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NOTE ON A PAPER BY STETKÆR-HANSEN CONCERNING ESSENTIAL SELFADJOINTNESS OF SCHROEDINGER OPERATORS

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Introduction.

By G we denote an open set in R_n, by $(u,v)=\int_G u\bar{v}\,dx$ the scalar product defined in the Hilbert space $L^2(G)$. By $H_{2,\mathrm{loc}}(G)$ we denote the space of all functions which are defined in G and possess locally square integrable derivatives up to the second order, and $Q_{\alpha,\mathrm{loc}}(G)$ is the set of all functions satisfying in G a local Stummel condition. (A description of this condition can be found, for example, in [5] and [3]. Atomic Coulomb potentials are included in $Q_{\alpha,\mathrm{loc}}$.) Let

$$a_{jk}(x) \in C^2(G), \qquad b_j(x) \in C^1(G), \qquad q(x) \in Q_{\alpha,\mathrm{loc}}(G)$$

be realvalued functions and (a_{ik}) a positive definite symmetric matrix. If we denote by A the symmetric operator defined in $C_0{}^2(G)$ by the differential expression

$$Du = \sum_{i,k=1}^{n} D_j a_{jk} D_k u + qu, \qquad D_j = i\frac{\partial}{\partial x_j} + b_j,$$

it is known (cf. [2], [3]) that the adjoint operator A^* has the domain of definition

$$(1) \qquad \mathfrak{D}(A^*) = \{u \mid u \in L^2(G) \cap H_{2,\mathrm{loc}}(G),\, Du \in L^2(G)\}.$$

Now choose nonnegative lipschitzean functions $\varrho(x)$ and $\sigma(x)$ with the following properties in G:

$$(2) \qquad \sum a_{jk}\varrho_{x_j}\varrho_{x_k} \leqq 1 \quad \text{a.e.},$$

$$(3) \qquad \sum a_{jk}\sigma_{x_j}\sigma_{x_k} \leqq e^{2\sigma} \quad \text{a.e.},$$

$$(4) \qquad \lim_{x\to\partial G}\{\varrho(x)+\sigma(x)\} = \infty\,. \quad {}^{*)}$$

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*) The condition (2) was first used by Jörgens [3], the conditions (3) and (4) are due to the author [7] resp. [8].

If (2) and (3) hold with $\varphi^2(\varrho)$ and $\psi^2(\sigma)$ respectively instead of 1 and $e^{2\sigma}$ respectively at the right side and if

THEOREM. *If δ is a positive number and*

$$(Au,u) \geqq (1+\delta)(e^{2\sigma}u,u) \quad \textit{for all } u \in C_0^2(G), \tag{5}$$

then A in $C_0^2(G)$ is essentially selfadjoint.

Clearly $\sigma(x)$ has to be chosen as small as possible to make condition (5) less restrictive.

The proof of our theorem (which in the case $\sigma \equiv 0$ is due to Stetkær-Hansen [4]) is a suitable generalization of a proof of Wienholtz [9]. In the case $\varrho \equiv 0$ Triebel [6] deduces a special result in a similar way.

Proof of the theorem.

Since A is bounded from below by 1, it is sufficient to show that $h \in L^2(G)$ and $(h, Au) = 0$ for all $u \in C_0^2(G)$ imply $h = \Theta$ (Θ denoting the zero element of $L^2(G)$); cf. [1, p. 159]. Making essential use of (1), we deduce from (5) in the same way as in [9, p. 60] and [4] that

$$\int_G |h|^2 (\sum a_{jk} \gamma_{x_j} \gamma_{x_k})\, dx \geqq (1+\delta) \int_G |h|^2 e^{2\sigma} \gamma^2\, dx \tag{6}$$

holds for all lipschitzean functions $\gamma(x)$ with a compact support in G.

Let $f(t)$, $g(t)$ be functions defined in $[0, \infty)$ with piecewise continuous first derivatives and compact support. Following an idea of Jörgens [3], we put $\gamma(x) = f(\varrho(x))g(\sigma(x))$. Because of (4), $\gamma(x)$ has compact support in G. We insert this γ into (6) and arrive at the inequality

$$\begin{aligned} (1+\varepsilon) \int_G |h|^2 e^{2\sigma} f^2 (g')^2\, dx + (1+1/\varepsilon) \int_G |h|^2 (f')^2 g^2\, dx \\ \geqq \tfrac{1}{2}\delta \int_G |h|^2 f^2 g^2\, dx + (1+\tfrac{1}{2}\delta) \int_G |h|^2 e^{2\sigma} f^2 g^2\, dx \end{aligned} \tag{7}$$

(for any $\varepsilon > 0$) by using (2), (3) and some easy estimates.

We now choose

$$f(t) = \begin{cases} 1 & \text{for } 0 \leqq t \leqq R, \\ \text{linear} & \text{for } R \leqq t \leqq R+1, \\ 0 & \text{for } R+1 \leqq t, \end{cases} \qquad g(t) = \begin{cases} e^{-t} - \alpha^{-1} e^{-\alpha t} & \text{for } 0 \leqq t \leqq \alpha, \\ 0 & \text{for } \alpha \leqq t. \end{cases}$$

It follows that

$$\int^{\infty} dt/\varphi(t) = \infty \quad \text{and} \quad \int^{\infty} dt/\psi(t) < \infty,$$

the new functions

$$r(x) = \int^{\varrho(x)} dt/\varphi(t) \quad \text{and} \quad s(x) = \operatorname{Max}\left\{0; -\log \int_{\sigma(x)}^{\infty} dt/\psi(t)\right\}$$

satisfy (2) and (3) respectively; cf. [4], [7].

$$f' \equiv 0 \quad \text{for } t \notin [R, R+1], \qquad |f'| \equiv 1 \quad \text{for } t \in (R, R+1),$$
$$f \leqq 1, \qquad g \leqq e^{-t}, \qquad |g'| \leqq (1+1/\alpha)e^{-t},$$

and $g(t)$ converges uniformly to e^{-t} as $\alpha \to \infty$. Inserting this into (7), taking into consideration that $h \in L^2(G)$ and letting $\alpha \to \infty$, we finally get the inequality

$$(1+\varepsilon)\int_G |h|^2 f^2\,dx + (1+1/\varepsilon)\int_{R \leqq \varrho \leqq R+1} |h|^2 e^{-2\sigma}\,dx$$
$$\geqq \tfrac{1}{2}\delta \int_{\varrho \leqq R} |h|^2 e^{-2\sigma}\,dx + (1+\tfrac{1}{2}\delta)\int_G |h|^2 f^2\,dx\,.$$

For $\varepsilon < \frac{1}{2}\delta$, $h \neq \Theta$ and R sufficiently large this is a contradiction.

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