,\ x_2{}^*x_2 \in K_A{}^+$, then for any $\varepsilon>0$ there exist $y_i \in K_A{}^+$, $i=1,2$, such that

$$\begin{aligned} \varepsilon + f(x_1{}^*x_1+x_2{}^*x_2) &\geqq f_1(y_1)+f_2(y_2)\ , \\ y_1+y_2 &= x_1{}^*x_1+x_2{}^*x_2\ . \end{aligned}$$

By proposition 1.1 there exist $z_{ij} \in A$, $i=1,2$, $j=1,2$, such that

$$y_i = z_{i1}^* z_{i1}+z_{i2}^* z_{i2}, \qquad x_j x_j{}^* = z_{1j} z_{1j}^* + z_{2j} z_{2j}^*\ ,$$

and since f_i are invariant we have

$$\varepsilon + f(x_1{}^*x_1+x_2{}^*x_2) \geqq \big(f_1(z_{11}z_{11}^*)+f_2(z_{21}z_{21}^*)\big) + \big(f_1(z_{12}z_{12}^*)+f_2(z_{22}z_{22}^*)\big),$$

hence

$$f(x_1{}^*x_1+x_2{}^*x_2) \geqq f(x_1x_1{}^*) + f(x_2x_2{}^*)\ ,$$

which proves that f is linear on positive elements and invariant. Hence it is an invariant C^* integral, clearly self-adjoint.

THEOREM 3.2. *The traces on A^+ form a complete lattice cone.*

Proof. If f_1 and f_2 are traces on A^+ with ideals of definition J_1 and J_2, then the proof of theorem 3.1 applies for any $x \in J_1 + J_2$ and shows that $f_1 \wedge f_2$ is a trace with $J_1 + J_2$ as ideal of definition.

Proposition 3.3. *If ϱ_1 and ϱ_2 are invariant convex functionals on A^+, then $\varrho_1 \wedge \varrho_2$ exists as an invariant convex functional.*

Proof. Again if J_1 and J_2 are the ideals of definition for ϱ_1 and ϱ_2, the proof of theorem 3.1 applies for any $x \in J_1 + J_2$ and gives an invariant convex functional with ideal of definition $J_1 + J_2$.

It follows from theorem 3.2 that for any state of A there is a minimal trace majorizing the state on A^+. In order to find conditions on the algebra which will assure that this trace is not trivial, we introduce the set J_A consisting of those $x \in A^+$ for which there exists a constant β_x such that for any finite set $\{x_n\}$,

$$\Sigma x_n x_n^* \leqq x \Rightarrow \|\Sigma x_n^* x_n\| \leqq \beta_x .$$

Proposition 3.4. *J_A is a strongly invariant order ideal of A^+.*

Proof. Clearly $\alpha \in \mathsf{R}^+$ and $0 \leqq y \leqq x \in J_A$ imply $\alpha y \in J_A$. If

$$\Sigma x_n x_n^* \leqq y_1^* y_1 + y_2^* y_2, \quad y_i y_i^* = J_A ,$$

then by corollary 1.2 there exist z_{ni} such that

$$x_n^* x_n = z_{n1}^* z_{n1} + z_{n2}^* z_{n2}, \quad \Sigma z_{ni} z_{ni}^* \leqq y_i y_i^* .$$

By definition of J_A we have $\|\Sigma z_{ni}^* z_{ni}\| \leqq \beta_i$, hence $\|\Sigma x_n^* x_n\| \leqq \beta_1 + \beta_2$ so that $y_1^* y_1 + y_2^* y_2 \in J_A$, and the theorem follows.

Theorem 3.5. *For any state of A the minimal majorizing trace is lower semi-continuous and finite on J_A.*

Proof. Let f be a state of A, and for any $x \in A^+$ define

$$\varphi(x) = \sup\{\Sigma f(y_n^* y_n) \mid \Sigma y_n y_n^* \leqq x\} .$$

We have $f(x) \leqq \varphi(x) \leqq \infty$; and since $x \in J_A$ implies $\|\Sigma y_n^* y_n\| \leqq \beta_x$, we have in this case $\varphi(x) \leqq \beta_x < \infty$. Clearly $\alpha \in \mathsf{R}^+$ implies $\varphi(\alpha x) = \alpha \varphi(x)$ for any x, and $\varphi(x_1) + \varphi(x_2) \leqq \varphi(x_1 + x_2)$. However for any $\alpha \in R^+$ smaller than $\varphi(x_1^* x_1 + x_2^* x_2)$ there exists a finite set $\{y_n\}$ such that

$$\alpha \leqq \sum f(y_n{}^*y_n), \quad \sum y_n y_n{}^* \leqq x_1{}^*x_1 + x_2{}^*x_2 ,$$

hence by corollary 1.2

$$y_n{}^*y_n = z_{n1}^* z_{n1} + z_{n2}^* z_{n2}, \quad \sum z_{ni} z_{ni}^* \leqq x_i x_i{}^* .$$

It follows that

$$\varphi(x_1{}^*x_1 + x_2{}^*x_2) \leqq \varphi(x_1 x_1{}^*) + \varphi(x_2 x_2{}^*)$$

which together with the above established conditions implies that φ is a trace on A^+.

In order to prove that φ is lower semi-continuous let $x \in A^+$ be a limit point of elements $y \in A^+$ with $\varphi(y) \leqq 1$. For any $\alpha \in \mathsf{R}^+$ smaller than $\varphi(x)$ there exists a set $\{x_1, x_2, \ldots x_n\}$ such that

$$\alpha \leqq \sum f(x_i{}^*x_i), \quad \sum x_i x_i{}^* \leqq x .$$

For any $\varepsilon > 0$ choose $y \in A^+$, $\varphi(y) \leqq 1$, $x \leqq y + \varepsilon$, and by corollary 1.2

$$x_i{}^*x_i = y_i{}^*y_i + z_i{}^*z_i, \quad \sum y_i y_i{}^* \leqq y, \quad \sum z_i z_i{}^* \leqq \varepsilon .$$

We have $\|z_i\|^2 \leqq \varepsilon$, hence

$$\alpha \leqq \sum f(x_i{}^*x_i) \leqq \sum f(y_i{}^*y_i) + n\varepsilon \leqq 1 + n\varepsilon .$$

Since ε and α are arbitrary, $\varphi(x) \leqq 1$, and the theorem follows.

Clearly the φ constructed above is the minimal trace majorizing f.

LEMMA 3.6. *If A is a C^* algebra of type* I, *then*

$$J_A = \{x \in A^+ \mid \sup_{\hat{A}} \operatorname{tr}\pi(x) < \infty\} .$$

PROOF. If $x \in A^+$ has bounded trace on $\hat{A}$ and $\sum x_n x_n{}^* \leqq x$, then by [6, Lemme 3.3.6]

$$\|\sum x_n{}^*x_n\| = \sup_{\hat{A}} \|\pi(\sum x_n{}^*x_n)\| \leqq \sup_{\hat{A}} \operatorname{tr}\pi(\sum x_n{}^*x_n) \leqq \sup_{\hat{A}} \operatorname{tr}\pi(x) ,$$

hence $x \in J_A$.

Conversely, if $x \in A^+$ does not have bounded trace, then either there exists $\pi \in \hat{A}$ with $\operatorname{tr}\pi(x) = \infty$, or we can find a distinct sequence $\{\pi_n\} \subset \hat{A}$ such that $\operatorname{tr}\pi_n(x) > 2^n$. Since A is of type I, $\pi(A)$ contains the compact operators; hence the minimal trace majorizing any pure state whose associated representation belongs to $\pi \in \hat{A}$ is $\operatorname{tr}\pi(\cdot)$. Hence if $\operatorname{tr}\pi(x) = \infty$, we conclude from theorem 3.5 that $x \notin J_A$. Considering the other case, let $\{f_n\}$ be a sequence of pure states such that for any n the representation associated with f_n is contained in π_n. If $f = \sum 2^{-n} f_n$, then since the sequence $\{\pi_n\}$ consists of pairwise inequivalent irreducible representations, we conclude from [6, Théorème 2.8.3] that the minimal trace majorizing f is given by $\varphi(y) = \sum 2^{-n} \operatorname{tr} \pi_n(y)$ for $y \in A^+$. By assumption we have $\varphi(x) = \infty$, hence $x \notin J_A$ by theorem 3.5.

PROPOSITION 3.7. *Any state of a C^* algebra with continuous trace is majorized by an invariant C^* integral.*

PROOF. By lemma 3.6 the set J_A is dense in A^+, hence contains $K_A{}^+$, and theorem 3.5 applies.

CONJECTURE: *The above statement holds for C^* integrals as well.*

For any positive element x in a von Neumann algebra B let c-supp$x$ denote the smallest element in the center of B majorizing x. If x is a projection, then c-supp$x$ is also a projection, hence coincides with the central support of x.

Let φ be a faithful, invariant positive C^* integral which represents A on H. If $\bar{A}$ denotes the weak closure of A, then φ has a unique faithful, weakly lower semi-continuous extension (again denoted φ) from $K_A{}^+$ to $\bar{A}^+$. The positive C^* integrals majorized by φ are denoted $M_\varphi{}^+$, and we have

THEOREM 3.8. 1) *The map $\Phi: \bar{A}_1{}^+ \to M_\varphi{}^+$ given by*

$$\Phi(a)(x) = \varphi(ax)$$

for $a \in \bar{A}_1{}^+$, $x \in K_A$, is an order-preserving, affine homeomorphism of $\bar{A}_1{}^+$ in the weak topology onto $M_\varphi{}^+$ in the weak topology.*

2) *The operators $a \in \bar{A}_1{}^+$ with $\varphi(a) = 1$ map onto the states of $M_\varphi{}^+$.*

3) *The operators $a \in \bar{A}_1{}^+$ with* c-supp$a = 1$ *map onto the C^* integrals for which φ is the minimal majorant.*

PROOF. Since for all $x \in K_A{}^+$

$$\Phi(a)(x) \leqq \|a\| \varphi(x) \leqq \varphi(x),$$

Φ is an order-preserving affine map of $\bar{A}_1{}^+$ into $M_\varphi{}^+$. If $\Phi(a) = 0$, then $\varphi(au_\lambda) = 0$, hence $au_\lambda = 0$ for an approximative unit $\{u_\lambda\} \subset K_A{}^+$ converging strongly up to 1; hence $a = 0$ and Φ is one-to-one. By [12, Theorem 2.5] the extended positive functionals (weights in the terminology of [1]) on A^+ form a weak* compact set. Hence the weak* closed, convex set of those majorized by φ, which actually consists of C^* integrals, is also weak* compact, and so Φ is a map between compact sets. For all $x \in K_A{}^+$ the functional $\varphi(\cdot x)$ is bounded and majorized by $\|x\|\varphi$, hence by [12, Theorem 2.4] it is a vector functional in the representation space H, hence it is weakly continuous on $\bar{A}$. It follows that if a net $\{a_i\} \subset \bar{A}_1{}^+$ converges weakly to a, then $a \in \bar{A}_1{}^+$ and for all $x \in K_A$

$$\lim \Phi(a_i)(x) = \lim \varphi(a_i x) = \varphi(ax) = \Phi(a)(x) ,$$

which proves that Φ is continuous and thus a homeomorphism.

Regarding $\Phi(\bar{A}_1{}^+)$ as a subset of the set of self-adjoint C^* integrals, the polar set of $\Phi(\bar{A}_1{}^+)$ consists of those $x \in K_A{}^R$ for which $\varphi(ax) \geqq -1$. If $x = x_+ - x_-$, we immediately have

$$\varphi([x_-]x) = -\varphi(x_-) \geqq -1 .$$

And conversely, for any $x \in K_A{}^R$ such that $\varphi(x_-) \leqq 1$

$$\varphi(ax) \geqq -\varphi(ax_-) \geqq -\|a\|\varphi(x_-) \geqq -1 .$$

Hence we have determined the polar set. Since $f \in M_\varphi{}^+$ and $\varphi(x_-) \leqq 1$ imply

$$f(x) \geqq -f(x_-) \geqq -\varphi(x_-) \geqq -1 ,$$

$M_\varphi{}^+$ is contained in the bi-polar of $\Phi(\bar{A}_1{}^+)$. However the latter set is already convex and compact, hence $\Phi(\bar{A}_1{}^+) = M_\varphi{}^+$ and (1) is proved.

If $\Phi(a)$ is a state, then since $a^{\frac{1}{2}} u_\lambda a^{\frac{1}{2}}$ converges strongly up to a for an approximative unit $\{u_\lambda\} \subset K_A{}^+$,

$$\varphi(a) = \lim \varphi(a^{\frac{1}{2}} u_\lambda a^{\frac{1}{2}}) = \lim \Phi(a)(u_\lambda) = \|\Phi(a)\| .$$

Conversely, if $\varphi(a) = 1$, the same equality proves that $\Phi(a)$ is a state, hence (2).

If $\Phi(a)$ is a trace, then for any $x \in K_A$ we have a polar decomposition $xa - ax = u|xa - ax|$, and therefore

$$\varphi(|xa - ax|) = \varphi(u^* xa - u^* ax) = \Phi(a)(u^* x - x u^*) = 0 .$$

Hence $xa = ax$ and a belongs to the center of $\bar{A}$. It follows that for any $a \in \bar{A}_1{}^+$, the smallest trace majorizing $\Phi(a)$ is $\Phi(\text{c-supp}\, a)$, which proves (3).

4. Baire operators.

For any *algebra A of operators on a Hilbert space H, let $\mathscr{B}_A{}^R$ denote the smallest monotone σ-class of operators in $B(H)$ containing A^R, that is, the smallest class containing A^R such that, for any bounded monotone sequence in the class, the weak limit is also in the class. From [10] we borrow the following facts: $\mathscr{B}_A{}^R$ is a uniformly closed Jordan algebra (a JC-algebra) in the weak closure $\bar{A}$ of A, and for any σ-finite projection

$p \in \bar{A}$ and any self-adjoint $x \in \bar{A}$ there exists $y \in \mathscr{B}_A{}^R$ such that $pxp = pyp$.

If A is a (concrete) C^* algebra of operators on a Hilbert space H, we shall refer to the elements of $\mathscr{B}_A{}^R$ as the (bounded, self-adjoint) Baire operators of A (on H). If A is an abstract C^* algebra, $\mathscr{B}_A{}^R$ will denote the set of Baire operators of A in its universal representation. In order to obtain complete consistency with the commutative case we probably ought to use the atomic representation of A rather than the universal, but for our applications the given definition is more convenient.

A C^* homomorphism of a C^* algebra A into a C^* algebra B is a self-adjoint linear map which preserves the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ on A^R. The map is necessarily positive and norm-decreasing, and since it is a homomorphism on commuting elements, it preserves spectral theory. (For further information see f.ex. [15].)

Lemma 4.1. *Let π be a C^* homomorphism between C^* algebras A and B, and let J be a JC-algebra in A^R. If $x_i \in J$, $i = 0, 1, 2$, with*

$$x_0 \leqq x_2, \quad \pi(x_0) \leqq \pi(x_1) \leqq \pi(x_2) ,$$

then there exists $x \in J$ such that

$$x_0 \leqq x \leqq x_2, \quad \pi(x) = \pi(x_1) .$$

Proof. We may obviously assume $x_0 = 0$. Define

$$y = x_2 - |x_2 - x_1| \leqq x_2 .$$

Since J is uniformly closed, we have $y_+ \in J$ and $y_- \in J$, and since π is homomorphic on commuting elements, we have

$$\pi(y_+) = \pi(y) = \pi(x_1), \quad \pi(y_-) = 0 .$$

We now repeat the proof of [14, Proposition 5], and define

$$x = \lim x_2^{\frac{1}{2}} (x_2 + y_-)^{\frac{1}{2}} (n^{-1} + x_2 + y_-)^{-1} y_+ (n^{-1} + x_2 + y_-)^{-1} (x_2 + y_-)^{\frac{1}{2}} x_2^{\frac{1}{2}} .$$

Since

$$aba = 2\big((a \circ b) \circ a - b \circ a^2\big) ,$$

we get $x \in J$, $0 \leqq x \leqq x_2$, and

$$\pi(x) = \lim \pi(x_2)\big(n^{-1} + \pi(x_2)\big)^{-1} \pi(x_1) \big(n^{-1} + \pi(x_2)\big)^{-1} \pi(x_2) = \pi(x_1) .$$

Proposition 4.2. *Let A and B be concrete C^* algebras with weak closures $\bar{A}$ and $\bar{B}$, respectively. If π is a normal C^* homomorphism of $\bar{A}$ into $\bar{B}$ taking A onto B, then $\pi(\mathscr{B}_A{}^R) = \mathscr{B}_B{}^R$.*

PROOF. Obviously $\pi(\mathscr{B}_A{}^R) \subset \mathscr{B}_B{}^R$. On the other hand $B \subset \pi(\mathscr{B}_A{}^R)$, hence it suffices to show that $\pi(\mathscr{B}_A{}^R)$ is a monotone class. Suppose

$$\{x_n\} \subset \mathscr{B}_A{}^R, \quad \pi(x_n) \nearrow y \in \bar{B} .$$

We may assume $y \leqq 1$, and since π preserves spectral theory and $\mathscr{B}_A{}^R$ is uniformly closed, we may assume $x_n \leqq 1$ for all n. Now define $p = \bigvee [x_n] \in \mathscr{B}_A{}^R$ with $x_n \leqq p$ for all n.

Put $z_1 = x_1$. By lemma 4.1 we can find $z_2 \in \mathscr{B}_A{}^R$ such that

$$z_1 \leqq z_2 \leqq p, \quad \pi(z_2) = \pi(x_2) .$$

By repetition of this procedure we get a sequence $\{z_n\} \subset \mathscr{B}_A{}^R$ which is monotone (increasing) and bounded above by p and hence converges to an element $x \in \mathscr{B}_A{}^R$. Since π is normal, we have

$$\pi(x) = \lim \pi(z_n) = \lim \pi(x_n) = y .$$

PROPOSITION 4.3. *If φ is a faithful, positive invariant C^* integral which represents A, then, for any positive x in the weak closure $\bar{A}$ of A,*

$$\varphi(x) < \infty \quad \textit{implies} \quad x \in \mathscr{B}_A{}^R .$$

PROOF. There exists a sequence $\{u_n\} \subset K_A{}^+$ such that

$$0 \leqq u_n \leqq 1, \quad \varphi(x) - n^{-1} \leqq \varphi(xu_n) \leqq \varphi(x) .$$

Define $p = \bigvee [u_n] \in \mathscr{B}_A{}^R$. To show that p is σ-finite we use that for any set $\{p_i\}$ of mutually orthogonal projections from $\bar{A}$

$$\sum p_i = p \Rightarrow \sum u_n{}^{\frac{1}{2}} p_i u_n{}^{\frac{1}{2}} = u_n \Rightarrow \sum \varphi(u_n{}^{\frac{1}{2}} p_i u_n{}^{\frac{1}{2}}) < \infty .$$

Hence, for each n, $\varphi(u_n p_i) = 0$ for all i but a countable number. Hence $\varphi(u_n p_i) = 0$ for all n and for all i but a countable number. But since φ is faithful on $\bar{A}^+$, this gives for all n

$$p_i \leqq 1 - [u_n] \Leftrightarrow p_i \leqq 1 - p \Rightarrow p_i = 0$$

for all i but a countable number.

By the results from [10] quoted above there exists $y \in \mathscr{B}_A{}^R$ such that $pxp = pyp$. Since $\mathscr{B}_A{}^R$ is a Jordan algebra $pyp \in \mathscr{B}_A{}^R$ and

$$\varphi((1-p)x) \leqq \varphi(x - x^{\frac{1}{2}} u_n x^{\frac{1}{2}}) \leqq n^{-1} ,$$

hence $pxp = x$ and the proposition follows.

THEOREM 4.4. *For any positive invariant C^* integral φ and any positive bounded functional $f \leqq \varphi$ there exists an element $b \in \mathscr{B}_A{}^+$ unique up to φ-equivalence such that for all $x \in A$*

$$f(x) = \varphi(bx) .$$

PROOF. By theorem 3.8 there exists a unique element $a \in \overline{\pi_\varphi(A)_1}^+$ such that $f(x)=\varphi(ax)$ for all $x \in A$. By proposition 4.3 a is a Baire operator relative to $\pi_\varphi(A)$ and since by [6, Proposition 12.1.5] π_φ extends to a normal representation of A'', by proposition 4.2 we can find $b \in \mathscr{B}_A{}^+$ such that $\pi_\varphi(b)=a$. If φ denotes also the normal extension of φ from A^+ to A''^+, then with abuse of notation we have $\varphi(bx)=\varphi(ax)=f(x)$ for all $x \in A$. If b' is another element of A''^+ satisfying the conditions of the theorem, then by the uniqueness assertion in theorem 3.8 we have $\pi_\varphi(b')=\pi_\varphi(b)$, hence

$$\varphi(|b-b'|) = \varphi(\pi_\varphi(|b-b'|)) = 0 .$$

It follows that b is determined up to φ-equivalence and the theorem is proved.

LEMMA 4.5. *For any $x \in \mathscr{B}_A{}^R$ there exists a separable C^* subalgebra $B \subset A$ such that $x \in \mathscr{B}_B{}^R$.*

PROOF. The class of operators satisfying the lemma contains A^R and is clearly monotone, hence contains $\mathscr{B}_A{}^R$. Notice that $\mathscr{B}_B{}^R$ is taken in the universal representation of A.

PROPOSITION 4.6. *For any $x \in \mathscr{B}_A{}^R$ and any (real) bounded Baire function f on $\hat{A}$ there exists $y \in \mathscr{B}_A{}^R$ such that for any $\pi \in \hat{A}$*

$$\pi(y) = f(\pi)\,\pi(x) .$$

PROOF. Suppose $x \in A^+$ and f (real) bounded and continuous on $\hat{A}$. Since the continuous functions on $\hat{A}$ and $\operatorname{Prim} A$ are the same, there exists by the result in [4] (see also a proof in [7]) an element $y \in A$ such that for any $\pi \in \hat{A}$

$$y-f(\pi)x \in \ker \pi ,$$

that is, $\pi(y)=f(\pi)\pi(x)$.

If $\{f_n\}$ is a bounded monotone sequence of functions satisfying the condition relative to x and converging pointwise to the bounded function f, then there exists a sequence $\{y_n\} \subset \mathscr{B}_A{}^R$ such that for all $\pi \in \hat{A}$

$$\pi(y_n) = f_n(\pi)\pi(x), \quad \|y_n\| \leqq \|f_n\|\,\|x\| .$$

It follows that $\{y_n\}$ is a bounded monotone sequence, hence it converges weakly to an element $y \in \mathscr{B}_A{}^R$. We have

$$\pi(y) = \lim \pi(y_n) = \lim f_n(\pi)\pi(x) = f(\pi)\pi(x) .$$

This shows that the class of functions satisfying the condition relative to some fixed $x \in A^+$ is a monotone class containing the bounded continuous functions, hence it contains the bounded Baire functions.

From this we see that the class of operators satisfying the condition for any positive bounded Baire function f contains A^R. If $\{x_n\}$ is a bounded monotone sequence of self-adjoint operators satisfying the condition relative to f, and converging weakly to the operator x, then there exists a sequence $\{y_n\} \subset \mathscr{B}_A{}^R$ such that for all $\pi \in \hat{A}$

$$\pi(y_n) = f(\pi)\pi(x_n), \quad \|y_n\| \leqq \|f\| \, \|x_n\| .$$

It follows that $\{y_n\}$ is a bounded monotone sequence, hence converges weakly to an element $y \in \mathscr{B}_A{}^R$. We have

$$\pi(y) = \lim \pi(y_n) = \lim f(\pi)\pi(x_n) = f(\pi)\pi(x) .$$

This shows that the class of operators satisfying the condition for any positive and hence, by decomposition, for any real bounded Baire function, is a monotone class containing A^R. Hence it contains $\mathscr{B}_A{}^R$, and the proposition follows.

5. Functionals on an algebra of type I.

PROPOSITION 5.1. *If A is a C^* algebra with continuous trace, then for any $x \in \mathscr{B}_A{}^R$ and $y \in K_A$, the function $\pi \to \operatorname{tr}\pi(xy)$ is a bounded Baire function on $\hat{A}$.*

PROOF. Since any $\pi \in \hat{A}$ extends to a normal representation of the enveloping von Neumann algebra, the function is well defined and bounded since

$$|\operatorname{tr}\pi(xy)| \leqq \|x\| \sup_{\hat{A}} \operatorname{tr}\pi(|y|) .$$

Assuming $y \geqq 0$ it is immediate that the class of operators satisfying the proposition is monotone, and since any $z \in K_A{}^R$ has a trace function in $K^R(\hat{A})$ by [12, Theorem 1.5], the class contains $K_A{}^R$, hence contains $\mathscr{B}_A{}^R$.

PROPOSITION 5.2. *If A is a C^* algebra with continuous trace, then for any $x \in \mathscr{B}_A{}^+$ and any $n \in \mathsf{N}$, the function $\pi \to n \wedge \operatorname{tr}\pi(x)$ is a Baire function on $\hat{A}$.*

PROOF. By lemma 4.5 there exists a separable C^* subalgebra $B \subset A$ such that $x \in \mathscr{B}_B{}^R$. Then a countable approximate identity $\{u_n\} \subset K_B{}^+ \subset K_A{}^+$ for B converges strongly up to a projection p, where $(1-p)B = 0$. Hence also $(1-p)x = 0$. It follows by proposition 4.5 that the Baire functions $\pi \to \operatorname{tr}\pi(xu_n)$ increase to the extended function $\pi \to \operatorname{tr}\pi(x)$, and the proposition follows.

For any C^* algebra A define

$$J_1 = \{x \in K_A{}^+ \mid \sup_{\hat{A}} \operatorname{tr}\pi(x) \leqq 1\} .$$

J_1 is hereditary-convex and strongly invariant, and for C^* algebras with continuous trace, it generates $K_A{}^+$ and the corresponding trace functions are all of $K_1{}^+(\hat{A})$ by [5, Lemme 23] combined with [12, Theorem 1.5].

PROPOSITION 5.3. *If A is a separable C^* algebra with continuous trace, then for any $x \in \mathscr{B}_A{}^R$ the function $\pi \to \|\pi(x)\|$ is a bounded Baire function on $\hat{A}$.*

PROOF. A change of x to $|x|$ does not affect the function, hence we may assume $x \geqq 0$. For any $\pi_0 \in \hat{A}$ and any $\varepsilon > 0$ there exists $y \in K_A{}^+$ such that $\pi_0(y)$ is a one-dimensional projection with $\|\pi_0(x)\| < \operatorname{tr}\pi_0(xy) + \varepsilon$. Define $f \in C(\hat{A})$ by

$$f(\pi) = (\operatorname{tr}\pi(y) \vee 1)^{-1} .$$

Then there exists $z \in A^+$ such that $\pi(z) = f(\pi)\pi(y)$ for all $\pi \in \hat{A}$. It follows that $z \in J_1$ and $\|\pi_0(x)\| < \operatorname{tr}\pi_0(xz) + \varepsilon$. Hence for any $\pi \in \hat{A}$ we have

$$\|\pi(x)\| = \sup_{J_1} \operatorname{tr}\pi(xz) .$$

Now let $\{z_n\} \subset J_1$ be a countable set which is dense in J_1. Since by lower semi-continuity of the trace

$$z_m \to z \ \Rightarrow\ \liminf \operatorname{tr}\pi(xz_m) \geqq \operatorname{tr}\pi(xz)$$

for any $\pi \in \hat{A}$, proposition 5.1 applies and the proof is complete.

In view of lemma 4.5 it is natural to ask whether there is a trick which removes the separability condition. At present the best we can do in this direction is the following

PROPOSITION 5.4. *If A is a C^* algebra with continuous trace, then for any $x \in \mathscr{B}_A{}^R$ the function $\pi \to \|\pi(x)\|$ is universally measurable on $\hat{A}$.*

PROOF. Let μ be a positive Radon measure on $\hat{A}$. We define an invariant C^* integral φ on K_A by

$$\varphi(a) = \int_{\hat{A}} \operatorname{tr}\pi(a)\, d\mu(\pi) .$$

If φ denotes also the extension of φ to a normal trace on A''^+ then a monotone class argument using proposition 5.1 shows that for $u \in K_A{}^+$ we have

$$\varphi(u^{\frac{1}{2}}bu^{\frac{1}{2}}) = \int_{\hat{A}} \operatorname{tr}\pi(u^{\frac{1}{2}}bu^{\frac{1}{2}})\, d\mu(\pi)$$

for all $b \in \mathscr{B}_A{}^R$. In particular if $b \in \mathscr{B}_A{}^+$, we can, as in proposition 5.2, choose $\{u_n\} \subset K_A{}^+$ such that $b^{\frac{1}{2}} u_n b^{\frac{1}{2}} \nearrow b$, hence, by normality of φ and σ-normality of the integral,

$$\varphi(b) = \int \operatorname{tr} \pi(b) \, d\mu(\pi) .$$

Since K_A is dense in the L_1-space of φ, which is nothing else but the pre-dual of $\pi_\varphi(A'')$, we can for any $\varepsilon > 0$ and any $x \in A''^+$ for which $\varphi(x) < \infty$ find $y \in K_A{}^+$ such that $\varphi(|x-y|) < \varepsilon$. In particular, if $x \in \mathscr{B}_A{}^+$, we have

$$\varphi(|x-y|) = \int_{\hat{A}} \operatorname{tr} \pi(|x-y|) \, d\mu(\pi) < \varepsilon .$$

Since $y \in K_A{}^+$, the function $\pi \to \|\pi(y)\|$ belongs to $K(\hat{A})$ and since we have the inequalities

$$\big| \, \|\pi(x)\| - \|\pi(y)\| \, \big| \leqq \|\pi(x-y)\| \leqq \operatorname{tr} \pi(|x-y|) ,$$

we conclude that the function $\pi \to \|\pi(x)\|$ is integrable.

For any $x \in \mathscr{B}_A{}^+$ we can find $\{u_n\} \in K_A{}^+$ such that $x^{\frac{1}{2}} u_n x^{\frac{1}{2}} = x_n \nearrow x$. We have $\varphi(x_n) < \infty$ and hence the function $\pi \to \|\pi(x_n)\|$ is integrable. But since $\|\pi(x_n)\| \nearrow \|\pi(x)\|$ for any $\pi \in \hat{A}$, we conclude that the function $\pi \to \|\pi(x)\|$ is measurable and the proposition follows.

Lemma 5.5. *If A is a C^* algebra with continuous trace then for any $x \in \mathscr{B}_A{}^+$ and any positive Radon measure μ on $\hat{A}$*

$$\int_{\hat{A}} \|\pi(x)\| \, d\mu(\pi) = \sup_{J_1} \int_{\hat{A}} \operatorname{tr} \pi(xz) \, d\mu(\pi) .$$

Proof. If $x \in K_A{}^+$, then for any $\varepsilon > 0$ there exists for each $\pi \in \hat{A}$ a $z \in J_1$ such that

$$\|\pi(x)\| < \operatorname{tr} \pi(xz) + \varepsilon .$$

Since A has continuous trace, this inequality holds in a neighbourhood O of π. As $x \in K_A$ the function $\pi \to \|\pi(x)\|$ vanishes outside a compact set $C \subset \hat{A}$.

Choose a finite covering $\{O_n\}$ of C and let $\{f_n\}$ be a partition of the unit on C with respect to this covering. Corresponding to each O_n we have a $z_n \in J_1$ and we define

$$z = \sum f_n \cdot z_n \in K_A{}^+ .$$

We have

$$\operatorname{tr} \pi(z) \leqq \sum f_n(\pi) \leqq 1$$

for all $\pi \in \hat{A}$, hence $z \in J_1$. Moreover

$$\operatorname{tr}\pi(xz) > \sum f_n(\pi)(\|\pi(x)\| - \varepsilon) = \|\pi(x)\| - \varepsilon$$

for all $\pi \in C$.

Since this shows that we can approximate the norm function uniformly from below by trace functions, the lemma is proved for $x \in K_A{}^+$. When $x \in \mathscr{B}_A{}^+$, we proceed by approximation in L_1 by elements from $K_A{}^+$ exactly as in the previous proposition.

THEOREM 5.6. *If A is a C^* algebra with continuous trace, and if f is a positive bounded functional on A, then there exists a unique bounded measure μ on $\hat{A}$ and an operator $b \in \mathscr{B}_A{}^+$ unique up to μ-equivalence, such that for all $x \in A$*

$$f(x) = \int_{\hat{A}} \operatorname{tr}\pi(bx)\, d\mu(\pi) ,$$

$$\|\mu\| = \|f\|, \qquad 1 \geqq \|b\| = \|\pi(b)\| \text{ a.e. } [\mu] .$$

PROOF. Let φ be the minimal invariant C^* integral majorizing f defined in proposition 3.7. By theorem 3.8 there is a unique element $a \in \overline{\pi_\varphi(A)_1{}^+}$, where π_φ denotes the representation associated with φ, such that $f(x) = \varphi(ax)$ for all $x \in A$. Since c-supp$\,a = 1$, we have in particular $\|a\| = 1$, and since $\varphi(a) = \|f\| < \infty$, we infer from proposition 4.3 that a is a Baire operator relative to $\pi_\varphi(A)$. Since any representation of A extends to a normal representation of the enveloping von Neumann algebra, we can by proposition 4.2 find an element $b_1 \in \mathscr{B}_A{}^R$ such that $\pi_\varphi(b_1) = a$. Since $\mathscr{B}_A{}^R$ is a uniformly closed Jordan algebra, we may assume $b_1 \geqq 0$, $\|b_1\| = 1$.

Now by [5, Théorème 1] there is a unique Radon measure μ_1 on $\hat{A}$ such that for any $x \in K_A$

$$\varphi(x) = \int_{\hat{A}} \operatorname{tr}\pi(x)\, d\mu_1(\pi) .$$

Since any function in $K_1{}^+(\hat{A})$ is the trace function of an element from J_1, we have $\|\mu_1\| = \sup_{J_1}\varphi(x)$. However J_1 is hereditary-convex and strongly invariant, hence by the construction of φ (see the proof of theorem 3.5) we have

$$\|\mu_1\| = \sup_{J_1}\varphi(x) = \sup_{J_1} f(x) \leqq \|f\| .$$

A normalization gives $\mu = \|\mu_1\|^{-1}\|f\|\mu_1$ and $b = \|\mu_1\|\|f\|^{-1}b_1$, where now $\|\mu\| = \|f\|$ and $\|b\| \leqq 1$.

By proposition 5.1 we may for any $x \in K_A{}^+$ define a functional g_x on $\mathscr{B}_A{}^R$ by

$$g_x(y) = \int_{\hat{A}} \operatorname{tr}\pi(xy)\, d\mu(\pi) .$$

For $y \in A^R$ we have $\varphi(xy) = \|b\| g_x(y)$, and since this equality is preserved under monotone limits, it holds for any $y \in \mathscr{B}_A{}^R$. In particular

$$f(x) = \|b\|^{-1} \varphi(bx) = g_x(b) = \int_{\hat{A}} \operatorname{tr}\pi(bx)\, d\mu(\pi) .$$

We notice that, by proposition 5.1, the trace function of b is measurable, and since f is bounded, the function is μ-integrable with

$$\int_{\hat{A}} \operatorname{tr}\pi(b)\, d\mu(\pi) = \|f\| .$$

By lemma 5.5 we have

$$\begin{aligned}\int_{\hat{A}} \|\pi(b)\|\, d\mu(\pi) &= \sup_{J_1} \int_{\hat{A}} \operatorname{tr}\pi(bz)\, d\mu(\pi) \\ &= \sup_{J_1} f(z) = \|\mu_1\| = \|b\|\, \|\mu\| ,\end{aligned}$$

hence $\|\pi(b)\| = \|b\|$ a.e. $[\mu]$.

To prove uniqueness of the decomposition let f be a functional defined by a positive bounded Radon measure ν on $\hat{A}$ with $\|\nu\| = \|f\|$ and an operator $d \in \mathscr{B}_A{}^+$ such that $\|\pi(d)\| = \|d\|$ a.e. $[\nu]$. By the result above there exists a decomposition μ, b of f such that the invariant C^* integral defined by $\|b\|\mu$ is the minimal trace majorizing f. Since f is clearly majorized by the trace defined by $\|d\|\nu$ and since the correspondence between invariant C^* integrals on A and Radon measures on $\hat{A}$ is one-to-one, we conclude that $\|b\|\mu \leqq \|d\|\nu$. However by lemma 5.5

$$\begin{aligned}\|b\|\, \|\mu\| &= \sup_{J_1} f(z) = \sup_{J_1} \int_{\hat{A}} \operatorname{tr}\pi(dz) d\nu(\pi) \\ &= \int_{\hat{A}} \|\pi(d)\| d\nu(\pi) = \|d\|\, \|\nu\| .\end{aligned}$$

Hence $\|b\|\mu = \|d\|\nu$, and since $\|\mu\| = \|\nu\| = \|f\|$, we have $\mu = \nu$.

If φ is the trace defined by μ, then $f = \varphi(b\,\cdot) = \varphi(d\,\cdot)$ hence by theorem 4.4

$$\int_{\hat{A}} \operatorname{tr}\pi(|b-d|)\, d\mu(\pi) = \varphi(|b-d|) = 0 ,$$

that is, $\pi(b) = \pi(d)$ a.e. $[\mu]$ and the theorem follows.

THEOREM 5.7. *If A is a separable C^* algebra of type* I, *then the results of theorem* 5.6 *hold for any positive bounded functional on A.*

PROOF. By [6, Théorème 4.5.5] A has an ascending chain of ideals I_α such that each I_α is the closure of the union of the preceding I_β if α is a limit ordinal, and $A_\alpha = I_\alpha / I_{\alpha-1}$ has continuous trace otherwise. Since A is separable, the chain is at most countable. To each I_α corresponds a central projection $p_\alpha \in A''$ such that $I_\alpha = p_\alpha A'' \cap A$ and p_α is upper strong limit of elements from I_α. Since each I_α is separable, we have $p_\alpha \in \mathscr{B}_A{}^R$. It follows that we have $I_\alpha{}^R = p_\alpha \mathscr{B}_A{}^R \cap A^R$ and hence to within isomorphism

$$\mathscr{B}_{I_\alpha}^R = p_\alpha \mathscr{B}_A{}^R$$

which implies

$$\mathscr{B}_{A_\alpha}^R = (p_\alpha - p_{\alpha-1}) \mathscr{B}_A{}^R .$$

Now let f be a positive bounded functional on A. If f denotes also the canonical extension to A'', then we can define functionals f_α on A by $f_\alpha(x) = f((p_\alpha - p_{\alpha-1})x)$. Each f_α may, in a canonical way, be regarded as a functional on A_α. Hence by theorem 5.6 there exist μ_α on $\hat{A}_\alpha$ and $b_\alpha \in \mathscr{B}_{A_\alpha}^+$ which decompose f_α. Since $\hat{A}_\alpha$ is homeomorphic with $\hat{I}_\alpha \backslash \hat{I}_{\alpha-1}$, we may regard μ_α as a Radon measure on $\hat{A}$, and by the above mentioned isomorphism we may regard b_α as an element of $(p_\alpha - p_{\alpha-1}) \mathscr{B}_A{}^+$. Define

$$\delta = \|f\|^{-1} \sum \|b_\alpha\| \, \|\mu_\alpha\| \leqq 1 ,$$
$$\mu = \delta^{-1} \sum \|b_\alpha\| \mu_\alpha ,$$
$$b = \delta \sum \|b_\alpha\|^{-1} b_\alpha .$$

Then μ is a Radon measure on $\hat{A}$ with $\|\mu\| = \|f\|$ and $b \in \mathscr{B}_A{}^+$ with $\|\pi(b)\| = \delta$ a.e. $[\mu]$. Moreover for any $x \in \mathscr{B}_A{}^+$

$$\int_{\hat{A}} \operatorname{tr} \pi(bx) \, d\mu(\pi) = \sum \int_{\hat{A}_\alpha} \operatorname{tr} \pi(b_\alpha x) \, d\mu_\alpha(\pi) = \sum f_\alpha(x) = f(x) .$$

If ν, d is another decomposition of f, let ν_α be the restriction of ν to $\hat{A}_\alpha$ and put $d_\alpha = (p_\alpha - p_{\alpha-1}) d$. Then for any $x \in A$

$$f_\alpha(x) = \int_{\hat{A}_\alpha} \operatorname{tr} \pi(d_\alpha x) \, d\nu_\alpha(\pi) .$$

Hence, by the uniqueness assertion in theorem 5.6, we have $\|d_\alpha\| \nu_\alpha = \|b_\alpha\| \mu_\alpha$ and $\|\nu_\alpha\| \pi(d_\alpha) = \|\mu_\alpha\| \pi(b_\alpha)$ a.e. $[\mu_\alpha]$. By the construction of d_α we have $\|\pi(d_\alpha)\| = \|d\|$ a.e. $[\nu_\alpha]$, and summing up we get

$$\|d\| \nu = \|d\| \sum \nu_\alpha = \sum \|b_\alpha\| \mu_\alpha = \delta \mu .$$

Since $\|\nu\| = \|\mu\| = \|f\|$, this gives $\|d\| = \delta$ and $\nu = \mu$. Then $\nu_\alpha = \delta^{-1}\|b_\alpha\|\mu_\alpha$. Hence $\pi(d) = \pi(b)$ a.e. $[\mu_\alpha]$ for every α, which gives $\pi(d) = \pi(b)$ a.e. $[\mu]$, and the proof is complete.

To see that a decomposition $f \sim (\mu, b)$ with $b \in \mathscr{B}_A{}^R$ is not necessarily possible when A is not separable, consider the following simple example: Let A be the C^* algebra consisting of the compact operators on a non-separable Hilbert space H with an identity adjoined. The enveloping von Neumann algebra of A can be faithfully represented on $H \oplus \mathbf{C}$, defining $x + \alpha \in A$ on $(\xi, \beta) \in H \oplus \mathbf{C}$ by $(x\xi + \alpha\xi, \alpha\beta)$. Clearly $\mathscr{B}_A{}^R$ consists of operators of the form $x + \alpha$ where x is an operator on H with countable dimensional range, whereas A'' contains every operator in $B(H) \oplus \mathbf{C}$. The two points π_0 and π_1 of $\hat{A}$ correspond to the projections p_0 and p_1 on $\mathbf{C}$ and H, respectively, and we have $p_0 + p_1 = 1$ but $p_0 \notin \mathscr{B}_A{}^R$. If g_0 and g_1 are the functionals on A defined by

$$g_0(x+\alpha) = \alpha, \quad g_1(x+\alpha) = \big((x+\alpha)\xi \mid \xi\big),$$

where ξ is a unit vector in H, then the measure in the decomposition of $f = g_0 + g_1$ is the sum of two point measures on π_0 and π_1, respectively. If we had a decomposition with $b \in \mathscr{B}_A{}^R$, then $b = \beta + b_1$, where $b_1 \in B(H)$ and b_1 had countable dimensional range. Then for any $x + \alpha \in A$

$$f(x+\alpha) = \operatorname{tr}\big(p_0(\beta + b_1)(\alpha + x)\big) + \operatorname{tr}\big(p_1(\beta + b_1)(\alpha + x)\big).$$

The last term of the sum forces $\beta = 0$ since f is finite, but then the first term is zero, a contradiction since μ is not concentrated on π_1.

It is highly tempting in this situation to draw the parallel to (commutative) measure theory, where we meet similar situations, functions which are measurable but not Baire functions when the spaces are too large. Copying the commutative procedure we may therefore define the Borel operators (relative to a C^* algebra A) as the elements of the smallest (sequential) monotone class of operators containing every difference between limits of increasing bounded nets of operators from A^R. This class will be a JC-algebra. Since we eventually hope to carry out a decomposition theory for C^* integrals, and since a C^* integral can be realized as a not necessarily countable sum of bounded functionals, we may be forced to use the concept of Borel operators.

REFERENCES

1. F. Combes, *Étude des représentations tracées d'une C* algèbre*, C. R. Acad. Sci. Paris 262 (1966), 114–117.
2. F. Combes, *Étude des poids définis sur une C* algèbre*, C. R. Acad. Sci. Paris 265 (1967), 340–343.

3. F. Combes, *Poids sur une C^* algèbre*, J. Math. pures et appl. 47 (1968), 57–100.
4. J. Dauns and K. H. Hofmann, *Representations of rings by continuous sections* (Mem. Amer. Math. Soc. 83), Amer. Math. Soc., Providence, R. I., 1968.
5. J. Dixmier, *Traces sur les C^* algèbres*, Ann. Inst. Fourier (Grenoble) 13 (1963), 219–262.
6. J. Dixmier, *Les C^* algèbres et leurs représentations*, Paris, 1964.
7. J. Dixmier, *Ideal center of a C^* algebra*, Duke Math. J. 35 (1968) 375–382.
8. H. Dye, *The Radon–Nikodym theorem for finite rings of operators*, Trans. Amer. Math. Soc. 72 (1952), 243–280.
9. E. G. Effros, *Order ideals in a C^* algebra and its dual*, Duke Math. J. 30 (1963), 391–412.
10. R. V. Kadison, *Unitary invariants for representations of operator algebras*, Ann. of Math. 66 (1957), 304–379.
11. L. H. Loomis, *An introduction to abstract harmonic analysis*, Princeton, N.J., 1953.
12. G. Kjærgård Pedersen, *Measure theory for C^* algebras*, Math. Scand. 19 (1966), 131–145.
13. G. Kjærgård Pedersen, *Measure theory for C^* algebras* II, Math. Scand. 22 (1968), 63–74.
14. G. Kjærgård Pedersen, *A decomposition theorem for C^* algebras*, Math. Scand. 22 (1968), 266–268.
15. E. Størmer, *On the Jordan structure of C^* algebras*, Trans. Amer. Math. Soc. 120 (1965), 438–447.
16. E. Størmer, *Two-sided ideals in C^* algebras*, Bull. Amer. Math. Soc. 73 (1967), 254–257.
17. E. Thoma, *Über unitäre Darstellungen abzählbarer diskreter Gruppen*, Math. Ann. 153 (1964), 111–138.

UNIVERSITY OF COPENHAGEN, DENMARK